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CONVERGENCE ANALYSIS FOR TOTAL ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN CONVEX METRIC SPACES WITH APPLICATIONS

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Abstract. The purpose of this paper is to study newly proposed finite-step iteration scheme in a convex metric space and establish some strong convergence theorems for two finite families of total asymptotically nonexpansive mappings. Also, we give some applications of our result. The results presented in this paper extend and generalize several results from the current existing literature.

1. Introduction and Preliminaries

Most of the problems in various disciplines of science are nonlinear in nature, whereas fixed point theory explored in the setting of Banach spaces mainly depends on the linear structure of the underlying space. One of the nonlinear framework for fixed point theory is a metric space embedded with a convex structure.

Definition 1.1. ([40]) Let (X,d) be a metric space. A mapping $W\colon X \times X \times [0,1] \to X$ is said to be a convex structure on X if for each $(x,y,\lambda) \in$

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$X \times X \times [0,1]$ and $u \in X$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1-\lambda) d(u, y).$$

A metric space X together with the convex structure W is called a *convex metric space*, which is denoted by (X, d, W).

Definition 1.2. ([40]) Let X be a convex metric space. A nonempty subset F of X is said to be convex if $W(x, y, \lambda) \in F$ whenever $(x, y, \lambda) \in F \times F \times [0,1]$.

Takahashi [40] has shown that open sphere $B(x, r) = \{y \in X : d(y, x) < r\}$ and closed sphere $B[x, r] = \{y \in X : d(y, x) \leq r\}$ are convex. Some examples of convex metric spaces are normed spaces and their convex subsets, Hadamard manifolds and CAT(0) spaces. But there are many examples of convex metric spaces which are not embedded in any normed space (see [40]).

Example 1.3. Let $(X, \|.\|)$ be a normed space. If the mapping $W \colon X \times X \times I \to X$ is defined by $W(x, y, \lambda) = \lambda x + (1-\lambda) y$ for each $x, y \in X$, $\lambda \in I$, then it is a convex structure on X.

Definition 1.4. A mapping $T \colon X \to X$ is called:

(1) nonexpansive if

$$d(Tx, Ty) \leq d(x, y), \ \forall\, x, y \in X.$$

(2) asymptotically nonexpansive [15] if there exists a sequence $u_n \in [0, \infty)$ with $\lim_{n\to\infty} u_n = 0$ such that

$$d(T^n x, T^n y) \leq (1 + u_n) d(x, y), \ \forall\, x, y \in X, \forall\, n \in \mathbb{N},$$

where $\mathbb{N}$ is the set of positive integers.

(3) asymptotically nonexpansive in the intermediate sense [5] if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y\in K} \Big(d(T^n x, T^n y) - d(x, y) \Big) \leq 0.$$

(4) generalized asymptotically nonexpansive mappings [17] if there exist sequences $\{r_n\}$, $\{s_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} r_n = 0 = \lim_{n\to\infty} s_n$ such that

$$d(T^n x, T^n y) \leq (1 + r_n) d(x, y) + s_n, \ \forall\, x, y \in X, \ \forall\, n \in \mathbb{N},$$

where $\mathbb{N}$ is the set of positive integers.

In 2006, Albert et al. [3] introduced the concept of total asymptotically nonexpansive mappings as the generalization of a few classes of mappings including above defined (1) nonexpansive mappings, (2) asymptotically nonexpansive mappings, (3) asymptotically nonexpansive mappings in the intermediate sense and (4) generalized asymptotically nonexpansive mappings.

Definition 1.5. A mapping $T\colon X \to X$ is said to be total asymptotically nonexpansive if there exist non-negative real sequences $\{k_n\}$ and $\{\nu_n\}$ with $k_n \to 0$ and $\nu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi\colon [0,\infty) \to [0,\infty)$ with $\psi(0) = 0$ such that

$$d(T^n x, T^n y) \leq d(x,y) + k_n \psi(d(x,y)) + \nu_n$$

for all $x, y \in X$ and $n \in \mathbb{N}$.

In particular, if $y \in F(T) = \{x \in X : T(x) = x\}$, then the mapping T is called total asymptotically quasi-nonexpansive.

Remark 1.6. From the above definition, it is clear that each asymptotically nonexpansive mapping is a total asymptotically nonexpansive with $\nu_n = 0$, $k_n = u_n$ for all $n \geq 1$, $\psi(t) = t$, $t \geq 0$.

Example 1.7. ([28]) We can easily check the following statements:

(i) Let $X = \mathbb{R}$, $K = [0,\infty)$ and $T\colon K \to K$ be defined by $T(x) = \sin x$. Then T is a total asymptotically nonexpansive mapping.

(ii) Let $X = \mathbb{R}$, $K = [-\frac{1}{\pi}, \frac{1}{\pi}]$ and $T\colon K \to K$ be defined by $T(x) = \lambda x \sin(\frac{1}{x})$, where $\lambda \in (0,1)$. Then T is a total asymptotically nonexpansive mapping.

(iii) Let $K = \{x := (x_1, x_2, \ldots, x_n, \ldots) | x_1 \leq 0, x_i \in \mathbb{R}, i \geq 2\}$ be a nonempty subset of $X = \ell^2$ with the norm $\|.\|$ defined as $\|x\| = \sqrt{\sum_{i=1}^{\infty} x_i^2}$, if $T\colon K \to K$ is defined as $T(x) = (0, 4x_2, 0, 0, 0, \ldots)$. Then T is an asymptotically nonexpansive mapping.

(iv) Let $X = \mathbb{R}$ and $K = [0,2]$. Let $T\colon K \to K$ be a mapping defined by

$$T(x) = \begin{cases} 1, & \text{if } x \in [0,1], \\ \frac{1}{\sqrt{3}}\sqrt{4 - x^2}, & \text{if } x \in [1,2]. \end{cases}$$

Then T is a total asymptotically nonexpansive mapping with $F(T) = \{1\}$. However, T is not a Lipschitzian and hence it is not an asymptotically nonexpansive mapping.

Definition 1.8. A mapping $T\colon X \to X$ is called semi-compact if for any bounded sequence $\{x_n\}$ in K with $d(x_n, Tx_n) \to 0$ as $n \to \infty$, there is a convergent subsequence of $\{x_n\}$.

Convergence results for the mappings in (1)-(4) in the setting of uniformly convex Banach spaces and CAT(0) spaces via different iterative schemes have been obtained by a number of researchers (e.g., [6, 13, 20, 21, 22, 24, 31, 29, 30, 32, 39, 43] and the references therein).

To know the importance of different iterative algorithms for the approximation of total asymptotically nonexpansive mappings in uniformly convex Banach spaces, CAT(0) spaces and hyperbolic spaces, we refer the interested reader to [1, 2, 7, 12, 14, 16, 23, 25, 28, 33, 34, 35, 36, 37, 42, 44].

Modified Mann Iteration([38]): In 1991, Schu [38] considered the following modified Mann iteration process:

$$\begin{cases} x_1 = x \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \ n \geq 1, \end{cases} \tag{1.1}$$

where $\{\alpha_n\}$ is a real sequence in $(0,1)$.

Modified Ishikawa Iteration([41]): In 1994, Tan and Xu [41] studied the modified Ishikawa iteration process which is a generalization of the Ishikawa iteration process:

$$\begin{cases} x_1 = x \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n, \ n \geq 1, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$. This iteration scheme reduces to the modified Mann iteration process when $\beta_n = 0$ for all $n \geq 1$.

Modified Noor Iteration([43]): In 2002, Xu and Noor [43] introduced a three-step iteration scheme as follows:

$$\begin{cases} x_1 = x \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n z_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n T^n x_n, \ n \geq 1, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0,1]$.

In 2008, Khan et al. [19] generalized the iterative processes (1.1)-(1.3) to the following iterative process for a finite family of mappings $\{T_i : i = 1, 2, \dots, r\}$.

Let $x_1 \in K$ and the iterative sequence $\{x_n\}$ is defined as follows:

$$\begin{cases} x_{n+1} = (1-\alpha_{rn})x_n + \alpha_{rn} T_r^n y_{(r-1)n}, \\ y_{(r-1)n} = (1-\alpha_{(r-1)n})x_n + \alpha_{(r-1)n} T_{r-1}^n y_{(r-2)n}, \\ \quad \vdots \\ y_{2n} = (1-\alpha_{2n})x_n + \alpha_{2n} T_2^n y_{1n}, \\ y_{1n} = (1-\alpha_{1n})x_n + \alpha_{1n} T_1^n y_{0n}, \ n \geq 1, \end{cases} \tag{1.4}$$

where $y_{0n} = x_n$ for all n and $\alpha_{in} \in [0,1]$, $n \geq 1$ and $i \in \{1, 2, \dots, r\}$.

Recently, Chen and Guo [8] introduced and studied a new finite-step iteration scheme for two finite families of asymptotically nonexpansive mappings as follows:

$$\begin{cases} x_1 = x \in K, \\ \quad x_n^{(0)} = x_n, \\ \quad x_n^{(1)} = \alpha_n^{(1)} T_1^n x_n^{(0)} + (1-\alpha_n^{(1)}) S_1^n x_n, \\ \quad x_n^{(2)} = \alpha_n^{(2)} T_2^n x_n^{(1)} + (1-\alpha_n^{(2)}) S_2^n x_n, \\ \qquad \vdots \\ \quad x_n^{(N-1)} = \alpha_n^{(N-1)} T_{N-1}^n x_n^{(N-2)} + (1-\alpha_n^{(N-1)}) S_{N-1}^n x_n, \\ \quad x_n^{(N)} = \alpha_n^{(N)} T_N^n x_n^{(N-1)} + (1-\alpha_n^{(N)}) S_N^n x_n, \\ x_{n+1} = x_n^{(N)}, \ \forall\, n \geq 1, \end{cases} \tag{1.5}$$

where $\{\alpha_n^{(i)}\} \subset [0,1]$ for all $i \in \{1,2,\dots,N\}$ and they proved weak convergence theorem for iteration scheme (1.5).

We need the following lemma to prove our main results.

Lemma 1.9. ([19]) *Let $\{p_n\}$, $\{q_n\}$, $\{r_n\}$ be three sequences of nonnegative real numbers satisfying the following conditions:*

$$p_{n+1} \leq (1+q_n)p_n + r_n,\ n \geq 0, \quad \sum_{n=0}^{\infty} q_n < \infty, \quad \sum_{n=0}^{\infty} r_n < \infty.$$

Then

(i) $\lim_{n\to\infty} p_n$ *exists.*
(ii) *In addition, if* $\liminf_{n\to\infty} p_n = 0$, *then* $\lim_{n\to\infty} p_n = 0$.

The purpose of this paper is to introduce the iterative process (1.5) in convex metric spaces and establish its strong convergence to a unique common fixed point of two finite families of total asymptotically nonexpansive mappings. The results presented in this paper extend and generalize some previous works from the current existing literature in the setting of convex metric spaces.

2. Main results

First we introduce the iterative process (1.5) in convex metric spaces as follows:

Let K be a convex subset of a convex metric space (X,d) and $x_0 \in K$. Suppose that $\{\alpha_n^{(i)}\} \subset [0,1]$ for all $n = 1,2,3,\dots$ and $i \in \{1,2,\dots,N\}$. Let

$\{S_i, T_i : i = 1, 2, \dots, N\}$ be two finite families of self-mappings of K. We translate (1.5) as follows:

$$\begin{cases} x_1 = x \in K, \\ x_n^{(0)} = x_n, \\ x_n^{(1)} = W(T_1^n x_n, S_1^n x_n, \alpha_n^{(1)}), \\ x_n^{(2)} = W(T_2^n x_n^{(1)}, S_2^n x_n, \alpha_n^{(2)}), \\ \quad \vdots \\ x_n^{(N-1)} = W(T_{N-1}^n x_n^{(N-2)}, S_{N-1}^n x_n, \alpha_n^{(N-1)}), \\ x_n^{(N)} = W(T_N^n x_n^{(N-1)}, S_N^n x_n, \alpha_n^{(N)}), \\ x_{n+1} = x_n^{(N)}, \ \forall\, n \geq 1. \end{cases} \tag{2.1}$$

Remark 2.1. (1) If $W(x, y, \lambda) = (1 - \lambda)\, x + \lambda\, y$ for all $(x, y, \lambda) \in X \times X \times [0, 1]$, then the iterative process (2.1) reduces to (1.5).

(2) It is easy to verify that Lemma 1.9(ii) holds under the hypothesis $\limsup_{n\to\infty} p_n = 0$ as well. Therefore, the condition (ii) in Lemma 1.9 can be reformulated as follows:
(ii)' If $\liminf_{n\to\infty} p_n = 0$ or $\limsup_{n\to\infty} p_n = 0$, then $\lim_{n\to\infty} p_n = 0$.

Now, we prove some lemmas to prove our strong convergence results. Assume that $\mathcal{N} = \{1, 2, 3, \dots, N\}$.

Lemma 2.2. *Let K be a nonempty, closed and convex subset of a convex complete metric space (X, d, W). For each $i \in \mathcal{N}$, let $T_i \colon K \to K$ be a $(\{k_{n_1}^i\}, \{\nu_{n_1}^i\}, \{\psi_1^i\})$-total asymptotically nonexpansive mapping with $\lim_{n\to\infty} k_{n_1}^i = 0$ and $\lim_{n\to\infty} \nu_{n_1}^i = 0$, and a strictly increasing function $\psi_1^i \colon [0, +\infty) \to [0, +\infty)$ satisfying $\psi_1^i(0) = 0$ and let $S_i \colon K \to K$ be a $(\{k_{n_2}^i\}, \{\nu_{n_2}^i\}, \{\psi_2^i\})$-total asymptotically nonexpansive mapping with $\lim_{n\to\infty} k_{n_2}^i = 0$ and $\lim_{n\to\infty} \nu_{n_2}^i = 0$, and a strictly increasing function $\psi_2^i \colon [0, +\infty) \to [0, +\infty)$ satisfying $\psi_2^i(0) = 0$. Assume that*

$$F := \bigcap_{i=1}^{N} F(S_i) \cap F(T_i) \neq \emptyset,$$

and for each $i \in \mathcal{N}$, the following conditions hold:

(i) $\sum_{n=1}^{\infty} k_{n_1}^i < +\infty$, $\sum_{n=1}^{\infty} k_{n_2}^i < +\infty$, $\sum_{n=1}^{\infty} \nu_{n_1}^i < +\infty$, *and* $\sum_{n=1}^{\infty} \nu_{n_2}^i < +\infty$.

(ii) *There exist constants $\mathcal{K}_i > 0$ and $\mathcal{K}_i' > 0$ such that*

$$\psi_1^i(r) \leq \mathcal{K}_i r, \ \psi_2^i(r) \leq \mathcal{K}_i' r, \ \forall\, r > 0.$$

Let $\{x_n\}$ *be the sequence defined by* (2.1)*, where* $\{\alpha_n^{(i)}\} \subset [0,1]$ *for all* $i \in \mathcal{N}$*. Then*

(i) $\lim_{n\to\infty} d(x_n, q)$ *for all* $q \in F$ *exists.*
(ii) $\lim_{n\to\infty} d(x_n, F)$ *exists, where* $d(x, F) = \inf\{d(x, z) : z \in F\}$*.*

Proof. (i) Let $k_n = \max_{i\in\mathcal{N}}\{k^i_{n_1}, k^i_{n_2}\}$, $\nu_n = \max_{i\in\mathcal{N}}\{\nu^i_{n_1}, \nu^i_{n_2}\}$, $\psi = \max_{i\in\mathcal{N}}\{\psi^i_1, \psi^i_2\}$ and $\mathcal{K} = \max_{i\in\mathcal{N}}\{\mathcal{K}_i, \mathcal{K}'_i\}$. By conditions (i) and (ii), we have $\sum_{n=1}^{\infty} k_n < +\infty$, $\sum_{n=1}^{\infty} \nu_n < +\infty$, $\psi(r) \le \mathcal{K}r$ for any $r > 0$. For every $q \in F$ and any $n \ge 1$, it follows from (2.1) that

$$\begin{aligned}
d(x_n^{(1)}, q) &= d(W(T_1^n x_n, S_1^n x_n, \alpha_n^{(1)}), q) \\
&\le \alpha_n^{(1)} d(T_1^n x_n, q) + (1-\alpha_n^{(1)}) d(S_1^n x_n, q) \\
&\le \alpha_n^{(1)}[d(x_n, q) + k^1_{n_1}\psi^1_1(d(x_n, q)) + \nu^1_{n_1}] \\
&\quad +(1-\alpha_n^{(1)})[d(x_n, q) + k^1_{n_2}\psi^1_2(d(x_n, q)) + \nu^1_{n_2}] \\
&\le \alpha_n^{(1)}[d(x_n, q) + k_n\psi(d(x_n, q)) + \nu_n] \\
&\quad +(1-\alpha_n^{(1)})[d(x_n, q) + k_n\psi(d(x_n, q)) + \nu_n] \\
&\le \alpha_n^{(1)}[d(x_n, q) + k_n\mathcal{K}_1 d(x_n, q) + \nu_n] \\
&\quad +(1-\alpha_n^{(1)})[d(x_n, q) + k_n\mathcal{K}'_1 d(x_n, q) + \nu_n] \\
&\le \alpha_n^{(1)}[d(x_n, q) + k_n\mathcal{K} d(x_n, q) + \nu_n] \\
&\quad +(1-\alpha_n^{(1)})[d(x_n, q) + k_n\mathcal{K} d(x_n, q) + \nu_n] \\
&= \alpha_n^{(1)}[(1 + k_n\mathcal{K}) d(x_n, q) + \nu_n] \\
&\quad +(1-\alpha_n^{(1)})[(1 + k_n\mathcal{K}) d(x_n, q) + \nu_n] \\
&= (1 + k_n\mathcal{K}) d(x_n, q) + \nu_n. \qquad (2.2)
\end{aligned}$$

Again using (2.1)-(2.2), we obtain

$$\begin{aligned}
d(x_n^{(2)}, q) &= d(W(T_2^n x_n^{(1)}, S_2^n x_n, \alpha_n^{(2)}), q) \\
&\le \alpha_n^{(2)} d(T_2^n x_n^{(1)}, q) + (1-\alpha_n^{(2)}) d(S_2^n x_n, q) \\
&\le \alpha_n^{(2)}[d(x_n^{(1)}, q) + k^2_{n_1}\psi^2_1(d(x_n^{(1)}, q)) + \nu^2_{n_1}] \\
&\quad +(1-\alpha_n^{(2)})[d(x_n, q) + k^2_{n_2}\psi^2_2(d(x_n, q)) + \nu^2_{n_2}] \\
&\le \alpha_n^{(2)}[d(x_n^{(1)}, q) + k_n\psi(d(x_n^{(1)}, q)) + \nu_n] \\
&\quad +(1-\alpha_n^{(2)})[d(x_n, q) + k_n\psi(d(x_n, q)) + \nu_n] \\
&\le \alpha_n^{(2)}[d(x_n^{(1)}, q) + k_n\mathcal{K}_2 d(x_n^{(1)}, q) + \nu_n] \\
&\quad +(1-\alpha_n^{(2)})[d(x_n, q) + k_n\mathcal{K}'_2 d(x_n, q) + \nu_n]
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n^{(2)}[d(x_n^{(1)},q)+k_n\mathcal{K}d(x_n^{(1)},q)+\nu_n] \\
&\quad +(1-\alpha_n^{(2)})[d(x_n,q)+k_n\mathcal{K}d(x_n,q)+\nu_n] \\
&= \alpha_n^{(2)}[(1+k_n\mathcal{K})d(x_n^{(1)},q)+\nu_n] \\
&\quad +(1-\alpha_n^{(2)})[(1+k_n\mathcal{K})d(x_n,q)+\nu_n] \\
&\leq \alpha_n^{(2)}(1+k_n\mathcal{K})[(1+k_n\mathcal{K})d(x_n,q)+\nu_n]+\alpha_n^{(2)}\nu_n \\
&\quad +(1-\alpha_n^{(2)})[(1+k_n\mathcal{K})d(x_n,q)+\nu_n] \\
&= \alpha_n^{(2)}(1+k_n\mathcal{K})^2d(x_n,q)+\alpha_n^{(2)}[(1+k_n\mathcal{K})\nu_n+\nu_n] \\
&\quad +(1-\alpha_n^{(2)})(1+k_n\mathcal{K})d(x_n,q)+(1-\alpha_n^{(2)})\nu_n \\
&\leq \alpha_n^{(2)}(1+k_n\mathcal{K})^2d(x_n,q)+\nu_n]+\alpha_n^{(2)}[(1+k_n\mathcal{K})\nu_n \\
&\quad +\nu_n]+(1-\alpha_n^{(2)})(1+k_n\mathcal{K})^2d(x_n,q)+(1-\alpha_n^{(2)})\nu_n \\
&= (1+k_n\mathcal{K})^2d(x_n,q)+[1+(1+k_n\mathcal{K})]\nu_n. \qquad (2.3)
\end{aligned}$$

Similarly, we can prove that

$$\begin{aligned}
d(x_n^{(3)},q) &\leq (1+k_n\mathcal{K})^3d(x_n,q)+[1+(1+k_n\mathcal{K}) \\
&\quad +(1+k_n\mathcal{K})^2]\nu_n \\
&= (1+k_n\mathcal{K})^3d(x_n,q)+\sum_{j=0}^{2}(1+k_n\mathcal{K})^j\nu_n. \qquad (2.4)
\end{aligned}$$

Continuing the above process, we get that

$$\begin{aligned}
d(x_n^{(N)},q) &= d(x_{n+1},q) \\
&\leq (1+k_n\mathcal{K})^Nd(x_n,q)+\sum_{j=0}^{N-1}(1+k_n\mathcal{K})^j\nu_n \\
&\leq [1+b_n^Nk_n]d(x_n,q)+\sum_{j=0}^{N-1}(1+k_n\mathcal{K})^j\nu_n \\
&\leq [1+\mathcal{R}_1k_n]d(x_n,q)+\mathcal{R}_2\nu_n, \qquad (2.5)
\end{aligned}$$

where $b_n^N=\binom{N}{1}\mathcal{K}+\binom{N}{2}(\mathcal{K})^2k_n+\cdots+\binom{N}{N}(\mathcal{K})^Nk_n^{N-1}$, and it follows from condition (i) that there exist positive constants $\mathcal{R}_1$ and $\mathcal{R}_2$ such that $b_n^N\leq\mathcal{R}_1$, $\sum_{j=0}^{N-1}(1+k_n\mathcal{K})^j\leq\mathcal{R}_2$ for each $n\geq 1$. By Lemma 1.9, the inequality (2.5) implies that $\lim_{n\to\infty}d(x_n,q)$ exists for each $q\in F$.

(ii) Taking infimum over all $q\in F$ in equation (2.5), we have that

$$d(x_{n+1},F) \leq [1+\mathcal{R}_1k_n]d(x_n,F)+\mathcal{R}_2\nu_n. \qquad (2.6)$$

Since $\sum_{n=1}^{\infty} k_n < \infty$ and $\sum_{n=1}^{\infty} \nu_n < \infty$, it follows from Lemma 1.9 that $\lim_{n\to\infty} d(x_n, F)$ exists. This completes the proof. $\square$

Theorem 2.3. *Let K be a nonempty, closed and convex subset of a convex complete metric space (X, d, W). For each $i \in \mathcal{N}$, let $T_i\colon K \to K$ be a $(\{k_{n_1}^i\}, \{\nu_{n_1}^i\}, \{\psi_1^i\})$-total asymptotically nonexpansive mapping with $\lim_{n\to\infty} k_{n_1}^i = 0$ and $\lim_{n\to\infty} \nu_{n_1}^i = 0$, and a strictly increasing function $\psi_1^i\colon [0,+\infty) \to [0,+\infty)$ satisfying $\psi_1^i(0) = 0$ and let $S_i\colon K \to K$ be a $(\{k_{n_2}^i\}, \{\nu_{n_2}^i\}, \{\psi_2^i\})$-total asymptotically nonexpansive mapping with $\lim_{n\to\infty} k_{n_2}^i = 0$ and $\lim_{n\to\infty} \nu_{n_2}^i = 0$, and a strictly increasing function $\psi_2^i\colon [0,+\infty) \to [0,+\infty)$ satisfying $\psi_2^i(0) = 0$. Assume that*

$$F := \bigcap_{i=1}^{N} F(S_i) \cap F(T_i) \neq \emptyset,$$

and for each $i \in \mathcal{N}$, the following conditions hold:

(i) $\sum_{n=1}^{\infty} k_{n_1}^i < +\infty$, $\sum_{n=1}^{\infty} k_{n_2}^i < +\infty$, $\sum_{n=1}^{\infty} \nu_{n_1}^i < +\infty$, *and*

$\sum_{n=1}^{\infty} \nu_{n_2}^i < +\infty$.

(ii) *There exist constants $\mathcal{K}_i > 0$ and $\mathcal{K}_i' > 0$ such that*

$$\psi_1^i(r) \le \mathcal{K}_i r,\ \psi_2^i(r) \le \mathcal{K}_i' r,\ \forall\, r > 0.$$

Let $\{x_n\}$ be the sequence defined by (2.1), where $\{\alpha_n^{(i)}\} \subset [0,1]$ for all $i \in \mathcal{N}$. Then $\{x_n\}$ converges strongly to some $q \in F$ if and only if

$$\liminf_{n\to\infty} d(x_n, F) = 0,$$

where $d(x, F) = \inf\{d(x, z) : z \in F\}$.

Proof. If $\{x_n\}$ converges strongly to $q \in F$, then $\lim_{n\to\infty} d(x_n, q) = 0$. Since $0 \le d(x_n, F) \le d(x_n, q)$, we have $\liminf_{n\to\infty} d(x_n, F) = 0$.

Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F) = 0$. It follows from Lemma 2.2 that $\lim_{n\to\infty} d(x_n, F)$ exists. Now $\liminf_{n\to\infty} d(x_n, F) = 0$ reveals that $\lim_{n\to\infty} d(x_n, F) = 0$ by Remark 2.1(2).

Next, we show that $\{x_n\}$ is a Cauchy sequence. By inequality (2.5) in the proof of Lemma 2.2, we know that

$$d(x_{n+1}, q) \quad\le\quad [1 + \mathcal{R}_1 k_n] d(x_n, q) + \mathcal{R}_2 \nu_n.$$

On account of $\sum_{n=1}^{\infty} k_n < +\infty$, $\sum_{n=1}^{\infty} \nu_n < +\infty$, set $e^{\mathcal{R}_1 \sum_{n=1}^{\infty} k_n} = \mathcal{R}_*$. Since $\lim_{n\to\infty} d(x_n, F) = 0$, for any given $\varepsilon > 0$, there exists a positive integer n_0

such that

$$d(x_{n_0}, F) < \frac{\varepsilon}{4(\mathcal{R}_* + 1)} \quad \text{and} \quad \sum_{n=n_0}^{\infty} \nu_n < \frac{\varepsilon}{2\mathcal{R}_*\mathcal{R}_2}. \tag{2.7}$$

The first inequality in (2.7) implies that there exists $q_0 \in F$ such that $d(x_{n_0}, q_0) < \frac{\varepsilon}{2(\mathcal{R}_*+1)}$. Hence, for any $n \geq n_0$ and $m \geq 1$, we have

$$\begin{aligned} d(x_{n_0+m}, x_{n_0}) &\leq d(x_{n_0+m}, q_0) + d(x_{n_0}, q_0) \\ &\leq \Big[e^{\mathcal{R}_1 \sum_{j=n_0}^{n_0+m-1} k_j} + 1\Big] d(x_{n_0}, q_0) + \mathcal{R}_2 \Big[\nu_{n_0+m-1} \\ &\quad + \nu_{n_0+m-2} e^{\mathcal{R}_1 k_{n_0+m-1}} + \nu_{n_0+m-3} e^{\mathcal{R}_1 \sum_{j=n_0+m-2}^{n_0+m-1} k_j} \\ &\quad + \cdots + \nu_{n_0} e^{\mathcal{R}_1 \sum_{j=n_0+1}^{n_0+m-1} k_j}\Big] \\ &\leq (\mathcal{R}_* + 1) d(x_{n_0}, q_0) + \mathcal{R}_*\mathcal{R}_2 \sum_{n=n_0}^{\infty} \nu_n \\ &< (\mathcal{R}_* + 1). \frac{\varepsilon}{2(\mathcal{R}_* + 1)} + \mathcal{R}_*\mathcal{R}_2 . \frac{\varepsilon}{2\mathcal{R}_*\mathcal{R}_2} \\ &= \varepsilon. \end{aligned} \tag{2.8}$$

This implies that $\{x_n\}$ is a Cauchy sequence in X. Since K is a closed subset of a convex metric space X, it is complete. We can assume that $\lim_{n\to\infty} x_n = z$, and $z \in K$. Suppose that $\liminf_{n\to\infty} d(x_n, F) = 0$. Then, we have from Lemma 1.9(ii) and Remark 2.1(2) that $\lim_{n\to\infty} d(x_n, F) = 0$. Moreover, since the set of common fixed points of two families of mappings is closed, so is F, thus $z \in F$ and so $\lim_{n\to\infty} d(x_n, F) = 0$. This shows that $\{x_n\}$ strongly converges to some $q \in F$. This completes the proof. □

Theorem 2.4. *Let K be a nonempty, closed and convex subset of a convex complete metric space (X, d, W). For each $i \in \mathcal{N}$, let $T_i \colon K \to K$ be a $(\{k_{n_1}^i\}, \{\nu_{n_1}^i\}, \{\psi_1^i\})$-total asymptotically nonexpansive mapping with $\lim_{n\to\infty} k_{n_1}^i = 0$ and $\lim_{n\to\infty} \nu_{n_1}^i = 0$, and a strictly increasing function $\psi_1^i \colon [0, +\infty) \to [0, +\infty)$ satisfying $\psi_1^i(0) = 0$ and let $S_i \colon K \to K$ be a $(\{k_{n_2}^i\}, \{\nu_{n_2}^i\}, \{\psi_2^i\})$-total asymptotically nonexpansive mapping with $\lim_{n\to\infty} k_{n_2}^i = 0$ and $\lim_{n\to\infty} \nu_{n_2}^i = 0$, and a strictly increasing function $\psi_2^i \colon [0, +\infty) \to [0, +\infty)$ satisfying $\psi_2^i(0) = 0$. Assume that*

$$F := \bigcap_{i=1}^{N} F(S_i) \cap F(T_i) \neq \emptyset,$$

and for each $i \in \mathcal{N}$, the following conditions hold:

(i) $\sum_{n=1}^{\infty} k_{n_1}^i < +\infty$, $\sum_{n=1}^{\infty} k_{n_2}^i < +\infty$, $\sum_{n=1}^{\infty} \nu_{n_1}^i < +\infty$, *and*

$\sum_{n=1}^{\infty} \nu_{n_2}^i < +\infty$.

(ii) *There exist constants* $\mathcal{K}_i > 0$ *and* $\mathcal{K}_i' > 0$ *such that*

$$\psi_1^i(r) \leq \mathcal{K}_i r,\ \psi_2^i(r) \leq \mathcal{K}_i' r,\ \forall\, r > 0.$$

Assume that $\lim_{n\to\infty} d(x_n, S_i x_n) = 0 = \lim_{n\to\infty} d(x_n, T_i x_n)$ *for all* $i \in \mathcal{N}$. *Let* $\{x_n\}$ *be the sequence defined by* (2.1), *where* $\{\alpha_n^{(i)}\} \subset [0,1]$ *for all* $i \in \mathcal{N}$. *Then* $\{x_n\}$ *converges strongly to a point in* F.

Proof. By hypothesis, $\lim_{n\to\infty} d(x_n, S_i x_n) = 0 = \lim_{n\to\infty} d(x_n, T_i x_n)$ for all $i \in \mathcal{N}$. Since K is compact so there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to q'$ (say) in K as $n_k \to \infty$. Continuity of S_i and T_i gives $S_i x_{n_k} \to S_i q'$ and $T_i x_{n_k} \to T_i q'$ as $n_k \to \infty$ for all $i \in \mathcal{N}$. Then by hypothesis of the theorem, we have

$$d(S_i q', q') = 0 = d(T_i q', q')$$

for all $i \in \mathcal{N}$. This yields $q' \in F = \bigcap_{i=1}^N F(S_i) \cap F(T_i)$ so that $\{x_n\}$ converges strongly to q' in $F = \bigcap_{i=1}^N F(S_i) \cap F(T_i)$. But by Lemma 2.2, $\lim_{n\to\infty} d(x_n, q)$ exists for all $q \in F = \bigcap_{i=1}^N F(S_i) \cap F(T_i)$, therefore $\{x_n\}$ must converges strongly to $q' \in F = \bigcap_{i=1}^N F(S_i) \cap F(T_i)$. This completes the proof. $\square$

3. Applications

As an application of Theorem 2.3, we establish some strong convergence results as follows.

Theorem 3.1. *Let* K *be a nonempty, closed and convex subset of a convex complete metric space* (X, d, W). *For each* $i \in \mathcal{N}$, *let* $T_i \colon K \to K$ *be a* $(\{k_{n_1}^i\}, \{\nu_{n_1}^i\}, \{\psi_1^i\})$*-total asymptotically nonexpansive mapping with* $\lim_{n\to\infty} k_{n_1}^i = 0$ *and* $\lim_{n\to\infty} \nu_{n_1}^i = 0$, *and a strictly increasing function* $\psi_1^i \colon [0, +\infty) \to [0, +\infty)$ *satisfying* $\psi_1^i(0) = 0$ *and let* $S_i \colon K \to K$ *be a* $(\{k_{n_2}^i\}, \{\nu_{n_2}^i\}, \{\psi_2^i\})$*-total asymptotically nonexpansive mapping with* $\lim_{n\to\infty} k_{n_2}^i = 0$ *and* $\lim_{n\to\infty} \nu_{n_2}^i = 0$, *and a strictly increasing function* $\psi_2^i \colon [0, +\infty) \to [0, +\infty)$ *satisfying* $\psi_2^i(0) = 0$. *Assume that*

$$F := \bigcap_{i=1}^{N} F(S_i) \cap F(T_i) \neq \emptyset,$$

and for each $i \in \mathcal{N}$, *the following conditions hold:*

(i) $\sum_{n=1}^{\infty} k_{n_1}^i < +\infty$, $\sum_{n=1}^{\infty} k_{n_2}^i < +\infty$, $\sum_{n=1}^{\infty} \nu_{n_1}^i < +\infty$, *and* $\sum_{n=1}^{\infty} \nu_{n_2}^i < +\infty$.

(ii) *There exist constants* $\mathcal{K} > 0$ *and* $\mathcal{K}' > 0$ *such that*

$$\psi_1^i(r) \leq \mathcal{K}_i r,\ \psi_2^i(r) \leq \mathcal{K}_i' r,\ \forall\, r > 0.$$

Assume that $\lim_{n\to\infty} d(x_n, S_i x_n) = 0 = \lim_{n\to\infty} d(x_n, T_i x_n)$ *for all* $i \in \mathcal{N}$. *Let* $\{x_n\}$ *be the sequence defined by* (2.1), *where* $\{\alpha_n^{(i)}\} \subset [0,1]$ *for all* $i \in \mathcal{N}$. *If there exists a* T_i *or* S_i, $i \in \mathcal{N}$, *which is semi-compact. Then the sequence* $\{x_n\}$ *converges to a point in* F.

Proof. Without loss of generality, we can assume that T_1 is semi-compact. From Lemma 2.2, we know that the sequence $\{x_n\}$ is bounded and by hypothesis of the theorem

$$\lim_{n\to\infty} d(x_n, S_i x_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(x_n, T_i x_n) = 0$$

for all $i \in \mathcal{N}$. Since T_1 is semi-compact and $\lim_{n\to\infty} d(x_n, T_1 x_n) = 0$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p^* \in K$. Thus

$$d(p^*, T_i p^*) = \lim_{j\to\infty} d(x_{n_j}, T_i x_{n_j}) = 0$$

and

$$d(p^*, S_i p^*) = \lim_{j\to\infty} d(x_{n_j}, S_i x_{n_j}) = 0$$

for all $i \in \mathcal{N}$. This implies that $p^* \in F = \bigcap_{i=1}^N F(S_i) \cap F(T_i)$ and so

$$\liminf_{n\to\infty} d(x_n, F) \leq \liminf_{j\to\infty} d(x_{n_j}, F) \leq \lim_{j\to\infty} d(x_{n_j}, p^*) = 0.$$

It follows from Theorem 2.3 that $\{x_n\}$ converges strongly to a point in F. This completes the proof. □

Theorem 3.2. *Let* K *be a nonempty, closed and convex subset of a convex complete metric space* (X, d, W). *For each* $i \in \mathcal{N}$, *let* $T_i \colon K \to K$ *be a* $(\{k_{n_1}^i\}, \{\nu_{n_1}^i\}, \{\psi_1^i\})$*-total asymptotically nonexpansive mapping with* $\lim_{n\to\infty} k_{n_1}^i = 0$ *and* $\lim_{n\to\infty} \nu_{n_1}^i = 0$, *and a strictly increasing function* $\psi_1^i \colon [0, +\infty) \to [0, +\infty)$ *satisfying* $\psi_1^i(0) = 0$ *and let* $S_i \colon K \to K$ *be a* $(\{k_{n_2}^i\}, \{\nu_{n_2}^i\}, \{\psi_2^i\})$*-total asymptotically nonexpansive mapping with* $\lim_{n\to\infty} k_{n_2}^i = 0$ *and* $\lim_{n\to\infty} \nu_{n_2}^i = 0$, *and a strictly increasing function* $\psi_2^i \colon [0, +\infty) \to [0, +\infty)$ *satisfying* $\psi_2^i(0) = 0$. *Assume that*

$$F := \bigcap_{i=1}^N F(S_i) \cap F(T_i) \neq \emptyset,$$

and for each $i \in \mathcal{N}$, *the following conditions hold:*

(i) $\sum_{n=1}^{\infty} k_{n_1}^i < +\infty$, $\sum_{n=1}^{\infty} k_{n_2}^i < +\infty$, $\sum_{n=1}^{\infty} \nu_{n_1}^i < +\infty$,

and $\sum_{n=1}^{\infty} \nu_{n_2}^i < +\infty$.

(ii) *There exist constants* $\mathcal{K} > 0$ *and* $\mathcal{K}' > 0$ *such that*

$$\psi_1^i(r) \leq \mathcal{K}_i r,\ \psi_2^i(r) \leq \mathcal{K}_i' r,\ \forall\, r > 0.$$

Let $\{x_n\}$ *be the sequence defined by* (2.1), *where* $\{\alpha_n^{(i)}\} \subset [0,1]$ *for all* $i \in \mathcal{N}$.

Assume that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$*; if the sequence* $\{z_n\}$ *in* K *satisfies* $\lim_{n\to\infty} d(z_n, z_{n+1}) = 0$*, then* $\liminf_{n\to\infty} d(z_n, F) = 0$*. Then* $\{x_n\}$ *converges to a unique point in* F*.*

Proof. By hypothesis, we have that $\liminf_{n\to\infty} d(x_n, F) = 0$. Therefore, we obtain from Theorem 2.3 that the sequence $\{x_n\}$ converges to a unique point in F. This completes the proof. □

For our next result, we need the following definition.

Definition 3.3. ([8]) A family $\{T_i : i = 1, 2, \ldots, m\}$ of m self-mappings of K with $F = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ is said to satisfy condition (B) on K if there is a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that

$$\max_{1\leq i\leq m} \{\|x - T_i x\|\} \geq f(d(x, F))$$

for all $x \in K$.

Theorem 3.4. *Let* K *be a nonempty, closed and convex subset of a convex complete metric space* (X, d, W)*. For each* $i \in \mathcal{N}$*, let* $T_i\colon K \to K$ *be a* $(\{k_{n_1}^i\}, \{\nu_{n_1}^i\}, \{\psi_1^i\})$*-total asymptotically nonexpansive mapping with* $\lim_{n\to\infty} k_{n_1}^i = 0$ *and* $\lim_{n\to\infty} \nu_{n_1}^i = 0$*, and a strictly increasing function* $\psi_1^i\colon [0,+\infty) \to [0,+\infty)$ *satisfying* $\psi_1^i(0) = 0$ *and let* $S_i\colon K \to K$ *be a* $(\{k_{n_2}^i\}, \{\nu_{n_2}^i\}, \{\psi_2^i\})$*-total asymptotically nonexpansive mapping with* $\lim_{n\to\infty} k_{n_2}^i = 0$ *and* $\lim_{n\to\infty} \nu_{n_2}^i = 0$*, and a strictly increasing function* $\psi_2^i\colon [0,+\infty) \to [0,+\infty)$ *satisfying* $\psi_2^i(0) = 0$*. Assume that*

$$F := \bigcap_{i=1}^{N} F(S_i) \cap F(T_i) \neq \emptyset,$$

and for each $i \in \mathcal{N}$*, the following conditions hold:*

(i) $\sum_{n=1}^{\infty} k_{n_1}^i < +\infty$, $\sum_{n=1}^{\infty} k_{n_2}^i < +\infty$, $\sum_{n=1}^{\infty} \nu_{n_1}^i < +\infty$,

and $\sum_{n=1}^{\infty} \nu_{n_2}^i < +\infty$.

(ii) *There exist constants* $\mathcal{K} > 0$ *and* $\mathcal{K}' > 0$ *such that*

$$\psi_1^i(r) \leq \mathcal{K}_i r, \ \psi_2^i(r) \leq \mathcal{K}'_i r, \ \forall\, r > 0.$$

Assume that $\lim_{n\to\infty} d(x_n, S_i x_n) = 0 = \lim_{n\to\infty} d(x_n, T_i x_n)$ *for all* $i \in \mathcal{N}$. *Let* $\{x_n\}$ *be the sequence defined by* (2.1), *where* $\{\alpha_n^{(i)}\} \subset [0,1]$ *for all* $i \in \mathcal{N}$. *If the family* $\{S_1, S_2, \dots, S_N, T_1, T_2, \dots, T_N\}$ *satisfies condition (B). Then* $\{x_n\}$ *converges strongly to a point in* F.

Proof. By hypothesis of the Theorem, we have $\lim_{n\to\infty} d(x_n, S_i x_n) = 0$, $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$ for all $i \in \mathcal{N}$, and so $\max_{1\leq i\leq N}\{d(x_n, S_i x_n), d(x_n, T_i x_n)\} \to 0$ as $n \to \infty$. It follows from condition (B) that

$$\lim_{n\to\infty} f(d(x_n, F)) = 0.$$

By Lemma 2.2(ii), we know that $\lim_{n\to\infty} d(x_n, F)$ exists. Since $f\colon [0,\infty) \to [0,\infty)$ is a nondecreasing function with $f(0) = 0$ and so $\lim_{n\to\infty} d(x_n, F) = 0$. Thus, $\liminf_{n\to\infty} d(x_n, F) = 0 = \limsup_{n\to\infty} d(x_n, F)$. By Theorem 2.3, $\{x_n\}$ converges strongly to a point in F. This completes the proof. $\square$

Now, we give an example in support of our result: take two mappings $T_1 = T_2 = \cdots = T_N = T$ and $S_1 = S_2 = \cdots = S_N = S$ as follows:

Example 3.5. Let $X = \mathbb{R}$ be the real line with the usual metric $d(x, y) = |x - y|$ and $K = [-1, 1]$. For each $x \in K$, define two mappings T, $S\colon K \to K$ by

$$T(x) = \begin{cases} -2\sin\frac{x}{2}, & \text{if } x \in [0,1], \\ \\ 2\sin\frac{x}{2}, & \text{if } x \in [-1,0). \end{cases}$$

and

$$S(x) = \begin{cases} \frac{x}{2}, & \text{if } x \neq 0, \\ \\ 0, & \text{if } x = 0. \end{cases}$$

Then T and S both are asymptotically nonexpansive mapping with constant sequence $\{k_n\} = \{1\}$ for all $n \geq 1$ and are uniformly L-Lipschtzian mappings with $L = \sup_{n\geq 1}\{k_n\}$ and hence are total asymptotically nonexpansive mapping by *remark* 1.6. Also $F(T) = \{0\}$ is the unique fixed point of T and $F(S) = \{0\}$ is the unique fixed point of S, that is, $F = F(S) \cap F(T) = \{0\}$ is the unique common fixed point of S and T.

Example 3.6. Let $X = \mathbb{R}$ be the real line with the usual metric $d(x, y) = |x - y|$ and $K = [0, 2]$. For each $x \in K$, define two mappings T, $S\colon K \to K$ by

$$T(x) = \begin{cases} 1, & \text{if } x \in [0,1], \\ \\ \sqrt{2-x}, & \text{if } x \in [1,2]. \end{cases}$$

and

$$S(x) = \begin{cases} 1, & \text{if } x \in [0,1], \\ \frac{1}{\sqrt{3}}\sqrt{4-x^2}, & \text{if } x \in [1,2]. \end{cases}$$

Then both T and S are total asymptotically nonexpansive and uniformly continuous mappings. Also $F(T) = \{1\}$ is the unique fixed point of T and $F(S) = \{1\}$ is the unique fixed point of S, that is, $F = F(S) \cap F(T) = \{1\}$ is the unique common fixed point of S and T.

4. Concluding remarks

In this paper, we introduce a new finite-step iteration scheme for two finite families of total asymptotically nonexpansive mappings in convex metric spaces and establish some strong convergence results. Also, we give some applications of our result in the setting of convex metric spaces. Our results extend and generalize the corresponding results of [4, 6, 8, 9, 10, 11, 18, 26, 27, 31, 41, 43] to the case of more general class of mappings, iteration schemes and spaces.

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