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PROPERTIES OF MEROMORPHIC FUNCTIONS DEFINED BY A CONVOLUTION OPERATOR

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Abstract. Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n,$$

which are analytic in the annulus $\mathbb{D} = \{z \in \mathbb{C} : 0 < |z| < 1\}$. In this paper, we introduce a convolution operator for functions f belonging to the class Σ and we obtain some mapping properties and argument estimates for meromorphic functions associated with this convolution operator.

1. Introduction

Let $\mathcal{H} = \mathcal{H}(\mathbb{U})$ denote the class of analytic functions in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

For $n \in \mathbb{N} = \{1, 2, \cdots\}$ and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}.$$

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Let f and F be members of $\mathcal{H}$. The function f is said to be subordinate to F if there exists a function w analytic in $\mathbb{U}$, with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = F(w(z)) \quad (z \in \mathbb{U}).$$

In such a case, we write

$$f \prec F \quad \text{or} \quad f(z) \prec F(z).$$

If the function F is univalent in $\mathbb{U}$, then we have (cf. [9])

$$f \prec F \quad \Longleftrightarrow \quad f(0) = F(0) \quad \text{and} \quad f(\mathbb{U}) \subset F(\mathbb{U}).$$

Let Σ denote the class of functions of the form [3]:

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n,$$

which are analytic in the annulus $\mathbb{D} = \mathbb{U}\backslash\{0\}$ with a simple pole at origin with residue one there. For functions

$$f_j(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_{n,j} z^n \quad (j = 1, 2; z \in \mathbb{D})$$

in the class Σ, we define the convolution of f_1 and f_2 [1] by

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_{n,1} a_{n,2} z^n \quad (z \in \mathbb{D}). \tag{1.1}$$

Making use of the convolution given by (1.1), we now define the following convolution operator D^α by

$$D^\alpha f(z) = \frac{1}{z(1-z)^{\alpha+1}} * f(z) \quad (\alpha > -1; f \in \Sigma; z \in \mathbb{D}). \tag{1.2}$$

It follows from (1.2) that

$$z(D^\alpha f(z))' = (\alpha+1) D^{\alpha+1} f(z) - (\alpha+2) D^\alpha f(z). \tag{1.3}$$

For $\alpha = n \in \mathbb{N}$, the operator D^α is introduced and studied by Ganigi and Uralegaddi [4] (see, also [14, 15]). Also, the operator D^α is closely related to Ruscheweyh derivative [11] for analytic functions defined in $\mathbb{U}$, which was extended by Goel and Sohi [5]. In the present paper, we shall derive certain interesting properties of the convolution operator D^α defined by (1.2).

2. Main results

To prove our results, we need the following lemmas.

Lemma 2.1. ([7]) Let h be analytic and convex in $\mathbb{U}$ with $h(0) = a$, $\gamma \neq 0$, $\mathrm{Re}\{\gamma\} \geq 0$. If $p \in \mathcal{H}[a, n]$ and

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z),$$

then

$$p(z) \prec q(z) \prec h(z),$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{(\gamma/n)-1}$$

and q is the best dominant.

Lemma 2.2. ([8]) Let Ω be a set in the complex plane $\mathbb{C}$ and let b be a complex number with $\mathrm{Re}\{b\} > 0$. Suppose that the function

$$\psi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$$

satisfies the condition:

$$\psi(ix, y; z) \notin \Omega,$$

for all real $x, y \geq -|b - ix|^2/(2\mathrm{Re}\{b\})$ and all $z \in \mathbb{U}$. If the function p is analytic in $\mathbb{U}$ with $p(0) = b$ and if

$$\psi(p(z), zp'(z); z) \in \Omega,$$

then

$$\mathrm{Re}\{p(z)\} > 0 \quad (z \in \mathbb{U}).$$

Lemma 2.3. ([13]) Let p be analytic in $\mathbb{U}$ with $p(0) = 1$ and $p(z) \neq 0$ for all $z \in \mathbb{U}$. If there exist two points $z_1, z_2 \in \mathbb{U}$ such that

$$-\frac{\pi}{2}\delta_1 = \arg\{p(z_1)\} < \arg\{p(z)\} < \arg\{p(z_2)\} = \frac{\pi}{2}\delta_2 \tag{2.1}$$

for some δ_1 and δ_2 ($\delta_1, \delta_2 > 0$) and for all z ($|z| < |z_1| = |z_2|$), then

$$\frac{z_1 p'(z_1)}{p(z_1)} = -i\left(\frac{\delta_1 + \delta_2}{2} m\right) \quad \text{and} \quad \frac{z_2 p'(z_2)}{p(z_2)} = i\left(\frac{\delta_1 + \delta_2}{2} m\right), \tag{2.2}$$

where

$$m \geq \frac{1-|b|}{1+|b|} \quad \text{and} \quad b = i\tan\left(\frac{\delta_2-\delta_1}{\delta_2+\delta_1}\right). \tag{2.3}$$

Theorem 2.4. *Let* $\alpha > -1,\ 0 \leq \lambda \leq 1$ *and* $\gamma > 1$. *If* $f \in \Sigma$, *then*

$$\operatorname{Re}\left\{(1-\lambda)\frac{D^{\alpha+1}f(z)}{D^{\alpha}f(z)} + \lambda\frac{D^{\alpha+2}f(z)}{D^{\alpha+1}f(z)}\right\} < \gamma \quad (z \in \mathbb{U}) \tag{2.4}$$

implies that

$$\operatorname{Re}\left\{\frac{D^{\alpha+1}f(z)}{D^{\alpha}f(z)}\right\} < \beta \quad (z \in \mathbb{U}), \tag{2.5}$$

where $\beta \in (1,\infty)$ *is the positive root of the equation:*

$$(2(\alpha+1)(1-\lambda)+2\lambda(\alpha+1))x^2 + (3\lambda - 2\gamma(\alpha+2))x - \lambda = 0. \tag{2.6}$$

Proof. Let

$$p(z) = \frac{1}{\beta-1}\left(\beta - \frac{D^{\alpha+1}f(z)}{D^{\alpha}f(z)}\right) \quad (z \in \mathbb{U}). \tag{2.7}$$

Then p is analytic in $\mathbb{U}$ and $p(0)=1$. Differentiating (2.7) and using (1.3), we obtain

$$\begin{aligned}
&(1-\lambda)\frac{D^{\alpha+1}f(z)}{D^{\alpha}f(z)} + \lambda\frac{D^{\alpha+2}f(z)}{D^{\alpha+1}f(z)} \\
&= (1-\lambda)\beta + \frac{\lambda(1+(\alpha+1)\beta)}{\alpha+2} - \left((1-\lambda)(\beta-1) + \frac{\lambda(\alpha+1)(\beta-1)}{\alpha+2}\right)p(z) \\
&\qquad - \frac{\lambda(\beta-1)zp'(z)}{(\alpha+2)(\beta-(\beta-1)p(z))} \\
&= \psi(p(z), zp'(z)),
\end{aligned}$$

where

$$\begin{aligned}
\psi(r,s) = (1-\lambda)\beta + \frac{\lambda(1+(\alpha+1)\beta)}{\alpha+2} - \left((1-\lambda)(\beta-1) + \frac{\lambda(\alpha+1)(\beta-1)}{\alpha+2}\right)r \\
- \frac{\lambda(\beta-1)s}{(\alpha+2)(\beta-(\beta-1)r)}
\end{aligned} \tag{2.8}$$

By virtue of (2.4) and (2.8), we have

$$\{\psi(p(z), zp'(z) : z \in \mathbb{U}\} \subset \Omega = \{w \in \mathbb{C} : \operatorname{Re}\{w\} < \gamma\}.$$

Now for all real $x, y \leq -(1+x^2)/2$, we have

$$\begin{aligned}\operatorname{Re}\{\psi(ix,y)\} &= (1-\lambda)\beta + \frac{\lambda(1+(\alpha+1)\beta)}{\alpha+2} - \frac{\lambda(\beta-1)\beta y}{(\alpha+2)(\beta^2+(\beta-1)^2x^2)} \\ &\geq (1-\lambda)\beta + \frac{\lambda(1+(\alpha+1)\beta)}{\alpha+2} + \frac{\lambda(\beta-1)\beta(1+x^2)}{2(\alpha+2)(\beta^2+(\beta-1)^2x^2)} \\ &\geq (1-\lambda)\beta + \frac{\lambda(1+(\alpha+1)\beta)}{\alpha+2} + \frac{\lambda(\beta-1)}{2(\alpha+2)\beta} = \gamma,\end{aligned}$$

where β is the positive root of the equation (2.6). Note that, if

$$g(x) = (2(1+\alpha)(1-\lambda) + 2\lambda(\alpha+1))x^2 + (3\lambda - 2\gamma(\alpha+2))x - \lambda,$$

then $g(0) = -\lambda < 0$ and $g(1) = 2((\alpha+1)(1-\gamma)) < 0$. This shows that $\beta \in (1,\infty)$. Hence for each $z \in \mathbb{U}$, $\psi(ix,y) \notin \Omega$. Therefore, by Lemma 2.2, $\operatorname{Re}\{p(z)\} > 0$ for $z \in \mathbb{U}$, which proves (2.5). □

Theorem 2.5. *Let $\lambda \geq 0$, $\gamma > 1$ and $0 \leq \delta < 1$. Suppose also that*

$$\operatorname{Re}\left\{\frac{D^\alpha g(z)}{D^{\alpha+1}g(z)}\right\} > \delta \quad (g \in \Sigma; z \in \mathbb{U}). \tag{2.9}$$

If $f \in \Sigma$ satisfies

$$\operatorname{Re}\left\{(1-\lambda)\frac{D^\alpha f(z)}{D^\alpha g(z)} + \lambda\frac{D^{\alpha+1}f(z)}{D^{\alpha+1}g(z)}\right\} < \gamma \quad (z \in \mathbb{U}), \tag{2.10}$$

then

$$\operatorname{Re}\left\{\frac{D^\alpha f(z)}{D^\alpha g(z)}\right\} < \frac{2\gamma(\alpha+1)+\lambda\delta}{2(\alpha+1)+\lambda\delta} \quad (z \in \mathbb{U}). \tag{2.11}$$

Proof. Let

$$\beta = \frac{2\gamma(\alpha+1)+\lambda\delta}{2(\alpha+1)+\lambda\delta} \quad (\beta > 1)$$

and

$$p(z) = \frac{1}{\beta-1}\left(\beta - \frac{D^\alpha f(z)}{D^\alpha g(z)}\right) \quad (z \in \mathbb{U}). \tag{2.12}$$

Then the function p is analytic in $\mathbb{U}$ and $p(0) = 1$. Setting

$$B(z) = \frac{D^\alpha g(z)}{D^{\alpha+1}g(z)} \quad (g \in \Sigma; z \in \mathbb{U}),$$

by assumption, we have

$$\operatorname{Re}\{B(z)\} > \delta \quad (z \in \mathbb{U}).$$

Differentiating (2.12) and using (1.3), we have

$$\begin{aligned}(1-\lambda)\frac{D^{\alpha}f(z)}{D^{\alpha}g(z)} &+ \lambda\frac{D^{\alpha+1}f(z)}{D^{\alpha+1}g(z)}\\ &= \beta-(\beta-1)p(z)-\frac{\lambda(\beta-1)B(z)zp'(z)}{\alpha+1}.\end{aligned}$$

Letting

$$\psi(r,s) = \beta-(\beta-1)r-\frac{\lambda(\beta-1)sB(z)}{\alpha+1} \quad (z \in \mathbb{U}),$$

we deduce from (2.10) that

$$\{\psi(p(z), zp'(z)); z \in \mathbb{U}\} \subset \Omega = \{w \in \mathbb{C} : \operatorname{Re}\{w\} < \gamma\}.$$

Now for all real $x, y \leq -(1+x^2)/2$, we have

$$\begin{aligned}\operatorname{Re}\{\psi(ix,y)\} &= \beta-\frac{\lambda(\beta-1)y}{\alpha+1}\operatorname{Re}\{B(z)\}\\ &\geq \beta+\frac{\lambda(\beta-1)\delta}{2(\alpha+1)}(1+x^2)\\ &\geq \beta+\frac{\lambda(\beta-1)\delta}{2(\alpha+1)} = \gamma,\end{aligned}$$

Hence for each $z \in \mathbb{U}$, $\psi(ix,y) \notin \Omega$. Thus by Lemma 2.2, $\operatorname{Re}\{p(z)\} > 0$ for $z \in \mathbb{U}$. Therefore we complete the proof of Theorem 2.5. □

Theorem 2.6. *Let Let $\alpha > -1$, $\beta \geq 1$ and $\gamma > 0$. If $f \in \Sigma$, then*

$$\operatorname{Re}\left\{\frac{D^{\alpha+1}f(z)}{D^{\alpha}f(z)}\right\} < \frac{\alpha+1+\gamma}{\alpha+1} \quad (z \in \mathbb{U}) \tag{2.13}$$

implies that

$$\operatorname{Re}\left\{(zD^{\alpha}f(z))^{-1/2\beta\gamma}\right\} > 2^{-1/\beta} \quad (z \in \mathbb{U}). \tag{2.14}$$

The bound $2^{-1/\beta}$ is the best possible.

Proof. From (1.3) and (2.13), we have

$$\operatorname{Re}\left\{\frac{z(D^{\alpha}f(z))'}{D^{\alpha}f(z)}\right\} < -1+\gamma \quad (z \in \mathbb{U}).$$

That is,

$$\frac{1}{2\gamma}\left(\frac{z(D^{\alpha}f(z))'}{D^{\alpha}f(z)}+1\right)\prec\frac{z}{1+z}\quad(z\in\mathbb{U}).\tag{2.15}$$

Let

$$p(z)=(zD^{\alpha}f(z))^{-1/2\gamma}\quad(z\in\mathbb{U}).$$

Then (2.15) may be written as

$$z\,(\log p(z))'\prec z\left(\log\frac{1}{1+z}\right)'\quad(z\in\mathbb{U}).\tag{2.16}$$

By using the well-known result [12] to (2.16), we obtain

$$p(z)\prec\frac{1}{1+z}\quad(z\in\mathbb{U}),$$

that is, that

$$(zD^{\alpha}f(z))^{-1/2\gamma\beta}=\left(\frac{1}{1+w(z)}\right)^{1/\beta}\quad(z\in\mathbb{U}),\tag{2.17}$$

where w is analytic function in $\mathbb{U}$, $w(0)=0$ and $|w(z)|<1$ for $z\in\mathbb{U}$. According to $\mathrm{Re}\{t^{1/\beta}\}\geq(\mathrm{Re}\{t\})^{1/\beta}$ for $\mathrm{Re}\{t\}>0$ and $\beta\geq 1$, (2.17) yields

$$\begin{aligned}\mathrm{Re}\left\{(zD^{\alpha}f(z))^{-1/2\gamma\beta}\right\}&\geq\left(\mathrm{Re}\left\{\frac{1}{1+w(z)}\right\}\right)^{1/\beta}\\&>2^{-1/\beta}\quad(z\in\mathbb{U}).\end{aligned}$$

To see that the bound $2^{-1/\beta}$ cannot be increased, we consider the function $g\in\Sigma$ such that

$$zD^{\alpha}g(z)=(1+z)^{2\gamma}\quad(z\in\mathbb{U}).$$

It is not so difficult to show that g satisfies (2.13) and

$$\mathrm{Re}\left\{(zD^{\alpha}g(z))^{-1/2\gamma\beta}\right\}\longrightarrow 2^{-1/\beta}$$

as $z=\mathrm{Re}\{z\}\to 1^{-}$. Therefore the proof of Theorem 2.6 is complete. $\square$

Theorem 2.7. *Let $\alpha>-1$, $\lambda\geq 0$ and $0<\delta_1,\delta_2\leq 1$. If $f\in\Sigma$ satisfies*

$$-\frac{\pi}{2}\delta_1<\arg\{(1-\lambda)zD^{\alpha}f(z)+\lambda zD^{\alpha+1}f(z)\}<\frac{\pi}{2}\delta_2,\tag{2.18}$$

then

$$-\frac{\pi}{2}\eta_1 < \arg\{zD^\alpha f(z)\} < \frac{\pi}{2}\eta_2, \tag{2.19}$$

where η_1 *and* η_2 *are the solutions of the equations*:

$$\delta_1 = \eta_1 + \frac{2}{\pi}\arctan\left\{\frac{\lambda(\eta_1+\eta_2)}{2(\alpha+1)}\left(\frac{1-|a|}{1+|a|}\right)\right\} \tag{2.20}$$

and

$$\delta_2 = \eta_2 + \frac{2}{\pi}\arctan\left\{\frac{\lambda(\eta_1+\eta_2)}{2(\alpha+1)}\left(\frac{1-|a|}{1+|a|}\right)\right\}, \tag{2.21}$$

when

$$a = i\tan\left\{\frac{\eta_2-\eta_1}{\eta_2+\eta_1}\right\}.$$

Proof. Let

$$p(z) = zD^\alpha f(z) \quad (z \in \mathbb{U}).$$

Then by using (1.3), we have

$$(1-\lambda)zD^\alpha f(z) + \lambda zD^{\alpha+1}f(z) = p(z) + \frac{\lambda}{\alpha+1}zp'(z). \tag{2.22}$$

Let h be the function which maps $\mathbb{U}$ onto the angular domain $\{w \in \mathbb{C} : -\pi\delta_1/2 < \arg\{w\} < \pi\delta_2/2\}$ with $h(0) = 1$. Then from (2.18) and (2.22), we get

$$p(z) + \frac{\lambda}{\alpha+1}zp'(z) \prec h(z).$$

Therefore an application of Lemma2.1 yields $\mathrm{Re}\{p(z)\} > 0$ for $z \in \mathbb{U}$ and hence $p(z) \neq 0$ for $z \in \mathbb{U}$.

Suppose that there exists two points $z_1, z_2 \in \mathbb{U}$ such that the condition (2.1) is satisfied. Then by Lemma 2.3 , we obtain (2.2) under the restriction (2.3). Therefore we have

$$\begin{aligned}
\arg\left\{p(z_1) + \frac{\lambda}{\alpha+1}z_1p'(z_1)\right\} &= \arg\{p(z_1)\} + \arg\left\{\alpha+1+\lambda\frac{z_1p'(z_1)}{p(z_1)}\right\} \\
&= -\frac{\pi}{2}\eta_1 + \arg\left\{\alpha+1-i\frac{\lambda(\eta_1+\eta_2)}{2}m\right\} \\
&\leq -\frac{\pi}{2}\eta_1 - \arctan\left\{\frac{\lambda(\eta_1+\eta_2)}{2(\alpha+1)}\left(\frac{1-|a|}{1+|a|}\right)\right\} \\
&= -\frac{\pi}{2}\delta_1
\end{aligned}$$

and

$$\arg\left\{p(z_2)+\frac{\lambda}{\alpha+1}z_2p'(z_2)\right\}\geq\frac{\pi}{2}\eta_1+\arctan\left\{\frac{\lambda(\eta_1+\eta_2)}{2(\alpha+1)}\left(\frac{1-|a|}{1+|a|}\right)\right\}$$
$$=\frac{\pi}{2}\delta_2,$$

which contradict the assumption (2.18). Therefore we have the assertion (2.19). □

For $\delta_1=\delta_2=\delta$ in Theorem 2.7, we have the following result.

Corollary 2.8. *Let $\alpha>-1$, $\lambda\geq 0$ and $0<\delta\leq 1$. If $f\in\Sigma$ satisfies*

$$|\arg\{(1-\lambda)zD^{\alpha}f(z)+\lambda zD^{\alpha+1}f(z)\}|<\frac{\pi}{2}\delta,$$

then

$$|\arg\{zD^{\alpha}f(z)\}|<\frac{\pi}{2}\eta,$$

where η is the solutions of the equation:

$$\delta=\eta+\frac{2}{\pi}\arctan\left\{\frac{\lambda}{\alpha+1}\right\}.$$

Now we consider the following integral operator F_c (see [2, 6, 9, 10]) defined by

$$F_c(f)(z)=\frac{c}{z^c+1}\int_0^z f(t)t^c dt\quad(\mathrm{Re}\{c\}\geq 0). \tag{2.23}$$

Theorem 2.9. *Let $\alpha>-1$, $c\geq 0$ and $0<\delta_1,\delta_2\leq 1$. If $f\in\Sigma$ satisfies*

$$-\frac{\pi}{2}\delta_1<\arg\{zD^{\alpha}f(z)\}<\frac{\pi}{2}\delta_2,$$

then

$$-\frac{\pi}{2}\eta_1<\arg\{zD^{\alpha}F_c(z)\}<\frac{\pi}{2}\eta_2,$$

where F_c is the integral operator defined by (2.23), *and η_1 and η_2 are the solutions of the equations* (2.20) *and* (2.21) *with $\alpha=c-1$ and $\lambda=1$.*

Proof. Let

$$p(z) = zD^{\alpha}F_c(z) \quad (z \in \mathbb{U}).$$

From the definition of F_c, it can be verified that

$$z(D^{\alpha}F_c(z))' = cD^{\alpha}f(z) - (c+1)D^{\alpha}F_c(z). \tag{2.24}$$

Therefore, using (2.24) and (1.3) for F_c, we have

$$zD^{\alpha}f(z) = p(z) + \frac{1}{c}zp'(z).$$

The remaining part of the proof is similar to that of Theorem 2.7 and so we omit for details. □

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References

[1] H.Kh. Abdullah, *Studt on meromorphic Hurwitz-Zeta function defined by linear operator*, Nonlinear Funct. Anal. Appl., **24**(1) (2019), 195-206.
[2] S.K. Bajpai, *A note on a class of meromorphic univalent functions,* Rev. Roumaine Math. Pures Appl., **22** (1977), 295-297.
[3] K.A. Challab, M. Darus and F. Ghanim, Înclusion properties of meromorphic functions associated with the extended Cho-Kwon-Srivastava operator by using hypergeometric function, Nonlinear Funct. Anal. Appl., **22**(5) (2017), 935-946.
[4] M.R. Ganigi and B.A. Uralegaddi, *New Criteria for meromorphic univalent functions,* Bull. Math. Soc. Sci. Math. R. S. Roumanie (N. S.) **33**(81) (1989), 9-13.
[5] R.M. Goel and N.S. Sohi, *A new criterion for p-valent functions,* Proc. Amer. Math. Soc., **78** (1980), 353-357.
[6] R.M. Goel and N.S. Sohi, *On a class of meromorphic functions, Glasnik Mat. Ser. III* **17**(37) (1981), 19-28.
[7] D.J. Hallenbeck and St. Ruscheweyh, *Subordination by convex functions,* Proc. Amer. Math. Soc., **52** (1975), 191-195.
[8] S.S. Miller and P.T. Mocanu, *Differential subordination and univalent functions,* Michigan Math. J., **28** (1981), 157-171.
[9] S.S. Miller and P.T. Mocanu, *Differential Subordination, Theory and Application* Marcel Dekker, Inc., New York, Basel, 2000.
[10] P.T. Mocanu and G.S. Salagean, *Integral operators and meromorphic starlike functions,* Mathematica, **32**(55) (1990), 147-152.
[11] St. Ruscheweyh, *New criteria for univalent functions,* Proc. Amer. Math. Soc., **49** (1975), 109-115.
[12] T.J. Suffridge, *Some remarks on convex maps of the unit disk,* Duke Math. J., **37** (1970), 775-777.

[13] N. Takahashi and M. Nunokawa, *A certain connection between starlike and convex functions,* Appl. Math. Lett., **16** (2003), 653-655.

[14] B.A. Uralegaddi and C. Somanatha, *Certain subclasses of meromorphic convex functions,* Indian J. Math., **32** (1990), 49-57.

[15] B.A. Uralegaddi and C. Somanatha, *On generalization of meromorphic Convex functions with negative coefficients,* Mathematica **35**(58) (1993), 99-107.