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EXISTENCE RESULTS FOR NEW GENERALIZED MIXED EQUILIBRIUM AND FIXED POINT PROBLEMS IN BANACH SPACES

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Abstract. In this paper, existence and uniqueness results for the solution of a new class of equilibrium problem is established. Using the KKM technique, we establish the existence and uniqueness of solutions of a new generalized mixed equilibrium problem (NGMEP) with trifunctions. Further, we propose an iterative algorithm for finding a common element in the solution set of the NGMEP and a fixed point set of a nonlinear mapping. We proved the strong convergence of the algorithm to a common element in the solution set of a system of NGMEP and a fixed point set of a countable family of totally quasi-ϕ-asymptotically nonexpansive mapping in the framework of a real uniformly convex and uniformly smooth Banach space. Our result generalize many other results obtained recently in this direction.

1. Introduction

The theory of variational inequality plays a very important role in many fields

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such as mechanics, physics, structural analysis, nonlinear programming, transportation, sciences and engineering. Because of their wide range of applications, variational inequality problems have been extensively studied, extended and generalized since its introduction in the sixties by Stampacchia [33]. One of the most important of such generalizations is the equilibrium problems.

Monotonicity and generalized monotonicity are very important tools in the study of equilibrium problems, mixed equilibrium problems and generalized mixed equilibrium problems. In recent years, there has been lots of research carried out on the existence results of the solutions of equilibrium problems, mixed equilibrium problems and generalized mixed equilibrium problems based on the different generalizations of monotonicity such as pseudomonotonicity, quasimonotonicity, relaxed monotonicity and relaxed semimonotonicity and so on (see [3, 4, 5, 14, 15, 16, 18, 35, 36]), and the references therein.

Karamardian and Schaible [20] introduced various kinds of generalized monotone mappings. This opened the door of many pieces of research papers aimed at extending the idea of Karamardian and Schaible for bifunctions to study equilibrium problem. In 2003, Fang and Huang [10] introduced the concept of variational-like inequalities with the relaxed η-α monotone and relaxed η-α semimonotone mappings. They obtained the existence results for variational-like inequalities with realxed η-α monotone and relaxed η-α semimonotone mappings in Banach spaces. Later, Bai et al extended this concept to η-α pseudomonotone mappings and obtained solutions for variational-like inequalities.

For equilibrium problems, Mahato and Nahak [24] introduced the concept of relaxed α-monotonicity bifunctions. They proved the existence of solutions for mixed equilibrium problems with the relaxed α-monotone bifunction in reflexive Banach space by using the KKM technique.

On the other hand, the iterative approximation of fixed point of totally quasi-ϕ-aymptotically nonexpansive mapping have been considered in the literature. In 2013, Saewan et al. [32], introduced an hybrid projection algorithm by the use of generalized f-projection for a countable family of totally quasi-ϕ-asymptotically nonexpansive mappings in a uniformly smooth and strictly convex Banach space. They proposed and proved the strong convergence of the following algorithm under appropriate conditions on the control parameters to a point $p = \pi^f_{\bigcap_{i=1}^{\infty} F(T_i)} x_1$ in the fixed point of T_i for all i :

$$\begin{cases} y_{n,i} = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T_i x_n), \\ C_{n+1,i} = \{u \in C_n : G(u, J y_{n,i}) \leq G(u, J x_n) + \gamma_n\}, \\ C_{n+1} = \bigcap_{i=1}^{\infty} C_{n,i}, \\ x_{n+1} = \pi^f_{C_{n+1}} x_1. \end{cases}$$

Furthermore, the problem of finding a common point in the set of solutions of different variations of an equilibrium problem and the fixed point set of a nonlinear mapping have been considered in several recent articles in the literature, see ([11, 12, 13, 21, 25, 26, 31, 34, 37, 39]) and the references therein.

In 2010, Petrot et al. [26] introduced an hybrid projection iterative algorithm for approximating a common element of the set of solutions of a generalized mixed equilibrium problem and the set of fixed points of two quasi-ϕ-nonexpansive mappings T and S in a real uniformly convex and uniformly smooth Banach space.

Recently, Saewan and Kumam [31] presented a new hybrid Ishikawa iteration process for finding a common solution of the fixed points for two countable families of weak relatively nonexpansive mappings and the set of solutions of the system of generalized equilibrium problems in a uniformly convex and uniformly smooth Banach space.

Very recently, Mahato et al [23] proposed a hybrid iterative algorithm by using a generalized f-projection operator to find a common element of the solutions of a system of trifunction equilibrium problems and the set of fixed points of an infinite family of quasi-ϕ-nonexpansive mappings. They obtained a strong convergence of the proposed method under generalized relaxed α-monotonicity of the trifunctions in the framework of a uniformly convex and uniformly smooth Banach space. To be precise, they proposed the following algorithm:

$$\begin{cases} x_0 = x \in C,\ C_0 = C,\ Q_0 = C, \\ z_n = J^{-1}\left(\alpha_{n_0} J x_n + \sum_{i=1}^{\infty} \alpha_{n_i} J T_i x_n\right), \\ y_n = J^{-1}(\delta_n J x_n + (1-\delta_n) J z_n), \\ u_n = T^{\psi_j}_{r_{j,n}} T^{\psi_{j-1}}_{r_{j-1,n}} \cdots T^{\psi_2}_{r_{2,n}} T^{\psi_1}_{r_{1,n}} y_{n,i}, \text{where} \\ T^{\psi_j}_{r_{j,n}} y_{n,i} = \{z \in C : \psi_j(y, z; z) + \dfrac{1}{r_{j,n}} \langle y - z, Jz - J y_n\rangle \geq 0, \forall y \in C\}, \\ C_n = \{w \in C_{n-1} : G(w, J u_n) \leq G(w, J y_n) \leq G(w, J x_n),\ n \geq 1\}, \\ Q_n = \{w \in Q_{n-1} : \langle x_n - w, Jx - J x_n\rangle + \rho f(w) - \rho f(x_n) \geq 0\},\ n \geq 1, \\ x_{n+1} = \pi^f_{C_n \cap Q_n} x. \end{cases} \tag{1.1}$$

They proved that $\{x_n\}$ converges to a point $\pi_F^f x$ under suitable conditions, where F is the set of solutions of the considered problem and J is the normalized duality mapping.

Motivated by the above results and the current research interest in this direction, our main purpose of this paper is as follows:

- to introduce a new class of generalized mixed equilibrium problem;
- to obtain the existence result for this new generalized mixed equilibrium problem using the KKM technique;
- to propose an iterative algorithm and prove its strong convergence to a common element in the solution set of a system of the equilibrium problems and fixed point set of a countable family of totally quasi-ϕ-asymptotically nonexpansive mappings.

Our results extend and improve the result of Mahato et al. [24] and other results in the literature.

2. Preliminaries

Let C be a nonempty, closed and convex subset of a real smooth, strictly convex and reflexive Banach space E with dual space E^*. Let $\varphi : C \to \mathbb{R} \cup \{\infty\}$ be a real valued function, $A : C \to E^*$ be a nonlinear mapping and $\psi : C \times C \times C \to \mathbb{R}$ be a trifunction. Then, for $r > 0$ and $z \in C$, the New Generalized Mixed Equilibrium Problem (NGMEP) consists of finding a point $x \in C$ such that

$$\psi(y, x; x) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x)\rangle \geq 0, \ \forall y \in C; \tag{2.1}$$

and find $x \in C$ such that

$$\psi(y, x; y) + \langle Ay, y - x\rangle + \varphi(y) - \phi(x) \geq \alpha(x, y), \ \forall y \in C. \tag{2.2}$$

The solution set of (2.1) will be denoted by Ω, that is

$$\Omega = \{x \in C : \Psi(y, x; x) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \ \forall y \in C\}. \tag{2.3}$$

If $\psi(\cdot, \cdot; \cdot) = \psi(\cdot, \cdot)$, then (2.1) reduces to the GMEP which is to find a point $x \in C$ such that

$$\psi(x, y) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \ \forall y \in C. \tag{2.4}$$

Let E be a real Banach space and $B = \{x \in E : ||x|| = 1\}$ be a unit sphere. The modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{1}{2}||x + y|| : ||x|| = ||y|| = 1, ||x - y|| \geq \epsilon\right\}. \tag{2.5}$$

Recall that E is said to be uniformly convex if $\delta_E(\epsilon) > 0$ for any $\epsilon \in (0, 2]$. E is said to be strictly convex if $\frac{||x+y||}{2} < 1$ for all $x, y \in E$ with $||x|| = ||y|| = 1$ and $x \neq y$. It is known that every uniformly convex Banach space is strictly convex and reflexive [27].

A Banach space E is said to be smooth if the limit

$$\lim_{t\to 0} \frac{||x+ty|| - ||x||}{t}$$

exists for all $x, y \in B$. It is said to be uniformly smooth if the limit is attained uniformly, $x, y \in B$. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = ||x||^2 = ||x^*||^2\}, \quad \forall x \in E.$$

It is widely known that if E is uniformly smooth, then the duality mapping J is norm-to-norm continuous on each bounded subset of E. The following are some important and useful properties of J, for further details, see [1, 36]:

- For every $x \in E$, Jx is nonempty, closed, convex and bounded subset of E^*.
- If E is smooth or E^* is strictly convex, then J is single valued. Also, if E is reflexive, then J is onto.
- If E is strictly convex, then J is strictly monotone, that is

$$\langle x - y, Jx - Jy \rangle > 0.$$

- If E is smooth, strictly convex and reflexive and $J^* : E^* \to 2^E$ is the normalized duality mapping on E^*, then $J^{-1} = J^*$, $JJ^* = I_{E^*}$ and $J^*J = I_E$, where I_E and I_{E^*} are the identity mappings on E and E^* respectively.
- If E is uniformly convex and uniformly smooth, then J is uniformly norm-to-norm continuous on bounded subsets of E and $J^* = J^{-1}$ is also uniformly norm-to-norm continuous on bounded subsets of E^*.

Next, we recall the concept of generalized f-projection operator.
Let $G : C \times E^* \to \mathbb{R}$ be a function defined as follows:

$$G(a, b) = ||a||^2 - 2\langle a, b \rangle + ||b||^2 + 2\rho f(a), \tag{2.6}$$

where $a \in C$ and $b \in E^*$, ρ is a positive number and $f : C \to \mathbb{R} \cup \{\infty\}$ is proper, convex and lower semi-continuous. From the definition of G and f, the following properties are easy to see:

(i) $G(a, \cdot)$ is convex and continuous in the second variable with a fixed;
(ii) $G(\cdot, b)$ is convex and lower semi-continuous in the first variable, with b fixed.

Definition 2.1. ([38]) Let C be a nonempty, closed and convex subset of a real Banach space E with its dual space E^*. The operator $\pi_C^f : E^* \to 2^C$ defined by

$$\pi_C^f b = \{u \in C : G(u,b) = \inf_{a \in C} G(a,b)\} \ \forall b \in E^*,$$

is called the generalized f-operator.

Wu and Huang [38] proposed and proved the following theorem for the generalized f-projection operator.

Lemma 2.2. *Let C be a nonempty, closed and convex subset of a real reflexive smooth Banach space E. Then the following statements hold:*

(i) *$\pi_C^f b$ is a nonempty, closed and convex subset of C for all $b \in E^*$.*

(ii) *If E is smooth, then for all $b \in E^*$, $x \in \pi_C^f b$ if and only if*

$$\langle x - y, b - Jx\rangle + \rho f(y) - \rho f(x) \geq 0, \ \forall y \in C.$$

(iii) *If E is strictly convex and $f : C \to \mathbb{R} \cup \{\infty\}$ is positive homogeneous i.e ($f(tx) = tf(x)$, for all $t > 0$ such that $tx \in C$ with $x \in C$), then π_C^f is a single valued mapping.*

Recall that the normalized duality mapping J is single valued when E is a smooth Banach space. There exists a unique element $b \in E^*$ such that $b = Jx$ where $x \in E$. This substitution in (2.6), yields

$$G(a, Jx) = ||a||^2 - 2\langle a, Jx\rangle + ||x||^2 + 2\rho f(a).$$

Now, we consider another type of generalized f-operator in Banach spaces given in [22].

Definition 2.3. Let C be a nonempty, closed and convex subset of real smooth Banach space E. We say that $\pi_C^f : E \to 2^C$ is generalized f-projection operator if

$$\pi_C^f x = \{u \in C : G(u, Jx) = \inf_{a \in C} G(a, Jx)\}, \ \forall x \in E.$$

Lemma 2.4. ([22]) *Let C be a nonempty, closed and convex subset of a real reflexive smooth Banach space E. Then the following statements hold:*

(i) *$\pi_C^f x$ is a nonempty, closed and convex subset C for all $x \in E$.*

(ii) *For all $x \in E$, $\hat{x} \in \pi_C^f x$ if and only if*

$$\langle \hat{x} - y, Jx - J\hat{x}\rangle + \rho f(y) - \rho f(x) \geq 0, \ \forall y \in C.$$

(iii) *If E is strictly convex, then π_C^f is a single valued mapping.*

Lemma 2.5. ([8]) *Let E be a Banach space and $f : E \to \mathbb{R} \cup \{\infty\}$ be a proper lower semi-continuous convex functional. Then there exists $x^* \in E^*$ and $\alpha \in \mathbb{R}$ such that*

$$f(x) \geq \langle x, x^* \rangle + \alpha.$$

Let C be a closed and convex subset of E and $T : C \to C$ be a mapping. A point $x \in C$ is called a fixed point of T, if $x = Tx$. A point $p \in C$ is called an asymptotic fixed point of T (see [30]), if C contains a sequence $\{x_n\}$ which converges weakly to p such that $||x_n - Tx_n|| \to 0$ as $n \to \infty$. We denote the set of fixed points and asymptotic fixed points of T by $F(T)$ and $\hat{F}(T)$ respectively.

A mapping $T : C \to C$ is said to be

(1) relatively nonexpansive, if $\hat{F}(T) = F(T)$ and

$$\phi(p, Tx) \leq \phi(p, x), \ \forall \ x \in C, \ p \in F(T); \tag{2.7}$$

(2) relatively asymptotically nonexpansive [2], if $\hat{F}(T) = F(T)$ and there exists a sequence $\{k_n\} \subset [1, +\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\phi(p, T^n x) \leq k_n \phi(p, x), \ \forall \ x \in C, \ p \in F(T); \tag{2.8}$$

(3) quasi-ϕ-asymptotically nonexpansive [40], if $F(T) \neq \emptyset$ and

$$\phi(p, Tx) \leq \phi(p, x), \ \forall \ x \in C, \ p \ \in F(T); \tag{2.9}$$

(4) quasi-ϕ-asymptotically nonexpansive [40], if $F(T) \neq \emptyset$ and

$$\phi(p, T^n x) \leq k_n \phi(p, x), \ \forall \ x \in C, \ p \in F(T); \tag{2.10}$$

(5) totally quasi-ϕ-asymptotically nonexpansive, if $F(T) \neq \emptyset$ and there exist nonnegative real sequences $\{\lambda_n\}$, $\{\mu_n\}$ with $\lambda_n \to 0$, $\mu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\vartheta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\vartheta(0) = 0$ such that

$$\phi(p, T^n x) \leq \phi(p, x) + \lambda_n \vartheta(\phi(p, x)) + \mu_n, \ \forall \ x \in C, \ p \in F(T). \tag{2.11}$$

Remark 2.6. It is clear that the class of relative nonexpansive mapping is contained in the class of relative quasi-nonexpansive mapping, the class of quasi-ϕ-nonexpansive mapping is contained in the class of quasi-ϕ-asymptotically nonexpansive mapping and the class of quasi-ϕ-asymptotically is contained in the class of totally quasi-ϕ-asymptotically nonexpansive mapping. The converses are not true.

If $f(y) > 0$ for all $y \in C$ with $f(0) = 0$, then the definition of totally quasi-ϕ-nonexpansive mapping T is equivalent to the following: If $F(T) \neq \emptyset$ and there exist nonnegative real sequences $\{\lambda_n\}, \{\mu_n\}$ such that $\lambda_n \to 0$, $\mu_n \to 0$

as $n \to \infty$ and a strictly continuous function $\vartheta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\vartheta(0) = 0$ such that:

$$G(p, JT^n x) \leq G(p, Jx) + \lambda_n \vartheta(G(p, Jx)) + \mu_n, \ \forall\, x \in C, \ p \in F(T).$$

Lemma 2.7. ([39]) *Let E be a uniformly convex Banach space, $r > 0$ be a positive number and $B_r(0)$ be a closed ball of E. Then, for any given $\{x_n\}_{n=1}^{\infty} \subset B_r(0)$ and a given sequence $\{\lambda_n\}_{n=1}^{\infty}$ of positive number with $\sum_{n=1}^{\infty} \lambda_n = 1$, there exists a continuous strictly increasing and convex function $g : [0, 2r] \to [0, \infty)$ with $g(0) = 0$ such that for any $i, j \in \mathbb{N}$, with $i < j$*

$$\left\| \sum_{n=1}^{\infty} \lambda_n x_n \right\|^2 \leq \sum_{n=1}^{\infty} \lambda_n ||x_n||^2 - \lambda_i \lambda_j g(||x_i - x_j||).$$

Lemma 2.8. ([19]) *Let C be a nonempty, closed and convex subset of a real reflexive smooth Banach space E, let $x \in E$ and $\hat{x} \in \pi_C^f x$. Then*

$$\phi(y, \hat{x}) + G(\hat{x}, Jx) \leq G(y, Jx), \ \forall y \in C. \tag{2.12}$$

Lemma 2.9. ([19]) *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences in E such that $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim\limits_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim\limits_{n\to\infty} ||x_n - y_n|| = 0$.*

Lemma 2.10. ([29]) *Let C be a nonempty, closed and convex subset of a uniformly convex and smooth Banach space and let T be a closed and quasi-ϕ-nonexpansive mapping from C into itself. Then $F(T)$ is a closed and convex subset of C.*

Recall that an operator $A : C \to E^*$ is κ-inverse strongly monotone, if

$$\langle x - y, Ax - Ay \rangle \geq \kappa ||Ax - Ay||^2, \tag{2.13}$$

for all $x, y \in C$ and $\kappa > 0$.

Throughout this paper, unless otherwise stated, C is a nonempty, closed and convex subset of a smooth strictly convex and reflexive Banach space E with dual E^*.

Definition 2.11. A function $\psi : C \times C \times C \to \mathbb{R}$ is said to be generalized relaxed α-monotone if for any $x, y \in C$, we have

$$\psi(y, x; y) \geq \psi(y, x; x) + \alpha(x, y), \tag{2.14}$$

where $\lim\limits_{t\to 0^+} \dfrac{\alpha(x, x - t(x - y))}{t} = 0$.

If $\alpha \equiv 0$ in Definition 2.11, we say that the function ψ is generalized monotone.

Remark 2.12. (i) If in Definition 2.11, $\alpha(x,y) = \beta(y-x)$, where $\beta : C \to \mathbb{R}$ with $\beta(tz) = t^p\beta(z)$, for $t > 0$, $p > 1$, then we say ψ is relaxed β-monotone.

For an example of a generalized relaxed α-monotone mapping, we refer the reader to [28].

Definition 2.13. A real valued function ψ on C is said to be hemicontinuous if $\lim_{t\to 0^+} \psi(y,x;x-t(x-y)) = \psi(y,x;x)$, for each $x, y \in C$.

Definition 2.14. Let $F : C \to 2^E$ be a multi-valued mapping. Then F is said to be a KKM mapping if for any sequence $\{y_i\}_{i=1}^n$ of C, we have $co\{y_i\}_{i=1}^n \subset \cup_{i=1}^n F(y_i)$, where $co\{y_i\}_{i=1}^n$ is the convex hull of $\{y_i\}_{i=1}^n$.

Lemma 2.15. ([9]) *Let M be a nonempty subset of a Hausdorff topological vector E and let $F : M \to 2^E$ be a KKM mapping. If $F(y)$ is closed in E for all $y \in M$, then $\cap_{y\in M} F(y)$ is nonempty.*

Lemma 2.16. ([29]) *Let C be a closed convex subset of a uniformly convex and smooth Banach space and T a closed and quasi-ϕ-nonexpansive mapping from C into itself. Then $F(T)$ is a closed and convex subset C.*

Let $T : C \to C$ be a nonlinear mapping. T is said to be uniformly asymptotically regular on C if

$$\lim_{n\to\infty} \left(\sup_{x\in C} ||T^{n+1}x - T^n x|| \right) = 0.$$

T is said to be closed, if for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = p$ and $\lim_{n\to\infty} Tx_n = q$, then $Tp = q$.

Lemma 2.17. ([7]) *Let C be a nonempty, closed and convex subset of a real uniformly smooth and uniformly convex Banach space E. Let $T : C \to C$ be a total quasi-ϕ-asymptotically nonexpansive mapping with nonnegative real sequences $\{\lambda_n\}, \{\mu_n\}$ such that $\lambda_n \to 0$, $\mu_n \to 0$ as $n \to \infty$. Let $\vartheta : \mathbb{R}^+ \to \mathbb{R}^+$ be a strictly increasing continuous function with $\vartheta(0) = 0$. If $\mu_1 = 0$, then $F(T)$ is a closed and convex subset of C.*

Assumption A: In what follows, the trifunction $\psi : C \times C \times C \to \mathbb{R}$ will be assumed to satisfy the following:

(i) $\psi(y,x;\cdot)$ is hemicontinuous;
(ii) $\psi(\cdot,x;z)$ is convex and lower semicontinuous;

(iii) $\psi(x, x; z) = 0$;
(iv) ψ is generalized relaxed α-monotone;
(v) $\psi(x, y; z) + \psi(y, x; z) = 0$;
(vi) $\alpha(\cdot, y)$ is lower semicontinuous.

3. Existence results

First, we prove the following lemma which shows the equivalence of Problem (2.1) and (2.2).

Lemma 3.1. *Let C be a nonempty, closed, convex and bounded subsets of a smooth, strictly convex and reflexive Banach space E. Let $\psi : C \times C \times C \to \mathbb{R}$ be a function satisfying (i)-(iv) of Assumption A, $A : C \to E^*$ be a nonlinear mapping and $\varphi : C \to \mathbb{R} \cup \{\infty\}$ be a real valued function. Then the Problems* (2.1) *and* (2.2) *are equivalent.*

Proof. Let $x \in C$ be a solution of (2.1). Then, from the generalized α-monotonicity of ψ, we have

$$\begin{aligned}\psi(y, x; y) + \langle Ay, y - x\rangle + \varphi(y) - \varphi(x) &\geq \psi(y, x; x) + \alpha(x, y) \\ &\quad + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x) \\ &\geq \alpha(x, y). \qquad (3.1)\end{aligned}$$

Hence $x \in C$ is a solution of (2.2). Conversely, let $x \in C$ be a solution of (2.2) and let $y \in C$ be any point with $\varphi(y) < \infty$. It follows from (2.2) that $\varphi(x) < \infty$. Let $y_t = (1-t)x + ty$, for all $t \in (0, 1)$. Then we have $y_t \in C$. Since $x \in C$ is a solution of (2.2), it follows that

$$\psi(y_t, x; y_t) + \langle Ay_t, y_t - x\rangle + \varphi(y_t) - \varphi(x) \geq \alpha(x, y_t). \qquad (3.2)$$

By conditions (i) and (ii), we have the following estimates:

$$\psi(y_t, x; y_t) \leq t\psi(y, x; y_t) + (1-t)\psi(x, x; y_t) = t\psi(y, x; y_t),$$

$$\psi(y_t, x; y_t) \leq \psi(y, x; y_t),$$

$$\begin{aligned}\langle Ay_t, y_t - x\rangle &= \langle Ay_t, (1-t)x + ty - x\rangle \\ &= t\langle Ay_t, y - x\rangle,\end{aligned}$$

$$\begin{aligned}\varphi(y_t) - \varphi(x) &= \phi((1-t)x + ty) - \phi(x) \\ &\leq (1-t)\varphi(x) + t\psi(y) - \varphi(x) \\ &= t(\psi(y) - \psi(x)). \qquad (3.3)\end{aligned}$$

From (3.2) and (3.3), we have

$$t\psi(y, x; y_t) + t\langle Ay_t, y - x\rangle + t(\varphi(y) - \varphi(x)) \geq \alpha(x, y_t).$$

Hence,

$$\psi(y,x;y_t)+\langle Ay_t,y-x\rangle+\varphi(y)-\varphi(x)\geq\frac{\alpha(x,y_t)}{t}.$$

By (i) and letting $t\to 0$, we get

$$\psi(y,x;x)+\langle Ax,y-x\rangle+\varphi(y)-\varphi(x)\geq 0. \tag{3.4}$$

Thus, $x\in C$ is a solution of (2.1). $\square$

Theorem 3.2. *Let C be a nonempty, bounded, closed and convex subsets of a smooth, strictly convex and reflexive Banach space E. Let $\psi: C\times C\times C\to\mathbb{R}$ be a trifunction satisfying $(i),(ii)$ and (iv)-(vi) of Assumption A, $A: C\to E^*$ be an inverse strongly monotone operator and $\varphi: C\to\mathbb{R}$ be a proper lower semicontinuous mapping. Then Problem (2.1) is solvable.*

Proof. Define two multi-valued mappings $F,G: C\to 2^{E^*}$ as follows:

$$F(y)=\{x\in C:\psi(y,x;x)+\langle Ax,y-x\rangle+\varphi(y)-\varphi(x)\geq 0,\ \forall y\in C\}$$

and

$$G(y)=\{x\in C:\psi(y,x;y)+\langle Ay,y-x\rangle+\varphi(y)-\varphi(x)\geq\alpha(x,y),\ \forall y\in C\}.$$

Observe that $x^*\in C$ is a solution of (2.1) if and only if $x^*\in\cap_{y\in C}F(y)$. Thus, it is sufficient to show that $\cap_{y\in C}F(y)\neq\emptyset$.

We claim that F is a KKM mapping. For if F is not a KKM mapping, then there exists $\{y_i\}_{i=1}^n\subset C$ such that $co\{y_i\}_{i=1}^n\not\subseteq\cup_{i=1}^nF(y_i)$, that means there exists a $x_0\in co\{y_i\}_{i=1}^n$, with $x_0=\sum\limits_{i=1}^n t_iy_i$ where $t_i\geq 0$, $i=1,2\cdots,n$ and $\sum\limits_{i=1}^n t_i=1$, but $x_0\notin\cup_{i=1}^nF(y_i)$. By the defintion F, we have

$$\psi(y_i,x_0;x_0)+\langle Ax_0,y_i-x_0\rangle+\varphi(y_i)-\varphi(x_0)<0,\quad i=1,2\cdots n. \tag{3.5}$$

From (ii) and (v), we have

$$\begin{aligned}0&=\psi(x_0,x_0;x_0)+\langle Ax_0,x_0-x_0\rangle+\varphi(x_0)-\varphi(x_0)\rangle\\&\leq\sum_{i=1}^n t_i\psi(y_i,x_0,x_0)+\sum_{i=1}^n t_i\langle Ax_0,y_i-x_0\rangle+\sum_{i=1}^n t_i\varphi(y_i)-\varphi(x_0)\\&=\sum_{i=1}^n t_i[\psi(y_i,x_0,x_0)+\langle Ax_0,y_i-x_0\rangle+\varphi(y_i)-\varphi(x_0)]<0,\end{aligned} \tag{3.6}$$

which is a contradiction. Hence F is a KKM mapping.

Next, we show that $F(y)\subset G(y)$ for all $y\in C$. For any given $y\in C$, let $x\in F(y)$. Then

$$\psi(y,x;x)+\langle Ax,y-x\rangle+\varphi(y)-\varphi(x)\geq 0.$$

Since ψ is a generalized relaxed α-monotone, we have

$$\begin{aligned}\psi(y,x;y)+\langle Ay,y-x\rangle+\varphi(y)-\varphi(x) &\geq \psi(y,x;x)+\alpha(x,y)\\ &\quad+\langle Ax,y-x\rangle+\varphi(y)-\varphi(x)\\ &\geq \alpha(x,y). \end{aligned}\tag{3.7}$$

Therefore $F(y)\subset G(y)$ for all $y\in C$. This implies that G is also a KKM mapping.

Let $\{x_n\}$ be any sequence in $G(y)$ such that $x_n \rightharpoonup x$ as $n\to\infty$. It follows that

$$\psi(y,x_n;y)+\langle Ay,y-x_n\rangle+\varphi(y)-\varphi(x_n)\rangle \geq \alpha(x_n,y). \tag{3.8}$$

By (ii), continuity of J, lower semicontinuity of $||\cdot||^2$ and α, we have

$$\begin{aligned}0 &\leq \limsup_{n\to\infty}[-\alpha(x_n,y)+\psi(y,x_n;y)+\langle Ay,y-x_n\rangle+\varphi(y)-\varphi(x_n)\\ &\leq \limsup_{n\to\infty}(-\alpha(x_n,y))+\limsup_{n\to\infty}\psi(y,x_n,y)\\ &\quad+\limsup_{n\to\infty}\langle Ay,y-x_n\rangle+\limsup_{n\to\infty}[\varphi(y)-\varphi(x_n)]\\ &\leq -\liminf_{n\to\infty}\alpha(x_n,y)+\limsup_{n\to\infty}\psi(y,x_n;y)\\ &\quad+\limsup_{n\to\infty}\langle Ay,y-x_n\rangle+\limsup_{n\to\infty}[\varphi(y)-\varphi(x_n)]\\ &\leq -\liminf_{n\to\infty}\alpha(x_n,y)+\limsup_{n\to\infty}\psi(y,x_n;y)\\ &\quad+\limsup_{n\to\infty}\langle Ay,y-x_n\rangle+\limsup_{n\to\infty}[\varphi(y)-\varphi(x_n)]\\ &\leq -\alpha(x,y)+\psi(y,x;y)+\langle Ay,y-x\rangle+\varphi(y)-\varphi(x),\end{aligned}$$

which implies

$$\psi(y,x;y)+\langle Ay,y-x\rangle+\varphi(y)-\varphi(x)\geq \alpha(x,y). \tag{3.9}$$

This shows that $x\in G(y)$, and hence $G(y)$ is closed for all $y\in C$.

Since C is a closed, bounded and convex subset of a reflexive Banach space E, it is weakly compact. Hence $G(y)$ is also weakly compact. Thus, all conditions of Lemma 2.15 are satisfied. Therefore, by Lemma 2.15 and (2.1), we get $\bigcap_{y\in C}F(y)=\bigcap_{y\in C}G(y)\neq\emptyset$. Hence the Problem (2.1) is solvable. □

Definition 3.3. Let C be a nonempty, bounded, closed and convex subset of a smooth, strictly convex and reflexive Banach space E. Let $\psi: C\times C\times C\to\mathbb{R}$ be a trifunction satisfying conditions (i),(ii) and (iv)-(vi), $A: C\to E^*$ be a nonlinear mapping and $\varphi: C\to\mathbb{R}\cup\{\infty\}$ be a proper lower semicontinuous function. Assume that (2.1) has a unique solution, for each $r>0$ and $x\in E$. This unique solution is T_r^{ψ} and is called the resolvent operator associated with

(ψ, A, φ) of order r at $x \in E$. In other words, the resolvent operator associated with (ψ, A, φ) is the multi-valued mapping $T_r^{\psi} : E \to 2^C$ defined by

$$T_r^{\psi}(x) := \{z \in C : \psi(y, z; z) + \langle Az, y - z\rangle + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \forall y \in C\}, \tag{3.10}$$

for all $x \in E$.

Under the assumptions of Theorem 3.2, we have the unique existence of $T_r^{\psi}(x)$. Hence, $T_r^{\psi}(x)$ is well defined. We give the following fundamental properties of the resolvent operator T_r^{ψ}. First, we give the following useful definition.

Definition 3.4. Let E be a Banach space. A nonlinear mapping $T : E \to E$ is said to be firmly nonexpansive type if

$$\langle T_r x - T_r y, JT_r y - Jy\rangle \geq \langle T_r x - T_r y, JT_r x - Jx\rangle,\ x, y \in E. \tag{3.11}$$

Lemma 3.5. *Let C be a nonempty, bounded, closed and convex subset of a smooth, strictly convex and reflexive Banach space E. For $r > 0$, $z \in C$, let $\psi : C \times C \times C \to \mathbb{R}$ be a trifunction satisfying the assumptions of Theorem* 3.2, *$A : C \to E^*$ be a κ-inverse strongly monotone operator and $\varphi : C \to \mathbb{R} \cup \{\infty\}$ be a proper lower semicontinuous mapping. In addition, let $\alpha(x, y) + \alpha(y, x) \geq 0$, for all $x, y \in C$ and $T_r^{\psi} : E \to 2^C$ be defined as* (3.10). *Then the following hold:*

(a) $T_r(x)$ is single valued;

(b) $T_r(x)$ is firmly nonexpansive, that is,

$$\langle T_r(x) - T_r(y), JT_r x - Jx\rangle \leq \langle T_r(x) - T_r(y), JT_r y - Jy\rangle;$$

(c) $F(T_r) = GMEP(\psi, A, \varphi) = \Omega$;

(d) $\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x),\ q \in F(T),\ x \in E$;

(e) $GMEP(\psi, A, \varphi)$ is closed and convex.

Proof. (a) For $x \in E$, let $z_1, z_2 \in T_r(x)$. Then,

$$\psi(z_2, z_1; z_1) + \langle Az_1, z_2 - z_1\rangle + \varphi(z_2) - \varphi(z_1) + \frac{1}{r}\langle z_2 - z_1, Jz_1 - Jx\rangle \geq 0$$

and

$$\psi(z_1, z_2; z_2)+\langle Az_2, z_1 - z_2\rangle+\varphi(z_1) - \varphi(z_2)+\frac{1}{r}\langle z_1 - z_2, Jz_2 - Jx\rangle \geq 0.$$

Adding the two inequalities above, we obtain

$$\begin{aligned}
\psi(z_2, z_1; z_1) + \psi(z_1, z_2; z_2) + \langle Az_2, z_1 - z_2\rangle + \langle Az_1, z_2 - z_1\rangle \\
+ \frac{1}{r}\langle z_2 - z_1, Jz_1 - Jx\rangle + \frac{1}{r}\langle z_1 - z_2, Jz_2 - Jx\rangle \geq 0, \\
- \psi(z_1, z_2; z_1) + \psi(z_1, z_2; z_2) + \langle Az_2 - Az_1, z_1 - z_2\rangle \\
+ \frac{1}{r}\langle z_2 - z_1, Jz_1 - Jz_2\rangle \geq 0, \\
\psi(z_1, z_2; z_2) - \psi(z_1, z_2; z_1) + \langle Az_2 - Az_1, z_1 - z_2\rangle \\
+ \frac{1}{r}\langle z_2 - z_1, Jz_1 - Jz_2\rangle \geq 0, \\
\psi(z_1, z_2; z_1) - \psi(z_1, z_2; z_2) + \langle Az_1 - Az_2, z_1 - z_2\rangle \\
- \frac{1}{r}\langle z_2 - z_1, Jz_1 - Jz_2\rangle \leq 0, \\
\alpha(z_2, z_1) + \langle Az_1 - Az_2, z_1 - z_2\rangle - \frac{1}{r}\langle z_2 - z_1, Jz_1 - Jz_2\rangle \leq 0,
\end{aligned}$$

it implies that

$$\alpha(z_2 - z_1) + \kappa||z_1 - z_2||^2 - \frac{1}{r}\langle z_1 - z_2, Jz_1 - Jz_2\rangle \leq 0.$$

Hence we have

$$\langle z_2 - z_1, Jz_1 - Jz_2\rangle \geq r\alpha(z_2, z_1). \tag{3.12}$$

By interchanging the role of z_1 and z_2 in (3.12), we get

$$\langle z_1 - z_2, Jz_2 - Jz_1\rangle \geq r\alpha(z_1, z_2),$$

that is,

$$\langle z_2 - z_1, Jz_1 - Jz_2\rangle \geq r\alpha(z_1, z_2). \tag{3.13}$$

Adding (3.12) and (3.13), we have

$$2\langle z_2 - z_1, Jz_1 - Jz_2\rangle \geq r(\alpha(z_2, z_1) + \alpha(z_1, z_2)),$$

it implies that

$$\langle z_2 - z_1, Jz_1 - Jz_2\rangle \geq 0.$$

Since J is monotone and E is strictly convex, we get $z_1 = z_2$. Hence $T_r(x)$ is single valued.

(b) For $x, y \in C$, let $z_1 = T_r x$ and $z_2 = T_r y$, we have

$$\psi(z_2, z_1; z_1) + \langle Az_1, z_2 - z_1\rangle + \varphi(z_2) - \varphi(z_1) + \frac{1}{r}\langle z_2 - z_1, Jz_1 - Jx\rangle \geq 0$$

and

$$\psi(z_1, z_2; z_2) + \langle Az_2, z_1 - z_2\rangle + \varphi(z_1) - \varphi(z_2) + \frac{1}{r}\langle z_2 - z_1, Jz_1 - Jy\rangle \geq 0.$$

By adding the two inequalities, we obtain

$$\begin{aligned}\psi(z_2, z_1; z_1) + \psi(z_1, z_2; z_2) + \langle Az_1, z_2 - z_1\rangle + \langle Az_2, z_1 - z_2\rangle \\ + \frac{1}{r}\langle z_2 - z_1, Jz_1 - Jx\rangle + \frac{1}{r}\langle z_1 - z_2, Jz_2 - Jy\rangle \geq 0,\end{aligned}$$

that is

$$\begin{aligned}\psi(z_1, z_2; z_2) + \psi(z_2, z_1; z_1) + \langle Az_2 - Az_1, z_1 - z_2\rangle \\ + \frac{1}{r}\langle z_2 - z_1, (Jz_1 - Jx) - (Jz_2 - Jy)\rangle \geq 0,\end{aligned}$$

which implies

$$\begin{aligned}\frac{1}{r}\langle z_2 - z_1, (Jz_1 - Jx) - (Jz_2 - Jy)\rangle \geq -\psi(z_1, z_2; z_2) - \psi(z-2, z_1; z_1) \\ + \langle Az_1 - Az_2, z_1 - z_2\rangle,\end{aligned}$$

that is,

$$\begin{aligned}&\frac{1}{r}\langle z_2 - z_1, (Jz_1 - Jx) - (Jz_2 - Jy)\rangle \\ &= \psi(z_2, z_1; z_2) - \psi(z_2, z_1, z_1) + \kappa||Az_1 - Az_2||^2 \\ &= \alpha(z_1, z_2) + \kappa||Az_1 - Az_2||^2 \\ &\geq \alpha(z_1, z_2). \qquad (3.14)\end{aligned}$$

By interchanging the role of z_1 and z_2 in (3.14), we obtain

$$\frac{1}{r}\langle z_1 - z_2, (Jz_2 - Jy) - (Jz_1 - Jx) \geq \alpha(z_2, z_1). \qquad (3.15)$$

Adding (3.14) and (3.15), we obtain

$$2\langle z_2 - z_1, (Jz_1 - Jx) - (Jz_2 - Jy)\rangle \geq r(\alpha(z_2, z_1) + \alpha(z_1, z_2)),$$

that is,

$$\langle z_2 - z_1, (Jz_1 - Jx) - (Jz_2 - Jy)\rangle \geq 0, \qquad (3.16)$$

which implies

$$\langle T_r y - T_r x, (JT_r x - Jx) - (JT_r y - Jy)\rangle \geq 0.$$

Hence,

$$\langle T_rx - T_ry, JT_rx - Jx\rangle \leq \langle T_rx - T_ry, JT_ry - Jy\rangle.$$

(c) Let $p \in F(T_r)$. Then, $p = T_rp$, so we have

$$\psi(y, p; p) + \langle Ap, y - p\rangle + \varphi(y) - \varphi(p) + \frac{1}{r}\langle y - p, Jp - Jp\rangle \geq 0, \ \ \forall y \in C,$$

it implies that

$$\psi(y, p; p) + \langle Ap, y - p\rangle + \varphi(y) - \varphi(p) \geq 0, \quad \forall y \in C.$$

Hence we have $p \in GMEP(\psi, A, \varphi)$. So $F(T_r) \subset GMEP(\psi, A, \varphi)$. By the same way, we can prove that $GMEP(\psi, A, \varphi) \subset F(T_r)$. Hence we have

$$F(T_r) = GMEP(\psi, A, \varphi). \tag{3.17}$$

(d) From (b), we have for $x, y \in C$,

$$\langle T_rx - T_ry, JT_rx - Jx\rangle \leq \langle T_rx - T_ry, JT_ry - Jy\rangle.$$

Moreover,

$$\begin{aligned}\phi(T_rx, T_ry)+\phi(T_ry, T_rx) &= 2||T_rx||^2-2\langle T_rx, JT_ry\rangle-2\langle T_ry, JT_rx\rangle+2||T_ry||^2\\ &= 2\langle T_rx, JT_rx - JT_ry\rangle + 2\langle T_ry, JT_ry - JT_rx\rangle\\ &= 2\langle T_rx - T_ry, JT_rx - JT_ry\rangle\end{aligned}$$

and

$$\begin{aligned}&\phi(T_rx, y) + \phi(T_ry, x) - \phi(T_rx, x) - \phi(T_ry, y)\\ &\quad = ||T_rx||^2 - 2\langle T_rx, Jy\rangle + ||y||^2 + ||T_ry||^2 - \langle T_ry, Jx\rangle\\ &\qquad + ||x||^2 - ||T_rx||^2 + 2\langle T_rx, Jx\rangle - ||x||^2\\ &\qquad - ||T_ry||^2 + 2\langle T_ry, Jy\rangle - ||y||^2\\ &\quad = 2\langle T_rx, Jx - Jy\rangle - 2\langle T_ry, Jx - Jy\rangle\\ &\quad = 2\langle T_rx - T_ry, Jx - Jy\rangle.\end{aligned}$$

Hence, we have

$$\begin{aligned}\phi(T_rx, T_ry) + \phi(T_ry, T_rx) \leq\ &\phi(T_rx, y) + \phi(T_ry, x)\\ &- \phi(T_rx, x) - \phi(T_ry, y).\end{aligned} \tag{3.18}$$

Set $y = q \in F(T_r)$, we obtain

$$\phi(q, T_rx) + \phi(T_rx, x) \leq \phi(q, x).$$

(e) Finally, we show that $GMEP(\psi, A, \varphi)$ is closed and convex. Observe from (3.18) that

$$\phi(T_rx, T_ry) + \phi(T_ry, T_rx) \leq \phi(T_rx, y) + \phi(T_ry, x). \tag{3.19}$$

Set $y = q \in F(T_r)$, we have

$$\phi(q, T_r x) \leq \phi(q, x), \tag{3.20}$$

which shows that T_r is quasi-ϕ-nonexpansive. Using Lemma 2.16, it follows that $F(T_r)$ is closed and convex. Consequently by (c), $GMEP(\psi, A, \phi)$ is closed and convex. □

4. Convergence analysis

In this section, we prove a strong convergence theorem for finding a common element in the solutions of a system of new generalized mixed equilibrium problem (NGMEP) and the set of fixed points of a countable family of closed and uniformly totally quasi-ϕ-asymptotically nonexpansive mapping in a uniformly smooth and uniformly convex Banach space.

Theorem 4.1. *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E. For any $j = 1, 2 \dots m$, let $\psi_j : C \times C \times C \to \mathbb{R}$ satisfy all the conditions of Assumption A and Lemma 3.5 and $\psi_j(y, \cdot; y)$ be continuous, let $A_j : C \to E^*$ be a κ-inverse strongly monotone mapping and $\varphi_j : C \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous and convex function. Let $T_i : C \to C$, $(i = 1, 2, \dots)$ be a countable family of closed and uniformly totally quasi-ϕ-asymptotically nonexpansive mappings with nonnegative real sequences $\{\lambda_n\}, \{\mu_n\}$ such that $\lambda_n \to 0$, $\mu_n \to 0$ as $n \to \infty$. Let $\vartheta : \mathbb{R}^+ \to \mathbb{R}^+$ be a strictly increasing continuous function with $\vartheta(0) = 0$ and asumme T_i is uniformly asymptotically regular for all $i \geq 1$ with $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and such that*

$$\Gamma := \cap_{i=1}^{\infty} F(T_i) \bigcap (\cap_{j=1}^{m} \Omega_j) \neq \emptyset.$$

Let $f : E \to \mathbb{R}$ be a convex lower semicontinuous mapping with $C \subset int(D(f))$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$, $C_0 = C$, $Q_0 = C$,

$$\begin{cases} y_{n,i} = J^{-1}(\alpha_{n_0} J x_n + \sum_{i=1}^{\infty} \alpha_{n_i} J T_i^n x_n), \\ z_n = J^{-1}(\beta_n J x_n + (1 - \beta_n) J y_{n,i}), \\ u_n = T_{r_{m_n}}^{\psi_m} T_{r_{m-1,n}}^{\psi_{m-1}} \cdots T_{r_{2,n}}^{\psi_2} T_{r_{1,n}}^{\psi_1} z_n, \\ C_n = \{w \in C_{n-1} : G(w, J u_n) \leq G(w, J z_n) \leq G(w, J x_n) + \gamma_n\}, n \geq 1, \\ Q_n = \{w \in Q_{n-1} : \langle x_n - w, J x - J x_n \rangle + \rho f(w) - \rho f(x_n) \geq 0\}, n \geq 1, \\ x_{n+1} = \pi_{C_n \cap Q_n}^f, \end{cases} \tag{4.1}$$

where $\gamma_n = \sum_{i=1}^{\infty} (\lambda_n \vartheta \sup(G(p, J x_n)) + \mu_n)$, $J : E \to E^*$ is the normalized duality mapping, $\{\beta_n\}$ and $\{\alpha_{n_i}\}_{i=0}^{\infty}$ are sequences in $(0, 1)$ such that

(a) $\sum_{i=0}^{\infty} \alpha_{n_i} = 1, \quad \forall\ n \geq 0$;

(b) $\limsup_{n \to \infty} \beta_n < 1$;

(c) $\liminf_{n\to\infty} \alpha_{n_0}\alpha_{n_i} > 0, \quad \forall i;$

(d) $\{r_{j,n}\} \subset [\epsilon, \infty)$, *for some* $\epsilon > 0$.

Then $\{x_n\}$ *converges strongly to* $\pi_\Gamma^f x$, *where* π_Γ^f *is the generalized* f*-projection of* E *onto* Γ.

Proof. Define $H(y,z;z) = \psi(y,z;z) + \langle Az, y-z\rangle + \varphi(y) - \varphi(z),\ y \in C$. It is easy to see that H satisfies the conditions of Assumption A and we define $T_r^H : E \to C$ by

$$T_r^H(x) = \{z \in C : H(y,z;z) + \frac{1}{r}\langle y-z, Jz-Jx\rangle \geq 0,\ \forall\ y \in C\}. \tag{4.2}$$

Therefore, the NGMEP (2.1) can be rewritten as the problem: find $z \in C$ such that

$$H(y,z;z) \geq 0,$$

for all $y \in C$. Furthermore, the resolvent (3.10) reduces to (4.2).

We shall split the rest of the proof into four steps.

Step 1: We first show that $C_n \cap Q_n$ is closed and convex for each $n \in \mathbb{N}$. From the definition of Q_n, it is clear that Q_n is closed and convex for each $n \in \mathbb{N}$. From the definition of C_0, it is clear that C_0 is closed and convex. Suppose C_k is closed and convex for a positive integer k. For any $w \in C_k$, we know that

$$G(w, Ju_k) \leq G(w, Jx_k) + \gamma_k$$

is equivalent

$$2\langle w, Ju_k - Jx_k\rangle \leq ||x_k||^2 - ||u_k||^2 + \gamma_k. \tag{4.3}$$

Hence C_{k+1} is closed and convex. This shows that C_n is closed and convex for all n. Therefore, the sequence $\{x_n\}$ given by Algorithm (4.1) is well defined.

Step 2: We show that the $\Gamma \subset C_n \cap Q_n$. Clearly, $\Gamma \subset C_0 = C$. Assume that $\Gamma \subset C_{n-1}$ for some $n \geq 1$. Let $p \in \Gamma$. Since $u_n = \Omega_n^m z_n$, where $\Omega_n^j = T_{r_{m_n}}^{\psi_m} T_{r_{m-1,n}}^{\psi_{m-1}} \cdots T_{r_{2,n}}^{\psi_2} T_{r_{1,n}}^{\psi_1}$, $j = 1, 2 \ldots m$ and $\Omega_n^0 = I$ then

$$\begin{aligned}
G(p, Ju_n) &= G(p, J\Omega_n^m z_n) \\
&\leq G(p, Jz_n) \\
&= G(p, \beta_n Jx_n + (1-\beta_n)Jy_{n,i}) \\
&= ||p||^2 - 2\langle p, \beta_n Jx_n + (1-\beta_n)Jy_{n,i}\rangle \\
&\quad + ||\beta_n Jx_n + (1-\beta_n)Jy_{n,i}||^2 + 2\rho f(p) \\
&= ||p||^2 - 2\beta_n\langle p, Jx_n\rangle - 2(1-\beta_n)\langle p, Jy_{n,i}\rangle \\
&\quad + \beta_n||x_n||^2 + (1-\beta_n)||y_{n,i}||^2 \\
&\quad - \beta_n(1-\beta_n)g(||Jx_n - Jy_{n,i}||) + 2\rho f(p) \\
&\leq \beta_n G(p, Jx_n) + (1-\beta_n)G(p, Jy_{n,i}),
\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}
G(p, Jy_{n,i}) &= G(p, \alpha_{n_0} Jx_n + \sum_{i=1}^{\infty} \alpha_{n_i} JT_i^n x_n) \\
&= ||p||^2 - 2\langle p, \alpha_{n_0} Jx_n + \sum_{i=1}^{\infty} \alpha_{n_i} JT_i^n x_n \rangle \\
&\quad + ||\alpha_{n_0} Jx_n + \sum_{i=1}^{\infty} \alpha_{n_i} JT_i^n x_n||^2 + 2\rho f(p) \\
&\leq ||p||^2 - 2\alpha_{n_0} \langle p, Jx_n\rangle - 2\sum_{i=1}^{\infty} \alpha_{n_i} \langle p, JT_i^n x_n\rangle \\
&\quad + \alpha_{n_0} ||x_n||^2 + \sum_{i=1}^{\infty} \alpha_{n_i} ||T_i^n x_n||^2 + 2\rho f(p) \\
&= \alpha_{n_0}(||p||^2 - 2\langle p, Jx_n\rangle + ||x_n||^2 + 2\rho f(p)) \\
&\quad + \sum_{i=1}^{\infty} \alpha_{n_i}(||p||^2 - 2\langle p, JT_i^n x_n\rangle + ||T_i^n x_n||^2 + 2\rho f(p)) \\
&= \alpha_{n_0} G(p, Jx_n) + \sum_{i=1}^{\infty} \alpha_{n_i} G(p, JT_i x_n) \\
&= \alpha_{n_0} G(p, Jx_n) + \sum_{i=1}^{\infty} \alpha_{n_i} (G(p, Jx_n) + \lambda_n \vartheta((Gp, Jx_n)) + \mu_n) \\
&\leq \alpha_{n_0} G(p, Jx_n) + \sum_{i=1}^{\infty} G(p, Jx_n) \\
&\quad + \sum_{i=1}^{\infty} \alpha_{n_i} (\lambda_n \vartheta(G(p, Jx_n)) + \mu_n) \\
&\leq G(p, Jx_n) + \gamma_n. \qquad (4.5)
\end{aligned}$$

This shows that $p \in C_n$. Since $p \in \Gamma$ is arbitrary, we have $\Gamma \subset C_n, \ \forall n \in \mathbb{N}$.

Next we show by induction that $\Gamma \subset C_n \cap Q_n$, for all $n \in \mathbb{N}$. From $Q_0 = C$, we have $\Gamma \subset C_0 \cap Q_0$. Suppose that $\Gamma \subset C_n \cap Q_n$ for some $n \in \mathbb{N}$. Then there exists $x_{n+1} \in C_n \cap Q_n$ such that

$$x_{n+1} = \pi^f_{C_n \cap Q_n} x,$$

that is,

$$\langle x_{n+1} - w, Jx - Jx_{n+1}\rangle + \rho f(w) - \rho f(x_{n+1}) \geq 0, \ \forall n \in C_n \cap Q_n,$$

it implies that

$$\langle x_{n+1} - p, Jx - Jx_{n+1} + \rho f(p) - \rho f(x_{n+1}) \geq 0,\ \forall p \in \Gamma \subset C_n \cap Q_n.$$

This means that

$$p \in Q_{n+1}.$$

So, we have $\Gamma \subset Q_{n+1}$. Therefore, we obtain $\Gamma \subset C_{n+1} \cap Q_{n+1}$. Thus, we have that $\Gamma \subset C_n \cap Q_n$, for all $n \in \mathbb{N}$.

Step 3: We show that the sequence $\{x_n\} \subset C$ is Cauchy and that

$$\lim_{n\to\infty} G(p, Jx) = a,$$

for some $a \in \mathbb{R}$. Since $f : E \to \mathbb{R}$ is convex and lower semicontinuous, from Lemma 2.5, we have that there exist $x^* \in E^*$ and $\alpha \in \mathbb{R}$ such that

$$f(u) \geq \langle u, x^* \rangle + \alpha,\ u \in E.$$

Since $x_n \in E$, we have

$$\begin{aligned} G(x_n, Jx) &= ||x_n||^2 - \langle x_n, Jx \rangle + ||x||^2 + 2\rho f(x_n) \\ &\geq ||x_n||^2 - 2\langle x_n, Jx \rangle + ||x_n||^2 + 2\rho \langle x_n, x^* \rangle + 2\rho\alpha \\ &= ||x_n||^2 - 2\langle x_n, Jx - \rho x^* \rangle + 2\rho\alpha \\ &\geq ||x_n||^2 - 2||x_n||||Jx - \rho x^*|| + 2\rho\alpha \\ &= (||x_n|| - ||Jx - \rho x^*||)^2 + ||x||^2 + 2\rho\alpha - ||Jx - \rho x^*||^2. \end{aligned}$$

For any $p \in \Gamma$, we have from $x_{n+1} = \pi^f_{C_n \cap Q_n} x_0$, that

$$(||x_n|| - ||Jx - \rho x^*||)^2 + ||x||^2 + 2\rho\alpha - ||Jx - \rho x^*||^2 \leq G(x_n, Jx) \leq G(p, Jx).$$

That is,

$$\begin{aligned} (||x_n|| - ||Jx - \rho x^*||)^2 &\leq G(p, Jx) - ||x||^2 - 2\rho\alpha + ||Jx - \rho x^*||^2 \\ &\leq G(p, Jx) + ||Jx - \rho x^*||^2. \end{aligned}$$

This implies that $\{x_n\}$ is bounded. Consequently $\{G(x_n, Jx)\}$, $\{y_{n,i}\}$, $\{u_n\}$ and $\{z_n\}$ are bounded. Since $x_{n+1} = \pi^f_{C_n \cap Q_n} x \in C_n \cap Q_n$ and $x_n \in \pi^f_{Q_n} x$, we have

$$G(x_n, Jx) \leq G(x_{n+1}, Jx),\ \forall n \in \mathbb{N}.$$

Therefore, $\{G(x_n, Jx)\}$ is nondecreasing. So there exists $a \in \mathbb{R}$ such that $\lim\limits_{n\to\infty} G(x_n, Jx) = a$. Using $x_n = \pi^f_{Q_n} x$ and Lemma 2.8, for any given $m \in \mathbb{N}$, we have

$$\phi(x_{n+m}, x_n) \leq G(x_{n+m}, Jx) - G(x_n, Jx),\ \forall n \in \mathbb{N}.$$

Letting $n \to \infty$ in the above, we obtain $\phi(x_{n+m}, x_n) \to 0$, $\forall m \geq 1$. By Lemma 2.9, we obtain

$$\lim_{n\to\infty} ||x_{n+m} - x_n|| = 0,$$

for all $m \in \mathbb{N}$. This implies $\{x_n\}$ is a Cauchy sequence in C. Without loss of generality, we can assume $\lim\limits_{n\to\infty} x_n = \bar{x}$.

Step 4: Next we show that

$$\bar{x} \in (\cap_{i=1}^{\infty} F(T_i)) \bigcap (\cap_{j=1}^{m} \Omega_j).$$

First, we show that $\bar{x} \in (\cap_{i=1}^{\infty} F(T_i))$. Taking $m = 1$ in Step 3, we have

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0.$$

By Lemma 2.9, we get

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{4.6}$$

Note that since $\{x_n\}$ is bounded, $\gamma_n \to 0$ as $n \to \infty$. Since $x_{n+1} = \pi^f_{C_n \cap Q_n} x \in C_n \cap Q_n \subset C_n$, we obtain by the definition of C_n, that

$$\begin{aligned} G(x_{n+1}, Ju_n) &\leq G(x_{n+1}, Jx_n) + \gamma_n, \\ &= ||x_{n+1}||^2 - 2\langle x_{n+1}, Jx_n \rangle \\ &\quad + ||x_n||^2 + 2\rho f(x_{n+1}) + \gamma_n, \\ &= \phi(x_{n+1}, x_n) + 2\rho f(x_{n+1}) + \gamma_n, \end{aligned}$$

Hence, we have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n) + \gamma_n.$$

Taking limit $n \to \infty$, we obtain $\phi(x_{n+1}, u_n) \to 0$, as $n \to \infty$ and by Lemma 2.9, we obtain

$$\lim_{n\to\infty} ||x_{n+1} - u_n|| = 0. \tag{4.7}$$

It follows immediately by using triangular inequality on (4.6) and (4.7), that

$$\lim_{n\to\infty} ||x_n - u_n|| = 0.$$

Since J is norm-to-norm continuous on bounded subsets of E, we have

$$\lim_{n\to\infty} ||Jx_n - Ju_n|| = 0. \tag{4.8}$$

Taking $T_0 = I$ (Identity element), we have from Lemma 2.7, that

$$
\begin{aligned}
G(p, Jy_{n,i}) &= G(p, \alpha_{n_0}Jx_n + \sum_{i=1}^{\infty} \alpha_{n_i} JT_i^n x_n) \\
&= ||p||^2 - 2\langle p, \alpha_{n_0}Jx_n + \sum_{i=1}^{\infty} \alpha_{n_i} JT_i^n x_n\rangle \\
&\quad + ||\alpha_{n_0}Jx_n + \sum_{i=1}^{\infty} \alpha_{n_i} JT_i^n x_n||^2 + 2\rho f(p) \\
&\leq ||p||^2 - 2\sum_{i=0}^{\infty} \alpha_{n_i}\langle p, JT_i^n x_n\rangle + \sum_{i=0}^{\infty} \alpha_{n_i}||JT_i^n x_n||^2 \\
&\quad - \alpha_{n_i}\alpha_{n_j} g(||JT_i^n x_n - JT_j^n x_n||) \\
&= \sum_{i=0}^{\infty} \alpha_{n_i} G(p, JT_i^n x_n) - \alpha_{n_i}\alpha_{n_j} g(||JT_i^n x_n - JT_j^n x_n||). \qquad (4.9)
\end{aligned}
$$

Substituting (4.9) in (4.4), we have

$$
\begin{aligned}
G(p, Ju_n) &\leq \beta_n G(p, Jx_n) \\
&\quad + (1-\beta_n)[G(p, Jx_n) + \gamma_n - \alpha_{n_i}\alpha_{n_j} g(||JT_i^n x_n - JT_j^n x_n||)],
\end{aligned}
$$

which implies

$$
\begin{aligned}
&(1-\beta_n)\alpha_{n_i}\alpha_{n_j} g(||JT_i^n x_n - JT_j^n x_n||) \\
&\leq \beta_n G(p, Jx_n) + (1-\beta_n)G(p, Jx_n) + \gamma_n(1-\beta_n) - G(p, Ju_n) \\
&\leq G(p, Jx_n) - G(p, Ju_n) + \gamma_n(1-\beta_n).
\end{aligned}
$$

Taking $j = 0$, for any i, we have

$$
\begin{aligned}
&(1-\beta_n)\alpha_{n_0}\alpha_{n_j} g(||JT_i^n x_n - Jx_n||) \\
&\leq G(p, Jx_n) - G(p, Ju_n) + \gamma_n(1-\beta_n). \qquad (4.10)
\end{aligned}
$$

Since

$$
\begin{aligned}
G(p, Jx_n) - G(p, Ju_n) &= ||x_n||^2 - ||u_n||^2 + 2\langle p, Jx_n - Ju_n\rangle \\
&\leq ||x_n - u_n||(||x_n|| + ||u_n||) + 2||p||||Jx_n - Ju_n||,
\end{aligned}
$$

we have from $||x_u - u_n|| \to 0$ and $||Jx_n - Ju_n|| \to 0$ as $n \to \infty$, that

$$\lim_{n\to\infty}\{G(p, Jx_n) - G(p, Ju_n)\} = 0, \qquad (4.11)$$

which implies that

$$\lim_{n\to\infty}\{\phi(p, x_n) - \phi(p, u_n)\} = 0. \qquad (4.12)$$

Hence, from condition (b), (4.10) and (4.11), we obtain

$$\lim_{n\to\infty}\{g(||JT_i^n x_n - Jx_n||) = 0\}. \tag{4.13}$$

Since g is continuous and strictly increasing, $g(0) = 0$ and J^{-1} is uniformly norm-to-norm continuous on any bounded subset of E, we obtain for each i, that

$$\lim_{n\to\infty} ||T_i^n x_n - x_n|| = 0. \tag{4.14}$$

Now,

$$||T_i^n x_n - \bar{x}|| \leq ||T_i^n x_n - x_n|| + ||x_n - \bar{x}|| \to 0, \text{ as } n \to \infty. \tag{4.15}$$

Using this and the uniformly asymptotically regularity of T, we have

$$||T_i^{n+1} x_n - \bar{x}|| \leq ||T_i^{n+1} x_n - T_i^n x_n|| + ||T_i^n x_n - \bar{x}|| \to 0, \text{ as } n \to \infty. \tag{4.16}$$

That is $T_i^{n+1} x_n = T_i(T_i^n) x_n \to \bar{x}$ as $n \to \infty$. From $T_i x_n \to \bar{x}$ as $n \to \infty$ and the closedness of T_i, we obtain $\bar{x} = T_i \bar{x}$, for each i. We see that $\bar{x} \in F(T_i)$ for all i, which implies $\bar{x} \in \bigcap_{i=1}^{\infty} F(T_i)$.

Next, we show that $\bar{x} \in (\bigcap_{j=1}^{m} \Omega_j)$. From $u_n = \Omega_n^m z_n$, where $\Omega_n^j = T_{r_{j,n}}^{\psi_j} T_{r_{j-1,n}}^{\psi_{j-1}} \cdots T_{r_{2,n}}^{\psi_2} T_{r_{1,n}}^{\psi_1}$, we get that

$$\begin{aligned}
\phi(p, u_n) &= \phi(p, \Omega_n^m z_n) \\
&\leq \phi(p, \Omega_n^{m-1} z_n) \\
&\leq \phi(p, \Omega_n^{m-2} z_n) \\
&\vdots \\
&\leq \phi(p, \Omega_n^j z_n).
\end{aligned}$$

Using Lemma 3.5 (d), it follows that

$$\begin{aligned}
\phi(\Omega_n^j z_n, z_n) &\leq \phi(p, z_n) - \phi(p, \Omega_n^j z_n) \\
&\leq \phi(p, x_n) - \phi(p, \Omega_n^j z_n) \\
&= \phi(p, x_n) - \phi(p, u_n).
\end{aligned}$$

By equation (4.12), we have that $\phi(\Omega_n^j z_n, z_n) \to 0$, as $n \to \infty$. Using Lemma 2.9, we obtain

$$\lim_{n\to\infty} ||\Omega_n^j z_n - z_n||, \ \forall j = 1, 2 \cdots m. \tag{4.17}$$

Since $x_{n+1} = \pi_{C_n \cap Q_n}^f x \in C_n \cap Q_n \subset C_n$, it follows that

$$G(x_{n+1}, Jz_n) \leq G(x_{n+1}, Jx_n) + \gamma_n,$$

which is equivalent to

$$\phi(x_{n+1}, z_n) \leq \phi(x_{n+1}, x_n) + \gamma_n.$$

Therefore, by using Lemma 2.9 and the boundedness of $\{x_n\}$, we obtain

$$\lim_{n\to\infty} ||x_{n+1} - z_n|| = 0.$$

We can easily obtain

$$\lim_{n\to\infty} ||x_n - z_n|| = 0, \tag{4.18}$$

using triangular inequality on $||x_n - z_n||$ and applying (4.6) and (4.18).

Since $x_n \to \bar{x}$ and $||x_n - z_n|| \to 0$, we obtain $z_n \to \bar{x}$ as $n \to \infty$. We have therefore by (4.17) and triangular inequality, that

$$\lim_{n\to\infty} ||\bar{x} - \Omega_n^j z_n|| \leq \lim_{n\to\infty} (||\Omega_n^j z_n - z_n|| + ||z_n - \bar{x}||) = 0, \tag{4.19}$$

for all $j = 1, 2 \cdots m$. Consequently,

$$\lim_{n\to\infty} ||\Omega_n^j z_n - \Omega_n^{j-1} z_n|| = 0, \ \forall j = 1, 2 \cdots, m. \tag{4.20}$$

Since $\{r_{j,n}\} \in [\epsilon, \infty)$, we have

$$\lim_{n\to\infty} \frac{||\Omega_n^j z_n - \Omega_n^{j-1} z_n||}{r_{j,n}} = 0. \tag{4.21}$$

Noticing that $u_n = T_{r_{j,n}}^{\psi_j} T_{r_{j-1,n}}^{\psi_{j-1}} \cdots T_{r_{2,n}}^{\psi_2} T_{r_{1,n}}^{\Psi_1} z_n$, we have

$$H_j(y, \Omega_n^j z_n; \Omega_n^j z_n) + \frac{1}{r_{j,n}} \langle y - \Omega_n^j z_n, J\Omega_n^j z_n - J\Omega_n^{j-1} z_n \rangle \geq 0, \ \forall y \in C,$$

for any j. Using the generalized relaxed α-monotonicity of H_j for each j, we obtain

$$\begin{aligned}
||y - \Omega_n^j z_n|| \frac{||J\Omega_n^j z_n - J\Omega_n^{j-1} z_n||}{r_{j,n}} &\geq \frac{1}{r_{j,n}} \langle y - \Omega_n^j z_n, J\Omega_n^j z_n - J\Omega_n^{j-1} z_n \rangle \\
&\geq -H_j(y, \Omega_n^j z_n; \Omega_n^{j-1} z_n) \\
&\geq \alpha_j(\Omega_n^j z_n, y) - H_j(y, \Omega_n^j; y).
\end{aligned}$$

Taking $n \to \infty$ in the above inequality and (4.21), we get

$$\alpha_j(\bar{x}, y) - H_j(y, \bar{x}; y) \leq 0, \ \forall y \in C.$$

For $t \in (0, 1]$ and $y \in C$, set $y_t = ty + (1 - t)\bar{x}$. Then $y_t \in C$, therefore

$$\alpha_j(\bar{x}, y_t) - H_j(y_t, \bar{x}; y_t) \leq 0,$$

it implies that,

$$\begin{aligned}
\alpha_j(\bar{x}, y_t) &\leq H_j(y_t, \bar{x}; y_t) \\
&\leq tH_j(y, \bar{x}; y) + (1 - t)H_j(\bar{x}, \bar{x}; y_t) \\
&= tH_j(y, \bar{x}; y_t)
\end{aligned}$$

Hence, we have

$$H_j(y, \bar{x}; y_t) \geq \frac{\alpha_j(\bar{x}, y_t)}{t}. \tag{4.22}$$

Since $\psi(y, x; \cdot)$ is hemicontinuous, taking $t \to 0$, we obtain

$$H_j(y, \bar{x}; \bar{x}) \geq 0.$$

Hence $\bar{x} \in (\cap_{j=1}^m \Omega_j)$. Therefore, $\bar{x} \in \Gamma = (\cap_{j=1}^m \Omega_j) \cap (\cap_{i=1}^\infty F(T_i))$. □

5. Some consequences of main results

In this section, we give some consequences of our main Theorem 4.1. We have the following result with $A = 0, \varphi = 0$.

Corollary 5.1. *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E. For any $j = 1, 2 \dots m$, let $\psi_j : C \times C \times C \to \mathbb{R}$ satisfy all the conditions of Assumption A and Lemma 3.5 and $\psi_j(y, \cdot; y)$ be continuous. Let $T_i : C \to C$, $(i = 1, 2, \dots)$ be a countable family of closed and uniformly totally quasi-ϕ-asymptotically nonexpansive mapping with nonnegative real sequences $\{\lambda_n\}, \{\mu_n\}$ such that $\lambda_n \to 0$, $\mu_n \to 0$ as $n \to \infty$. Let $\vartheta : \mathbb{R}^+ \to \mathbb{R}^+$ be a strictly increasing continuous function with $\vartheta(0) = 0$ and asumme T_i is uniformly asymptotically regular for all $i \geq 1$ with $\cap_{i=1}^\infty F(T_i) \neq \emptyset$ and such that*

$$\Gamma = \cap_{i=1}^\infty F(T_i) \bigcap (\cap_{j=1}^m \Omega_j) \neq \emptyset.$$

Let $f : E \to \mathbb{R}$ be a convex lower semicontinuous mapping with $C \subset int(D(f))$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$, $C_0 = C$, $Q_0 = C$,

$$\begin{cases} y_{n,i} = J^{-1}(\alpha_{n_0} J x_n + \sum_{i=1}^\infty \alpha_{n_i} J T_i^n x_n), \\ z_n = J^{-1}(\beta_n J x_n + (1 - \beta_n) J y_{n,i}), \\ u_n = T_{r_{m_n}}^{\psi_m} T_{r_{m-1,n}}^{\psi_{m-1}} \cdots T_{r_{2,n}}^{\psi_2} T_{r_{1,n}}^{\psi_1} z_n, \\ C_n = \{w \in C_{n-1} : G(w, J u_n) \leq G(w, J z_n) \leq G(w, J x_n) + \gamma_n\}, n \geq 1, \\ Q_n = \{w \in Q_{n-1} : \langle x_n - w, J x - J x_n \rangle + \rho f(w) - \rho f(x_n) \geq 0\}, n \geq 1, \\ x_{n+1} = \pi_{C_n \cap Q_n}^f, \end{cases} \tag{5.1}$$

where $\gamma_n = \sum_{i=1}^\infty (\lambda_n \vartheta \sup(G(p, J x_n)) + \mu_n)$, $J : E \to E^*$ is the normalized duality mapping, β_n and $\{\alpha_{n_i}\}_{i=0}^\infty$ are sequences in $[0, 1]$ such that

(a) $\sum_{i=0}^\infty \alpha_{n_i} = 1, \quad \forall\ n \geq 0$;
(b) $\limsup_{n\to\infty} \beta_n < 1$;
(c) $\liminf_{n\to\infty} \alpha_{n_0} \alpha_{n_i} > 0, \quad \forall i$;
(d) $\{r_{j,n}\} \subset [\epsilon, \infty)$, *for some* $\epsilon > 0$.

Then, $\{x_n\}$ converges strongly to $\pi_\Gamma^f x$, where π_Γ^f is the generalized f-projection of E onto Γ.

The following result coincides with Theorem 4.1 given in [23], with T_i being quasi-ϕ-nonexpansive for all i, $A = 0$, $\varphi = 0$ and $\gamma_n = 0$ in Theorem 4.1.

Corollary 5.2. *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E. For any $j = 1, 2 \dots m$, let $\psi_j : C \times C \times C \to \mathbb{R}$ satisfy all the conditions of Assumption A and Lemma 3.5 and $\psi_j(y, \cdot; y)$ be continuous. Let $T_i : C \to C$, $(i = 1, 2, \dots)$ be a countable family of closed quasi-ϕ-nonexpansive mapping. Assume*

$$\Gamma = \cap_{i=1}^{\infty} F(T_i) \bigcap (\cap_{j=1}^{m} \Omega_j) \neq \emptyset.$$

Let $f : E \to \mathbb{R}$ be a convex lower semicontinuous mapping with $C \subset int(D(f))$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$, $C_0 = C$, $Q_0 = C$,

$$\begin{cases} y_n = J^{-1}(\alpha_{n_0} Jx_n + \sum_{i=1}^{\infty} \alpha_{n_i} JT_i x_n), \\ z_n = J^{-1}(\beta_n Jx_n + (1 - \beta_n) Jy_n), \\ u_n = T_{r_{m_n}}^{\psi_m} T_{r_{m-1,n}}^{\psi_{m-1}} \cdots T_{r_{2,n}}^{\psi_2} T_{r_{1,n}}^{\psi_1} z_n, \\ C_n = \{w \in C_{n-1} : G(w, Ju_n) \leq G(w, Jz_n) \leq G(w, Jx_n)\}, n \geq 1, \\ Q_n = \{w \in Q_{n-1} : \langle x_n - w, Jx - Jx_n \rangle + \rho f(w) - \rho f(x_n) \geq 0\}, n \geq 1, \\ x_{n+1} = \pi_{C_n \cap Q_n}^f, \end{cases} \tag{5.2}$$

where $J : E \to E^*$ is the normalized duality mapping, β_n and $\{\alpha_{n_i}\}_{i=0}^{\infty}$ are sequences in $[0, 1]$ such that

(a) $\sum_{i=0}^{\infty} \alpha_{n_i} = 1, \quad \forall\ n \geq 0;$
(b) $\limsup_{n\to\infty} \beta_n < 1;$
(c) $\liminf_{n\to\infty} \alpha_{n_0} \alpha_{n_i} > 0, \quad \forall i;$
(d) $\{r_{j,n}\} \subset [\epsilon, \infty)$, *for some* $\epsilon > 0$.

Then, $\{x_n\}$ converges strongly to $\pi_\Gamma^f x$, where π_Γ^f is the generalized f-projection of E onto Γ.

6. Conclusion

This work aimed at theoretically and analytically study the existence of solutions for new generalized mixed equilibrium problem (NGMEP) under relaxed α-monotonicity in real uniformly smooth and uniformly convex Banch space. We proposed an iterative algorithm for approximating a common element in the solution of the NGMEP and fixed point set of a countable family of totally quasi-ϕ-asymptotically nonexpansive mapping. Our results generalize the one considered in [23] and many other related results in literature.

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References

[1] R.P. Agarwal, D. O'Regan and D.R. Saha, *Fixed point theory for Lipschitizian-type mappings with Applications*, Springer, New York, **6** (2009).

[2] R.P. Agarwal, Y.J. Cho and X.Qin, *Generalized projection algorithms for nonlinear operators*, Numer. Funct. Anal. Optim., **28** (2007), 1197–1215.

[3] M.R. Bai, S.Z. Zhou and G.Y. Ni, *Variational-like inequalities with relaxed $\eta - \alpha$ psuodomonotone mappings in Banach spaces*, Appl. Math. Lett., **19**(6) (2006), 547–554.

[4] M. Bianchi and S. Schaible, *Equilibrium problems under generalized convexity and generalized monotonicity*, J. Global Optim., **30**(2-3) (2004), 121–134.

[5] M. Bianchi and S. Schaible, *Generalized monotone bifunctions and equilibriums*, J. Optim. Theory Appl., **90**(1) (1996), 31–43.

[6] L.C. Ceng and J.C. Yao, *On generalized variational-like inequalities with generalized monotone multivalued mappings*, Appl. Math. Lett., **22**(3) (2009), 428–434.

[7] S.S. Chang, H.W.J. Lee, C.K. Chan and L. Yang, *Approximation theorems for total quasi-ϕ-asymptotically nonexpansive mappings with applications*, Appl. Math. Comput., **218**(6) (2011), 2921–2931.

[8] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, (1985).

[9] K. Fan, *A minimax inequality and applications*, Inequalities **3** (1972), 103-113.

[10] Y.P. Fang and N.J. Huang, *Variational-like inequalities with generalized monotone mappings in Banach spaces*, J. Optim. Theory Appl., **118**(2) (2003), 327–338.

[11] C. Izuchukwu, K.O. Aremu, A.A. Mebawondu and O.T. Mewomo, *A viscosity iterative technique for equilibrium and fixed point problems in a Hadamard space*, Appl. Gen. Topol., **20**(1) (2019), 193–210.

[12] C. Izuchukwu, A.A. Mebawondu, K.O. Aremu, H.A. Abass and O.T. Mewomo, *Viscosity iterative techniques for approximating a common zero of monotone operators in a Hadamard space*, Rend. Circ. Mat. Palermo (2), (2019), https://doi.org/10.1007/s12215-019-00415-2.

[13] C. Izuchukwu, G.C. Ugwunnadi, O.T. Mewomo, A.R. Khan, and M. Abbas, *Proximal-type algorithms for split minimization problem in p-uniformly convex metric space*, Numer. Algorithms, **82**(3) (2019), 909–935.

[14] L.O. Jolaoso, T.O. Alakoya, A. Taiwo, and O.T. Mewomo, *A parallel combination extragradient method with Armijo line searching for finding common solution of finite families of equilibrium and fixed point problems*, Rend. Circ. Mat. Palermo II, (2019), DOI:10.1007/s12215-019-00431-2.

[15] L.O. Jolaoso, F.U. Ogbuisi and O.T. Mewomo, *An iterative method for solving minimization, variational inequality and fixed point problems in reflexive Banach spaces*, Adv. Pure Appl. Math., **9**(3) (2018), 167–184.

[16] L.O. Jolaoso, K.O. Oyewole, C.C. Okeke and O.T. Mewomo, *A unified algorithm for solving split generalized mixed equilibrium problem and fixed point of nonspreading mapping in Hilbert space*, Demonst. Math., **51** (2018), 211-232.

[17] L.O. Jolaoso, A. Taiwo, T.O. Alakoya and O.T. Mewomo, *A self adaptive inertial subgradient extragradient algorithm for variational inequality and common fixed point of multivalued mappings in Hilbert spaces*, Demonstr. Math., **52** (2019), 183-203.

[18] L.O. Jolaoso, A. Taiwo, T.O. Alakoya and O.T. Mewomo, *A unified algorithm for solving variational inequality and fixed point problems with application to the split equality problem*, Comput. Appl. Math., 2019, DOI: 10.1007/s40314-019-1014-2.

[19] S. Kamimura and W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space*, *SIAM J. Optim.,* **13** (2002), 938–945.

[20] S. Karamardian and S. Schaible, *Seven kinds of monotone maps*, J. Optim. Theory Appl., **66**(1) (1990), 37–46.

[21] J.K. Kim and Salahuddin, *Extragradient methods for generalized mixed equilibrium problems and fixed point problems in Hilbert spaces,* Nonlinear Funct. Anal. Appl., **22**(4) (2017), 693-709.

[22] X. Li, N. Huang and D, O'Regan, *Strong convergence theorems for relatively nonexpansive mappings in Banach spaces with applications*, Comput. Math. Appl., **60**(5) (2010), 1322–1331.

[23] N.K. Mahato, M.A. Noor and N.K Sahu, *Existence results for trifunction equilibrium problems and fixed point problems*, Anal. Math. Phys., **9** (2019), 323–347.

[24] N.K. Mahato and C. Nahak, *Hybrid projection methods for the general variational-like inequality problems*, J. Adv. Math. Stud., **6**(1) (2013), 143–158.

[25] O.K. Oyewole, L.O. Jolaoso, C. Izuchukwu and O.T. Mewomo, *On approximation of common solution of finite family of mixed equilibrium problems involving μ-α relaxed monotone mapping in Banach space*, Politehn. Univ. Bucharest Sci. Bull. Ser. A . Appl. Math. Phys., **81**(1), (2019), 19–34.

[26] N. Petrot, K. Wattanawitoon and P. Kumam, *A hybrid projection method for generalized mixed equilibrium problems and fixed point problems in Banach spaces*, Nonlinear Anal. Hybrid Syst., **4**(4) (2010), 631–643.

[27] B.J. Pett, *A proof that every uniformly convex Banach space is reflexive*, Duke Math. J., **5**(2) (1939), 249–253.

[28] V. Preda, M. Beldiman and A. Bătătorescu, *On variational-like inequalities with generalized monotone mapppings*, In: Generalized convexity and Related Topics. Lecture Notes in Economics and Mathematical systems, Springer, Berlin, **583** (2007), 415–431.

[29] X. Qin, Y.J. Cho and S.M. Kang, *Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces*, J. Comput. Appl. Math., **255**(1) (2009), 20–30.

[30] S. Reich, *A weak convergence theorem for alternating method with Bregman distance*, in: A.G. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Accretive and Monotone type, Marcel Dekker, New York, 1996, 313-318.

[31] S. Saewan and P. Kumam, *A generalized f-projection method for countable families of weak relatively nonexpansive mappings and the system of generalized Ky Fan inequalities*, J. Glob. Optim., **56** (2012), 1–23.

[32] S. Saewan, P. Kanjanasamrawong, P. Kumam and Y.J. Cho, *The modified Mann type iterative algorithm for a countable family of totally quasi-ϕ-asymptotically nonexpansive mappings by the hybrid generalized f-projection method*, Fixed Point Theory Appl., **2013**, (2013), Article 63, doi: 10.1186/1687-1812-2013-63.

[33] G. Stampacchia, *Forms bilineaires coercitives sur les ensembles convexes*, C R. Acad. Sci. Paris., **258** (1964), 4414–4416.

[34] A. Taiwo, L.O. Jolaoso and O.T. Mewomo, *A modified Halpern algorithm for approximating a common solution of split equality convex minimization problem and fixed point problem in uniformly convex Banach spaces*, Comput. Appl. Math., **38**(2) (2019), Article 77.

[35] A. Taiwo, L.O. Jolaoso and O.T. Mewomo, *Parallel hybrid algorithm for solving pseudomonotone equilibrium and Split Common Fixed point problems*, *Bull. Malays. Math. Sci. Soc.*, (2019), DOI: 10.1007/s40840-019-00781-1.

[36] A. Taiwo, L.O. Jolaoso and O.T. Mewomo, *General alternative regularization method for solving Split Equality Common Fixed Point Problem for quasi-pseudocontractive mappings in Hilbert spaces*, Ricerche Mat., (2019), DOI: 10.1007/s11587-019-00460-0.

[37] Z.M. Wang, *Convergence theorems based on the shrinking projection method for hemi-relatively nonexpansive mappings variational inequalities and equilibrium problems,* Nonlinear Funct. Anal. Appl., **22**(3) (2017), 459-483.

[38] K. Wu and N. Huang, *The generalised f-projection operator with an application*, Bull. Aust. Math. Soc., **73**(2) (2006), 307–318.

[39] F. Yang, L. Zhao and J.K. Kim, *Hybrid projection method for generalized mixed equilibrium problem and fixed point problem of infinite family of asymptotically quasi-ϕ-nonexpansive mappings in Banach spaces*, Appl. Math. Comput., **218**(10) (2012), 6072–6082.

[40] H. Zhou, G. Gao and B. Tan, *Convergence theorems of a modified hybrid algorithm for a family of quasi-ϕ-asymptotically nonexpansive mappings*, J. Appl. Math. Comput., **32** (2010), 453–464.