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A FIXED POINT APPROACH TO THE STABILITY OF THE FUNCTIONAL EQUATIONS RELATED TO AN ADDITIVE AND QUARTIC MAPPING

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Abstract. In this paper, we prove the stability of functional equations related to an additive-quartic mapping by using the fixed-point theory, which L. C ădariu and V. Radu used as a way to prove the stability of functional equations.

1. Introduction

In this paper, let V and W be real vector spaces, Y be a real Banach space, and k be a fixed real number such that $k \neq 0, \pm 1$. For a given mapping $f : V \to W$, we use the following abbreviations:

$$
\begin{aligned}
Af(x,y) :=\ & f(x+y) - f(x) - f(y),\\
Qf(x,y) :=\ & f(x+2y) - 4f(x+y) + 6f(x) - 4f(x-y) + f(x-2y) - 24f(y),\\
D_1 f(x,y) =\ & f(kx+y) + f(kx-y) - k^2 f(x+y) - k^2 f(x-y)\\
& + \frac{2(k^2-1)}{k^4-k}(f(ky) - kf(y)) - 2f(kx) + 2k^2 f(x),\\
D_2 f(x,y) =\ & f(kx+y) + f(kx-y) - k^2 f(x+y) - k^2 f(x-y)\\
& + (k^2-1)(f(y)+f(-y)) - (k^4 - 2k^2 + k)f(x) - (k^4-k)f(-x),
\end{aligned}
$$

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$$\begin{aligned}
D_3 f(x,y) =& f(x+ky)+f(x-ky)-k^2 f(x+y)-k^2 f(x-y)+2(k^2-1)f(x)\\
&-f(ky)-\frac{k^4-2k^2-k}{2}f(y)-\frac{k^4-2k^2+k}{2}f(-y),\\
D_4 f(x,y) =& f(x+ky)+f(x-ky)-k^2 f(x+y)-k^2 f(x-y)\\
&-\frac{2(k^3-k)}{k^3-1}(f(ky)-kf(y))+2(k^2-1)f(x),\\
D_5 f(x,y) =& f(x+ky)+f(x-ky)-f(kx+y)-f(kx-y)+f(kx)\\
&+\frac{k^4+k-4}{2}f(x)+\frac{k^4-k}{2}f(-x)-(k^4-1)[f(y)+f(-y)]
\end{aligned}$$

for all $x, y \in V$. The stability problem of a group homomorphism was first raised by Ulam [19]. In the next year, Hyers [10] first gave a partial solution to Ulam's question for the case of the functional equation related to additive mappings. Hyers' result has greatly influenced the study of the stability problem of the various functional equations. His result was generalized by Rassias [17] and Găvruta [8].

A solution mapping of the functional equation $Af(x,y) = 0$ and a solution mapping of the functional equation $Qf(x,y) = 0$ are called an additive mapping and a quartic mapping, respectively. A mapping f is called an additive and quartic mapping if f is expressed by the sum of an additive mapping and a quartic mapping. A functional equation is called an additive and quartic functional equation provided that each solution of that equation is an additive and quartic mapping and every additive and quartic mapping is a solution of that equation. Many mathematicians investigated the stability of the additive and quartic functional equations [1, 2, 7, 9, 11, 13, 14, 15, 16, 18]. They [1, 2, 3, 9, 11, 12, 13, 16, 18] showed that the functional equations $D_i f(x,y) = 0$, $(i = 1, 2, 4, 5)$ are additive and quartic functional equations when $k = 2$. They proved the stability of the additive and quartic functional equations $D_i f(x,y) = 0$ when $k = 2$ by handling the additive part and the quartic part of the given mapping f, respectively.

In this paper, we will show that every solution of the functional equation $D_i f(x,y) = 0$ is an additive and quartic mapping and prove the stability of the function equation $D_i f(x,y) = 0$ when k is a real number, which is a generalization of the function equation $D_i f(x,y) = 0$ when $k = 2$. Also we will not split the given mapping $f : V \to Y$ into two parts to prove the stability of the functional equations $D_i f(x,y) = 0$. We will prove the stability of the functional equations $D_i f(x,y) = 0$, $(i = 1, 2, 3, 4, 5)$ at once by using the fixed point theory in the sense of Cădariu and Radu [4, 5].

In other words, for a given mapping f that approximately satisfies the functional equation $D_i f(x,y) = 0$, we will show that the mapping F, which is the solution of the functional equation $D_iF(x,y) = 0$, can be constructed using the formula

$$F(x) = \lim_{n\to\infty}\left(\frac{f(k^n x) + f(-k^n x)}{2\cdot k^{4n}} + \frac{f(k^n x) - f(-k^n x)}{2\cdot k^n}\right)$$

or

$$F(x) = \lim_{n\to\infty}\left(\frac{k^n}{2}\left(f\left(\frac{x}{k^n}\right) - f\left(-\frac{x}{k^n}\right)\right) + \frac{k^{4n}}{2}\left(f\left(\frac{x}{k^n}\right) + f\left(-\frac{x}{k^n}\right)\right)\right)$$

for all $x \in V$, and we will prove that the mapping F exists uniquely near the mapping f.

2. Main results

To prove the main theorem, we need to mention the following fixed-point theorem of Margolis and Diaz:

Theorem 2.1. ([6]) *Suppose that a complete generalized metric space (X, d), which means that the metric d may assume infinite values, and a strictly contractive mapping $J : X \to X$ with the Lipschitz constant $0 < L < 1$ are given. Then, for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x) = +\infty,\ \forall n \in \mathbb{N}\cup\{0\},$$

or there exists a nonnegative integer k such that:

(1) *$d(J^n x, J^{n+1}x) < +\infty$ for all $n \geq k$;*
(2) *the sequence $\{J^n x\}$ is convergent to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in $Y := \{y \in X, d(J^k x, y) < +\infty\}$;*
(4) *$d(y, y^*) \leq (1/(1-L))d(y, Jy)$ for all $y \in Y$.*

Lemma 2.2. *Let i be a fixed integer such that $i \in \{1,2,3,4,5\}$. Then the equalities*

$$f(x) - \frac{k^3+1}{2k^4}f(kx) + \frac{k^3-1}{2k^4}f(-kx) = E_i(x), \tag{2.1}$$

$$f(x) - \frac{k^4+k}{2}f\left(\frac{x}{k}\right) - \frac{k^4-k}{2}f\left(\frac{-x}{k}\right) = E_i'f(x) \tag{2.2}$$

hold for all $x \in V$*, where* $f(0) = 0$ *and the mappings* $E_i f, E'_i f : V \to W$ *are given by*

$$\begin{aligned}
E_1 f(x) =& \frac{-(k^2+k+1)(k^3+1)D_1 f(0,x) + (k^2+k+1)(k^3-1)D_1 f(0,-x)}{4(k+1)k^3}, \\
E_2 f(x) =& \frac{-(k^3+1)D_2 f(x,0) + (k^3-1)D_2 f(-x,0)}{4k^4}, \\
E_3(x) =& \frac{(k^3-1)D_3 f(0,x) - (k^3+1)D_3 f(0,-x)}{2k^4}, \\
E_4(x) =& \frac{(k^2+k+1)(k^2-k-1)D_4 f(0,x)}{4(k^5+k^4)} \\
& - \frac{(k^2+k+1)(k^2+k+1)D_4 f(0,-x)}{4(k^5+k^4)}, \\
E_5(x) =& \frac{(k^3+1)D_5 f(x,0) - (k^3-1)D_5 f(-x,0)}{2k^4}, \\
E'_1 f(x) =& \frac{(k^4-k)D_1 f\left(0, \frac{x}{k}\right)}{2(k^2-1)}, \\
E'_2 f(x) =& \frac{D_2 f\left(\frac{x}{k}, 0\right)}{2}, \\
E'_3 f(x) =& D_3 f\left(0, \frac{-x}{k}\right), \\
E'_4 f(x) =& \frac{(k^2+k+1)(k^2+k-1)D_4 f\left(0, \frac{x}{k}\right)}{4(k^2+k)} \\
& + \frac{(k^2+k+1)(k^2+k+1)D_4 f\left(0, \frac{-x}{k}\right)}{4(k^2+k)}, \\
E'_5 f(x) =& -D_5 f\left(\frac{x}{k}, 0\right)
\end{aligned}$$

for all $x \in V$*.*

Now we can prove some stability results of the functional equation $D_i F(x,y) = 0$ $(i = 1,2,3,4,5)$ by using the fixed point theory.

Theorem 2.3. *Let* i *be a fixed integer such that* $i \in \{1,2,3,4,5\}$ *and let* $\varphi : V^2 \to [0,\infty)$ *be a given function and* $|k| > 1$*. Suppose that the mapping* $f : V \to Y$ *satisfies the inequality*

$$\|D_i f(x,y)\| \le \varphi(x,y) \tag{2.3}$$

for all $x, y \in V$ *and* $f(0) = 0$*. If there exists a constant* $0 < L < 1$ *such that* φ *has the property*

$$\varphi(kx,ky) \le kL\varphi(x,y) \tag{2.4}$$

for all $x, y \in V$, then there exists a unique mapping $F : V \to Y$ such that

$$\|f(x) - F(x)\| \le \frac{\Phi_i(x)}{1-L} \tag{2.5}$$

for all $x \in V$ and $D_iF(x,y) = 0$ for all $x, y \in V$, where $\varphi_e : V^2 \to [0,\infty)$ and Φ_i are defined by

$$\begin{aligned}
\varphi_e(x,y) &:= \frac{\varphi(x,y)+\varphi(-x,-y)}{2},\\
\Phi_1 f(x) &:= \frac{|k^2+k+1|(|k|^3+1)\varphi_e(0,x)}{2|k+1||k|^3},\\
\Phi_2 f(x) &:= \frac{(|k|^3+1)\varphi_e(x,0)}{2k^4},\\
\Phi_3(x) &:= \frac{(|k|^3+1)\varphi_e(0,x)}{k^4},\\
\Phi_4(x) &:= \frac{(k^2+|k|+1)^2\varphi_e(0,x)}{2|k^5+k^4|},\\
\Phi_5(x) &:= \frac{(|k|^3+1)\varphi_e(x,0)}{k^4}.
\end{aligned}$$

In particular, F is represented by

$$F(x) = \lim_{n\to\infty}\left(\frac{f(k^n x)+f(-k^n x)}{2\cdot k^{4n}} + \frac{f(k^n x)-f(-k^n x)}{2\cdot k^n}\right) \tag{2.6}$$

for all $x \in V$.

Proof. Let $S = \{g : V \to Y \mid g(0) = 0\}$ and introduce a generalized metric on S by

$$d(g,h) = \inf\{K \in \mathbb{R}^+ \mid \|g(x)-h(x)\| \le K\Phi_i(x),\ x \in V\}.$$

It is easy to show that (S,d) is a generalized complete metric space. Now we consider the mapping $J : S \to S$, which is defined by

$$Jg(x) := \frac{g(kx)-g(-kx)}{2\cdot k} + \frac{g(kx)+g(-kx)}{2\cdot k^4}$$

for all $x \in V$. Let $g, h \in S$ and let $K \in [0,\infty]$ be an arbitrary constant with $d(g,h) \le K$. From the definition of d, we have

$$\begin{aligned}
\|Jg(x)-Jh(x)\| &\le \frac{|k^3+1|}{2\cdot|k|^4}\|g(kx)-h(kx)\| + \frac{|k^3-1|}{2\cdot|k|^4}\|g(-kx)-h(-kx)\|\\
&\le \frac{1}{|k|}K\Phi_i(kx)\\
&\le KL\Phi_i(x)
\end{aligned}$$

for all $x \in V$, which implies that

$$d(Jg, Jh) \leq Ld(g, h)$$

for any $g, h \in S$. That is, J is a strictly contractive self-mapping of S with the Lipschitz constant L. Moreover, by (2.1) and (2.3), we see that

$$\|f(x) - Jf(x)\| = \|E_i f(x)\| \leq \Phi_i(x)$$

for all $x \in V$. It means that $d(f, Jf) \leq 1 < \infty$ by the definition of d. Therefore according to Theorem 2.1, the sequence $\{J^n f\}$ converges to the unique fixed point $F : V \to Y$ of J in the set $T = \{g \in S \mid d(f, g) < \infty\}$, which is represented by (2.3) for all $x \in V$. Notice that

$$d(f, F) \leq \frac{1}{1-L} d(f, Jf) \leq \frac{1}{1-L}$$

which implies (2.5). By the definitions of F, together with (2.3) and (2.4), we have that

$$\begin{aligned}
\|D_i f(x, y)\| = &\lim_{n\to\infty} \Big\| \frac{D_i(k^n x, k^n y) - D_i f(-k^n x, -, k^n y)}{2 \cdot k^n} \\
&\qquad + \frac{D_i f(k^n x, k^n y) + D_i f(-k^n x, -k^n y)}{2 \cdot k^{4n}} \Big\| \\
\leq &\lim_{n\to\infty} \frac{|k|^{3n} + 1}{2 \cdot |k|^{4n}} \left(\varphi(k^n x, k^n y) + \varphi(-k^n x, -k^n y) \right) \\
\leq &\lim_{n\to\infty} \frac{(|k|^{3n} + 1)L^n}{2 \cdot |k|^{3n}} \left(\varphi(x, y) + \varphi(-x, -y) \right) \\
= &0
\end{aligned}$$

for all $x, y \in V$. So F satisfies $D_i F(x, y) = 0$ for all $x, y \in V$. Notice that if F satisfies the functional equation $D_i F(x, y) = 0$, then the equality $F(x) - JF(x) = E_i F(x)$ implies that F is a fixed point of J. Hence F is unique mapping satisfying (2.5). □

Theorem 2.4. *Let i be a fixed integer such that $i \in \{1, 2, 3, 4, 5\}$ and let $\varphi : V^2 \to [0, \infty)$ be a given function and $|k| < 1$. Suppose that the mapping $f : V \to Y$ satisfies the inequality* (2.3) *for all $x, y \in V$ and $f(0) = 0$. If there exists a constant $0 < L < 1$ such that φ has the property*

$$\varphi(kx, ky) \leq |k|^4 L \varphi(x, y) \tag{2.7}$$

for all $x, y \in V$, then there exists a unique mapping $F : V \to Y$ satisfying (2.5) *for all $x \in V$ and $D_i F(x, y) = 0$ for all $x, y \in V$. In particular, F is represented by* (2.3) *for all $x \in V$.*

Proof. Let the set (S, d) and the mapping $J : S \to S$ be as in the proof of Theorem 2.3. Let $g, h \in S$ and let $K \in [0, \infty]$ be an arbitrary constant with $d(g, h) \le K$. From the definition of d, we have

$$\|Jg(x) - Jh(x)\| \le \frac{1}{|k|^4} K\Phi_i(x) \le KL\Phi_i(x)$$

for all $x \in V$, which implies that

$$d(Jg, Jh) \le Ld(g, h)$$

for any $g, h \in S$. That is, J is a strictly contractive self-mapping of S with the Lipschitz constant L. Moreover, by (2.1) and (2.3), we see that

$$\|f(x) - Jf(x)\| = \|E_i f(x)\| \le \Phi_i(x)$$

for all $x \in V$. It means that $d(f, Jf) \le 1 < \infty$ by the definition of d. Therefore according to Theorem 2.1, the sequence $\{J^n f\}$ converges to the unique fixed point $F : V \to Y$ of J in the set $T = \{g \in S \mid d(f, g) < \infty\}$, which is represented by (2.3) for all $x \in V$. Notice that

$$d(f, F) \le \frac{1}{1 - L} d(f, Jf) \le \frac{1}{1 - L}$$

which implies (2.5). By the definitions of F, together with (2.3) and (2.7), we have that

$$\begin{aligned} \|D_i f(x, y)\| &\le \lim_{n\to\infty} \frac{|k|^{3n} + 1}{2 \cdot |k|^{4n}} \left(\varphi(k^n x, k^n y) + \varphi(-k^n x, -k^n y)\right) \\ &\le \lim_{n\to\infty} \frac{(|k|^{3n} + 1)L^n}{2} \left(\varphi(x, y) + \varphi(-x, -y)\right) \\ &=0 \end{aligned}$$

for all $x, y \in V$. So F satisfies $D_i F(x, y) = 0$ for all $x, y \in V$. Notice that if F satisfies the functional equation $D_i F(x, y) = 0$, then the equality $F(x) - JF(x) = E_i F(x)$ implies that F is a fixed point of J. Hence F is unique mapping satisfying (2.5). □

We continue our investigation with the next result.

Theorem 2.5. *Let i be a fixed integer such that $i \in \{1, 2, 3, 4, 5\}$ and let $\varphi : V^2 \to [0, \infty)$ and k be a real number such that $|k| > 1$. Suppose that $f : V \to Y$ satisfies the inequality (2.3) for all $x, y \in V$ and $f(0) = 0$. If there exists $0 < L < 1$ such that the mapping φ has the property*

$$L\varphi(kx, ky) \ge |k|^4 \varphi(x, y) \tag{2.8}$$

for all $x, y \in V$, then there exists a unique mapping $F : V \to Y$ satisfying $D_kF(x,y) = 0$ for all $x, y \in V$ and

$$\|f(x) - F(x)\| \le \frac{\Psi_i(x)}{1-L} \tag{2.9}$$

for all $x \in V$, where Ψ_i is defined by

$$\begin{aligned}
\Psi_1 f(x) :=& \frac{(k^4+|k|)\varphi_e\left(0, \frac{x}{k}\right)}{|k^2-1|},\\
\Psi_2 f(x) :=& \varphi_e\left(\frac{x}{k}, 0\right),\\
\Psi_3 f(x) :=& 2\varphi_e\left(0, \frac{x}{k}\right),\\
\Psi_4 f(x) :=& \frac{(k^2+|k|+1)^2\varphi_e\left(0, \frac{x}{k}\right)}{2|k^2+k|},\\
\Psi_5 f(x) :=& 2\varphi_e\left(\frac{x}{k}, 0\right).
\end{aligned}$$

In particular, F is represented by

$$F(x) = \lim_{n\to\infty}\left(\frac{k^n}{2}\left(f\left(\frac{x}{k^n}\right) - f\left(-\frac{x}{k^n}\right)\right) + \frac{k^{4n}}{2}\left(f\left(\frac{x}{k^n}\right) + f\left(-\frac{x}{k^n}\right)\right)\right) \tag{2.10}$$

for all $x \in V$.

Proof. Let S be the set of all mappings $g : V \to Y$ with $g(0) = 0$ and introduce a generalized metric on S by

$$d(g,h) = \inf\{K \in \mathbb{R}^+ \mid \|g(x) - h(x)\| \le K\Psi_i(x),\ x \in V\}.$$

Now we consider the mapping $J : S \to S$ defined by

$$Jg(x) := \frac{k}{2}\left(g\left(\frac{x}{k}\right) - g\left(-\frac{x}{k}\right)\right) + \frac{k^4}{2}\left(g\left(\frac{x}{k}\right) + g\left(-\frac{x}{k}\right)\right)$$

for all $g \in S$ and $x \in V$. Let $g, h \in S$ and let $K \in [0,\infty]$ be an arbitrary constant with $d(g,h) \le K$. From the definition of d, we have

$$\begin{aligned}
\|Jg(x) - Jh(x)\| &\le \frac{|k+k^4|}{2}\left\|g\left(\frac{x}{k}\right) - h\left(\frac{x}{k}\right)\right\| + \frac{|k-k^4|}{2}\left\|g\left(-\frac{x}{k}\right) - h\left(-\frac{x}{k}\right)\right\|\\
&\le |k|^4 K\Psi_i\left(\frac{x}{k}\right)\\
&\le LK\Psi_i(x)
\end{aligned}$$

for all $x \in V$. So

$$d(Jg, Jh) \le Ld(g,h)$$

for any $g, h \in S$. That is, J is a strictly contractive self-mapping of S with the Lipschitz constant L. Moreover, by (2.2) and (2.3), we see that

$$\|f(x) - Jf(x)\| = \left\|E_i' f(x)\right\| \leq \Psi_i(x)$$

for all $x \in V$, which implies that $d(f, Jf) \leq 1 < \infty$. Therefore according to Theorem 2.1, the sequence $\{J^n f\}$ converges to the unique fixed point F of J in the set $T := \{g \in S \mid d(f, g) < \infty\}$, which is represented by (2.10) for all $x \in V$.

Notice that

$$d(f, F) \leq \frac{1}{1-L} d(f, Jf) \leq \frac{1}{1-L}$$

which implies (2.9). From the definition of $F(x)$, (2.3), and (2.8), we have

$$\begin{aligned}
\|D_i F(x, y)\| &= \lim_{n\to\infty} \left\| \frac{k^n}{2} \left(D_i f\left(\frac{x}{k^n}, \frac{y}{k^n}\right) - D_i f\left(-\frac{x}{k^n}, -\frac{y}{k^n}\right)\right)\right. \\
&\quad \left. + \frac{k^{4n}}{2} \left(D_i f\left(\frac{x}{k^n}, \frac{y}{k^n}\right) + D_i f\left(-\frac{x}{k^n}, -\frac{y}{k^n}\right)\right) \right\| \\
&\leq \lim_{n\to\infty} \frac{|k|^n + |k|^{4n}}{2} \left(\varphi\left(\frac{x}{k^n}, \frac{y}{k^n}\right) + \varphi\left(-\frac{x}{k^n}, -\frac{y}{k^n}\right)\right) \\
&\leq \lim_{n\to\infty} \frac{(|k|^{3n} + 1)L^n}{2 \cdot |k|^{3n}} \left(\varphi(x, y) + \varphi(-x, -y)\right) \\
&= 0
\end{aligned}$$

for all $x, y \in V$. So F satisfies $D_i F(x, y) = 0$ for all $x, y \in V$. Notice that if F satisfies the functional equation $D_i F(x, y) = 0$, then the equality

$$F(x) - JF(x) = E_i' F(x)$$

implies that F is a fixed point of J. □

Theorem 2.6. *Let i be a fixed integer such that $i \in \{1, 2, 3, 4, 5\}$ and let $\varphi : V^2 \to [0, \infty)$ and k be a real number such that $|k| > 1$. Suppose that $f : V \to Y$ satisfies the inequality* (2.3) *for all $x, y \in V$ and $f(0) = 0$. If there exists $0 < L < 1$ such that the mapping φ has the property*

$$L\varphi(kx, ky) \geq |k|\varphi(x, y) \tag{2.11}$$

for all $x, y \in V$, then there exists a unique mapping $F : V \to Y$ satisfying $D_i F(x, y) = 0$ for all $x, y \in V$ and (2.9) *for all $x \in V$. In particular, F is represented by* (2.10) *for all $x \in V$.*

Proof. Let the set (S, d) and the mapping $J : S \to S$ be as in the proof of Theorem 2.5. Let $g, h \in S$ and let $K \in [0, \infty]$ be an arbitrary constant with $d(g, h) \leq K$. From the definition of d, we have

$$\|Jg(x) - Jh(x)\| \leq |k| K \Psi_i\left(\frac{x}{k}\right) \leq LK\Psi_i(x)$$

for all $x \in V$. So

$$d(Jg, Jh) \leq Ld(g, h)$$

for any $g, h \in S$. That is, J is a strictly contractive self-mapping of S with the Lipschitz constant L. Moreover, by (2.2) and (2.3), we see that

$$\|f(x) - Jf(x)\| = \left\|E_i' f(x)\right\| \leq \Psi_i(x)$$

for all $x \in V$, which implies that $d(f, Jf) \leq 1 < \infty$. Therefore according to Theorem 2.1, the sequence $\{J^n f\}$ converges to the unique fixed point F of J in the set $T := \{g \in S | d(f, g) < \infty\}$, which is represented by (2.10) for all $x \in V$.

Notice that

$$d(f, F) \leq \frac{1}{1-L} d(f, Jf) \leq \frac{1}{1-L}$$

which implies (2.9). From the definition of $F(x)$, (2.3), and (2.11), we have

$$\begin{aligned} \|D_i F(x,y)\| &\leq \lim_{n\to\infty} \frac{|k|^n + |k|^{4n}}{2} \left(\varphi\left(\frac{x}{k^n}, \frac{y}{k^n}\right) + \varphi\left(-\frac{x}{k^n}, -\frac{y}{k^n}\right)\right) \\ &\leq \lim_{n\to\infty} \frac{(|k|^{3n}+1)L^n}{2} \left(\varphi(x,y) + \varphi(-x,-y)\right) \\ &= 0 \end{aligned}$$

for all $x, y \in V$. So F satisfies $D_i F(x, y) = 0$ for all $x, y \in V$. Notice that if F satisfies the functional equation $D_i F(x, y) = 0$, then the equality $F(x) - JF(x) = E_i' F(x)$ implies that F is a fixed point of J. □

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