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A CONTRACTION PRINCIPLE IN WEAKLY CAUCHY NORMED SPACES

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Abstract. In this paper we establish a general contraction principle in any normed space X which is weakly Cauchy (see [1], [5] and [6]) but not necessarily complete and (possibly) not reflexive. Without using continuity (see Thorem 2.3 below), we prove the existence of a unique common fixed point for two weakly compatible self-mappings of a closed convex subset satisfying a contraction condition. In particular, our result generalizes and improves the main result of [5].

1. Introduction

Fixed point theory plays an important role in many disciplines, including variational and linear inequalities, optimization and has many applications in the field of approximation theory and minimum norm problems.

Now, Fixed point theory has a very rich literature providing a large amount of research papers devoted to many different topics and applications.

Through this paper, X denotes a normed space equipped with a norm denoted by $\|.\|$. By $\mathbb{N}$, we designate the set of natural numbers.

Let C be subset of X. A mapping $T : C \to X$ is said to be contraction (on C) if and only if there is a nonnegative real number $\alpha < 1$ with the property that

$$\|T(x) - T(y)\| \leq \alpha \|x - y\|, \forall x, y \in C.$$

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T is said to be non-expansive if and only if

$$\|T(x) - T(y)\| \leq \|x - y\|, \forall x, y \in C.$$

The following is the well known Banach contraction principle.

Theorem 1.1. *Let X be a complete metric space and let T be a contraction of X into itself. Then T has a unique fixed point in the sense that $T(x) = x$, for some $x \in X$. Moreover, the sequence $\{T^n(y)\}$ is strongly convergent to x for every y in X.*

The Banach contraction principle is one the fundamental results in the fixed point theory. It has inspired many authors, and has seen many generalizations and extensions (see for more informations [3], [4] and [7]).

In [1] (see also [5] and [6]), the following definition was introduced.

Definition 1.2. *A normed space X is said to be a weakly Cauchy space if and only if every Cauchy sequence $\{x_n\}$ in X is weakly convergent to an element $x \in X$.*

Remark. One can observe that every reflexive Banach space is a weakly Cauchy space.

The concept of weakly Cauchy normed spaces was used in [1] to prove the existence of the best approximation for elements in a convex closed subsets in normed spaces not necessarily reflexive Banach spaces or Hilbert spaces. In [5] the same concept was used to establish the following result:

Theorem 1.3. *([5]) Let X be a weakly Cauchy normed space, C be a closed convex subset of X and T a contraction mapping from C into C. Then T has a unique fixed point $y \in C$. Moreover the sequence $\{T^n(x)\}$ is weakly convergent to y for every $x \in C$, $w - \lim_{n\to\infty} T^n(x) = y$, for every $x \in C$.*

Thus, Theorem 1.3 provides a generalization of the Banach contraction principle.

The aim of this paper is extend Theorem 1.3 to the case of f-contractions. Moreover, the proof of our main result (see Theorem 2.3) will show that in fact the sequence $\{T^n(x)\}$ involved in Theorem 1.3 converges strongly to the unique fixed point y of T. Thus our result provides an improvement (see Corollary 2.4) to the main result of [5].

2. The results

Throughout the sequel, let X be a normed space equipped with a norm denoted by $\|\cdot\|$. Let C be subset of X. Let $f : C \to C$ be a mapping.

Definition 2.1. *A mapping $T : C \to C$ is said to be an f-contraction (on C) if and only if there is a nonnegative real number $\alpha < 1$ with the property that*

$$||T(x) - T(y)|| \leq \alpha||f(x) - f(y)||, \quad \forall x, y \in C. \tag{2.1}$$

The notion of weakly compatible mappings was introduced in 1998 by G. Jungck and B.E. Rhoades in the paper [2].

Definition 2.2. *Let C be a subset of X and let $T, f : C \to C$ two self-mappings. The pair $\{T, f\}$ is said to be weakly compatible if T and f commute on their incidence points (i.e., for every $x \in C$, if $Tx = fx$ then $Tfx = fTx$).*

The main result of this paper is the following.

Theorem 2.3. *Let X be a weakly Cauchy normed space, C be a (non empty) closed convex subset of X. Let $f, T : C \to C$ be two selfmappings of C. We suppose the following assumptions:*
(A1) T is an f-contraction.
(A2) $\overline{T(C)} \subset f(C)$, (where $\overline{T(C)}$ means the closure of $T(C)$).
(A3) The pair $\{T, f\}$ is weakly compatible.
Then T and f have a unique common fixed point z in C.

Proof. Let $x \in C$. We set $u_0(x) = x$. Since $v_1(x) := Tx \in f(C)$, we can find $u_1(x) \in C$ such that $v_1(x) := Tx = fu_1(x)$. By induction, we can construct two sequences $\{u_n(x)\}$ and $\{v_n(x)\}$ (depending on x) satisfying

$$u_0(x) = x, \quad v_0(x) = f(x) \quad \text{and} \quad v_{n+1}(x) := Tu_n(x) = fu_{n+1}(x), \quad \forall n \in \mathbb{N}. \tag{2.2}$$

Since T is an f-contraction, we have the inequalities

$$||v_{n+1}(x) - v_n(x)|| \leq \alpha^n||T(x) - f(x)||, \quad \forall n \in \mathbb{N}. \tag{2.3}$$

The inequality (2.3) insures that, if $0 \leq k \leq m$, we have

$$||v_m(x) - v_k(x)|| \leq \frac{\alpha^k}{1-\alpha}||T(x) - f(x)||. \tag{2.3}$$

To simplify the notations, we set $y_n := v_n(x)$ for every non negative integer n. With this notation, (2.3) shows that the sequence $\{y_n\}$ is a Cauchy sequence in X. Since X is weakly Cauchy, the sequence $\{y_n\}$ is weakly convergent to some element (say) z in X. Since C is convex and closed, then it contains all the weak limits as well as all the strong limits of the sequence $\{y_n\}$. Hence $z \in C$. We shall prove that z is a fixed point of T. To this end, let us denote $\{v_n\}$ and $\{u_n\}$ the pair of sequences associated to the point z via the process defined by the relations (2.2).

Let j be any arbitrary natural number and define the function ϕ_j, by setting

$$\phi_j(x) := \|v_j - x\|. \tag{2.4}$$

It is obvious that ϕ_j is a proper real valued lower semi-continuous convex on C.

For any real number $r \in \mathbb{R}$, we put

$$C_{j,r} := \{y \in C : \phi_j(y) \leq r\}. \tag{2.5}$$

The set $C_{j,r}$ is a closed convex subset of C. In particular, for every number $\epsilon > 0$, the set

$$D_{j,\epsilon} := \{y \in C : \|v_j - y\| \leq \|v_j - z\| - \epsilon\}.$$

is closed and convex. Therefore the complement

$$W_{j,\epsilon} := \{y \in C : \|v_j - y\| > \|v_j - z\| - \epsilon\}.$$

of the set $D_{j,\epsilon}$ is weakly open subset of C which contains the point z. Then there is a neighborhood $N_j(z)$ of z such that $N_j(z) \subset W_{j,\epsilon}$. By using the weak convergence of the sequence $\{v_n(x)\}$ to the point z, we can find a natural number n_0 such that

$$v_n(x) \in N_j(z), \quad \forall n \geq n_0.$$

It follows that

$$\|v_j - z\| - \epsilon < \|v_j - v_n(x)\|, \quad \forall n \geq n_0.$$

We shall distinguish two cases.

The first case is when $n_0 \leq j$. For such a case, we have

$$\|v_j - z\| - \epsilon < \|v_j - v_j(x)\|,$$

by which we get

$$\|v_j - z\| - \epsilon < \|v_j - v_j(x)\| \leq \alpha^j \|f(z) - f(x)\|. \tag{2.6}$$

The second case, is when $j < n_0$. In this case, for every integer $n \geq n_0$, we have the following inequalities

$$\begin{aligned}\|v_j - v_n(x)\| &\leq \|v_j - v_j(x)\| + \|v_j(x) - v_n(x)\| \\ &\leq \alpha^j \left[\|f(z) - f(x)\| + \frac{1}{1-\alpha} \|T(x) - f(x)\|\right]. \end{aligned} \tag{2.7}$$

Using (2.6) and (2.7), we see that in both cases, taking the limit as $j \to \infty$, we obtain

$$\limsup_{j\to\infty} \|v_j - z\| \leq \epsilon.$$

Since ϵ is arbitrary, we get

$$\lim_{n\to\infty} \|v_n - z\| = 0.$$

Thus the sequence $\{v_n\}$ converges strongly to z.

Now, we show that the sequence $\{v_n(x)\}$ also converges strongly to z.

Let $j \in \mathbb{N}$ be a given integer. Let n be any integer such that $n \geq j$. Then (by using (2.7)) we have the following inequalities:

$$\|v_n(x) - z\| \leq \|v_n(x) - v_j\| + \|v_j - z\|$$
$$\leq \alpha^j \left[\|f(z) - f(x)\| + \frac{1}{1-\alpha} \|T(x) - f(x)\| \right] + \|v_j - z\| . \tag{2.8}$$

From (2.8), we obtain

$$\limsup_{n\to\infty} \|v_n(x) - z\| \leq \alpha^j \left[\|f(z) - f(x)\| + \frac{1}{1-\alpha} \|T(x) - f(x)\| \right] + \|v_j - z\| ,$$

from which we deduce that

$$\limsup_{n\to\infty} \|v_n(x) - z\| = 0,$$

which is equivalent to say that sequence $\{v_n(x)\}$ converges strongly to z.

Thus, we have proved that

$$z = \lim_{n\to\infty} T(u_n(x)) = \lim_{n\to\infty} f(u_n(x)). \tag{2.9}$$

By assumption (A2), we know that $z \in f(C)$. Therefore, we can write $z = f(w)$ for some $w \in C$. Since T is an f-contraction, we have

$$\|T(u_n(x)) - T(w)\| \leq \alpha \|f(u_n(x)) - z\| . \tag{2.10}$$

By letting $n \to \infty$ in (2.10), we get $z = Tw = fw$. Since T and f are weakly compatible, we obtain $fz = Tz$. Since T is an f-contraction, for all nonnegative integer n, we have

$$\|T(z) - T(u_n(x))\| \leq \alpha \|f(z) - f(u_n(x))\| . \tag{2.11}$$

By letting $n \to \infty$ in (2.11), we get

$$\|T(z) - z\| \leq \alpha \|T(z) - z\| ,$$

which implies that $Tz = z$. We conclude that z is a common fixed point for T and f. The uniqueness of z is evident. This completes the proof. □

From the proof of the Theorem 2.3, we have the following corollary.

Corollary 2.4. *Let X be a weakly Cauchy normed space, C be a closed convex subset of X and T a contraction mapping from C into C. Then T has a unique fixed point $z \in C$. Moreover the sequence $\{T^n(x)\}$ is strongly convergent to z for every $x \in C$, i.e., $\lim_{n\to\infty} \|T^n(x) - z\| = 0$ for every x in C.*

Corollary 2.4 provides an improvement to the main result of [5].

Corollary 2.5. *Let X be a reflexive normed space, C be a closed convex subset of X and T a contraction mapping from C into C. Then T has a unique fixed point $z \in C$. Moreover the sequence $\{T^n(x)\}$ is strongly convergent to z for every $x \in C$, i.e., $\lim_{n\to\infty} \|T^n(x) - z\| = 0$ for every $x \in C$.*

Corollary 2.6. *Let X be a weakly Cauchy normed space. Let f, T be two weakly compatible mappings on X into itself such that $\overline{T(X)} \subset f(X)$. Suppose that there exists an $\alpha \in [0,1)$ such that*

$$\|Tx - Ty\| \leq \alpha \, \|fx - fy\|, \quad \forall x, y \in X.$$

Let $\mathcal{I}(T,f)$ be the collection of all closed convex subsets C of X such that $\overline{T(C)} \subset f(C) \subset C$. Then the intersection $\bigcap_{C \in \mathcal{I}(T,f)} C$ is not empty.

Proof. Using Theorem 2.3, the intersection $\bigcap_{C \in \mathcal{I}(T,f)} C$ is a set containing at least the unique common fixed point of T and f. □

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