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EXISTENCE OF SOLUTIONS FOR BOUNDARY VALUE PROBLEMS VIA F-CONTRACTION MAPPINGS IN METRIC SPACES

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Abstract. The purpose of this paper is to present some sufficient conditions for the existence and uniqueness of solutions of the nonlinear Hammerstein integral equations and the two-point boundary value problems for nonlinear second-ordinary differential equations. To establish this, we introduce the generalized Suzuki-(α, β)-F-contraction and the generalized (α, β)-F-contraction in the framework of a metric space and establish some fixed point results. The results obtained in this work provide extension as well as substantial generalization and improvement of several well-known results on fixed point theory and its applications.

1. Introduction and Preliminaries

The concept of the Banach Contraction Principle is a well-known result in the theory of nonlinear analysis. Due to its usefulness for showing the existence and uniqueness theorems for nonlinear differential and integral equations, this concept has been generalized in term of space and nonlinear mappings (see [3, 5, 7, 8, 11] and the references therein). One of the interesting generalization was introduced by Berinde [3, 4], he gave the following definition:

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Definition 1.1. Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be a generalized almost contraction if there exist $\delta \in [0,1)$ and $L \geq 0$ such that

$$d(Tx,Ty) \leq \delta d(x,y) + L\min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\},$$

for all $x, y \in X$.

Furthermore, in 2008, Suzuki [21] introduced a class of mappings satisfying condition (C) which is also known as the Suzuki-type generalized nonexpansive mapping and he proved some fixed point theorems for this class of mappings.

Definition 1.2. Let (X,d) be a metric space. A mapping $T : X \to X$ is said to satisfy condition (C) if for all $x, y \in X$,

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \Rightarrow d(Tx,Ty) < d(x,y).$$

Theorem 1.3. *Let (X,d) be a compact metric space and $T : X \to X$ be a mapping satisfying condition (C) for all $x, y \in X$. Then T has a unique fixed point.*

In the art of generalizing the contractive definition in the sense of Banach and other existing contraction mappings, Wardowski [23] introduced the notion of F-contractions. This class of mappings is defined as follows:

Definition 1.4. Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be a F-contraction if there exists $\tau > 0$ such that for all $x, y \in X$;

$$d(Tx,Ty) > 0 \quad \Rightarrow \quad \tau + F(d(Tx,Ty)) \leq F(d(x,y)), \tag{1.1}$$

where $F : \mathbb{R}^+ \to \mathbb{R}$ is a mapping satisfying the following conditions:

- (F_1) F is strictly increasing;
- (F_2) for all sequences $\{\alpha_n\} \subseteq \mathbb{R}^+$, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
- (F_3) there exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

He also established the following result:

Theorem 1.5. *Let (X,d) be a complete metric space and $T : X \to X$ be an F-contraction. Then T has a unique fixed point $x^* \in X$ and for each $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to x^*.*

Remark 1.6. ([23]) If we suppose that $F(t) = \ln t$, the F-contraction mapping becomes the Banach contraction mapping.

We denote by $\mathcal{F}_1$ the family of all functions satisfying $(F_1), (F_2)$ and (F_3). In 2014, Minak et al. [12] introduced and studied some fixed point results for the generalized F-contractions including the Ćirić-type generalized F-contraction and almost F-contraction on a complete metric space.

Definition 1.7. Let (X, d) be a metric space and $T : X \to X$ be a mapping. Then T is said to be the Ćirić type generalized F-contraction if $F \in \mathcal{F}_1$ and there exist $L \geq 0$ and $\tau > 0$ such that for all $x, y \in X$;

$$d(Tx, Ty) > 0 \quad \Rightarrow \quad \tau + F(d(Tx, Ty)) \leq F(M(x, y)), \tag{1.2}$$

where $M(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$.

Definition 1.8. Let (X, d) be a metric space and $T : X \to X$ be a mapping. Then T is said to be the Ćirić-type generalized F-contraction if $F \in \mathcal{F}_1$ and there exists $\tau > 0$ such that for all $x, y \in X$;

$$d(Tx, Ty) > 0 \quad \Rightarrow \quad \tau + F(d(Tx, Ty)) \leq F(d(x, y) + Ld(x, Ty)). \tag{1.3}$$

In the same year, Cosentino et al. [6] introduced and studied some fixed point results for the F-contraction of Hardy-Rogers type on a complete metric space.

Definition 1.9. Let (X, d) be a metric space and $T : X \to X$ be a mapping. Then T is said to be the F-contraction Hardy-Rogers type if $F \in \mathcal{F}_1$ and there exists $\tau > 0$ such that for all $x, y \in X$;

$$\begin{aligned} & d(Tx, Ty) > 0 \\ & \Rightarrow \quad \tau + F(d(Tx, Ty)) \leq F(\alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) \\ & \qquad\qquad + \delta d(y, Ty) + Ld(y, Tx)), \end{aligned} \tag{1.4}$$

where $\alpha + \beta + \gamma + 2\delta = 1, \gamma \neq 1$ and $L \geq 0$.

In [16], Piri et al. used the continuity condition instead of condition (F_3) and proved the following result:

Theorem 1.10. *Let X be a complete metric space and $T : X \to X$ be a selfmap of X. Assume that there exists $\tau > 0$ such that for all $x, y \in X$ with $Tx \neq Ty$,*

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \quad \Rightarrow \quad \tau + F(d(Tx, Ty)) \leq F(d(x, y)), \tag{1.5}$$

where $F : \mathbb{R}^+ \to \mathbb{R}$ is continuous strictly increasing and $\inf F = -\infty$. *Then T has a unique fixed point $z \in X$, and for every $x \in X$, the sequence $\{T^n x\}$ converges to z.*

Secelean in [19] proved the following lemma.

Lemma 1.11. ([19]) *Let $F : \mathbb{R}^+ \to \mathbb{R}$ be an increasing mapping and $\{\alpha_n\}$ be a sequence of positive integers. Then the following assertion hold:*

(1) *if* $\lim_{n\to\infty} F(\alpha_n) = -\infty$ *then* $\lim_{n\to\infty} \alpha_n = 0$;
(2) *if* $\inf F = -\infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$ *then* $\lim_{n\to\infty} F(\alpha_n) = -\infty$.

Furthermore, the authors in [19] replaced the condition F_2 in the definition of the F-contraction with the following condition.

(F_*) $\inf F = -\infty$

or, also by

(F_{**}) there exists a sequence $\{\alpha_n\}$ of positive real numbers such that

$$\lim_{n\to\infty} F(\alpha_n) = -\infty.$$

We denote by $\mathcal{F}$ the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ which satisfy conditions:

$(F_1^{'})$ F is strictly increasing,
$(F_2^{'})$ $\inf F = -\infty$,

or, also by,

$(F_3^{'})$ there exists a sequence $\{\alpha_n\}$ of positive real numbers such that

$$\lim_{n\to\infty} F(\alpha_n) = -\infty,$$

$(F_4^{'})$ F is continuous on $(0, \infty)$.

Samet et al. [18] introduced the notion of the α-admissible mapping and obtained some fixed point results for this class of mappings.

Definition 1.12. ([18]) Let $\alpha : X \times X \to [0, \infty)$ be a function. We say that a self mapping $T : X \to X$ is α-admissible if for all $x, y \in X$,

$$\alpha(x, y) \geq 1 \;\;\Rightarrow\;\; \alpha(Tx, Ty) \geq 1.$$

Definition 1.13. ([8]) Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be mappings. We say that T is a triangular α-admissible if

(1) T is α-admissible and
(2) $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1 \;\;\Rightarrow\;\; \alpha(x, z) \geq 1$ for all $x, y, z \in X$.

Theorem 1.14. ([18]) *Let (X, d) be a complete metric space and $T : X \to X$ be an α-admissible mapping. Suppose that the following conditions hold:*

(1) *for all* $x, y \in X$, *we have*

$$\alpha(x,y)d(Tx,Ty) \leq \psi(d(x,y)),$$

where $\psi : [0,\infty) \to [0,\infty)$ *is a nondecreasing function such that* $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ *for all* $t > 0$;

(2) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$;

(3) *either* T *is continuous or for any sequence* $\{x_n\}$ *in* X *with* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \geq 0$ *and* $x_n \to x$ *as* $n \to \infty$, *then* $\alpha(x_n, x) \geq 1$.

Then T *has a fixed point.*

In 2016, Chandok et al. [5] introduced another class of mappings, called the TAC-contraction mapping and established some fixed point results in the framework of a complete metric space.

Definition 1.15. Let $T : X \to X$ be a mapping and let $\alpha, \beta : X \to \mathbb{R}^+$ be two functions. Then T is called a cyclic (α,β)-admissible mapping, if

(1) $\alpha(x) \geq 1$ for some $x \in X$ implies that $\beta(Tx) \geq 1$,

(2) $\beta(x) \geq 1$ for some $x \in X$ implies that $\alpha(Tx) \geq 1$.

Definition 1.16. Let (X,d) be a metric space and let $\alpha, \beta : X \to [0,\infty)$ be two mappings. We say that T is a TAC-contractive mapping, if for all $x, y \in X$,

$$\alpha(x)\beta(y) \geq 1 \quad \Rightarrow \quad \psi(d(Tx,Ty)) \leq f(\psi(d(x,y)), \phi(d(x,y))),$$

where ψ is a continuous and nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$, ϕ is continuous with $\lim_{n\to\infty} \phi(t_n) = 0 \Rightarrow \lim_{n\to\infty} t_n = 0$ and $f : [0,\infty)^2 \to \mathbb{R}$ is continuous, $f(a,t) \leq a$ and $f(a,t) = a \Rightarrow a = 0$ or $t = 0$ for all $s, t \in [0,\infty)$.

Theorem 1.17. *Let* (X,d) *be a complete metric space and let* $T : X \to X$ *be a cyclic* (α,β)*-admissible mapping. Suppose that* T *is a TAC contraction mapping. Assume that there exists* $x_0 \in X$ *such that* $\alpha(x_0) \geq 1, \beta(x_0) \geq 1$ *and either of the following conditions hold:*

(1) T *is continuous,*

(2) *if for any sequence* $\{x_n\}$ *in* X *with* $\beta(x_n) \geq 1$, *for all* $n \geq 0$ *and* $x_n \to x$ *as* $n \to \infty$, *then* $\beta(x) \geq 1$.

In addition, if $\alpha(x) \geq 1$ *and* $\beta(y) \geq 1$ *for all* $x, y \in F(T)$ *(where* $F(T)$ *denotes the set of fixed points of* T*), then* T *has a unique fixed point.*

Definition 1.18. ([11]) Let X be a nonempty set, $T : X \to X$ be a mapping and $\alpha, \beta : X \times X \to \mathbb{R}^+$ be two functions. We say that T is an (α,β)-cyclic admissible mapping, if for all $x, y \in X$,

(1) $\alpha(x,y) \geq 1 \;\;\Rightarrow\;\; \beta(Tx,Ty) \geq 1$,
(2) $\beta(x,y) \geq 1 \;\;\Rightarrow\;\; \alpha(Tx,Ty) \geq 1$.

Lemma 1.19. ([11]) *Let X be a nonempty set and $T : X \to X$ be an (α,β)-cyclic admissible mapping. Suppose that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$. Define the sequence $x_{n+1} = Tx_n$, then $\alpha(x_m, x_{m+1}) \geq 1$ implies that $\beta(x_n, x_{n+1}) \geq 1$ and $\beta(x_m, x_{m+1}) \geq 1$ implies that $\alpha(x_n, x_{n+1}) \geq 1$, for all $n, m \in \mathbb{N} \cup \{0\}$ with $m < n$.*

Lemma 1.20. ([2]) *Suppose that (X,d) is a metric space and $\{x_n\}$ is a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exists an $\epsilon > 0$ and sequences of positive integers $\{x_{m_k}\}$ and $\{x_{n_k}\}$ with $n_k > m_k \geq k$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon, d(x_{m_k}, x_{n_{k-1}}) < \epsilon$ and*

(1) $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon$,
(2) $\lim_{k\to\infty} d(x_{n_k}, x_{m_{k+1}}) = \epsilon$,
(3) $\lim_{k\to\infty} d(x_{m_{k-1}}, x_{n_k}) = \epsilon$,
(4) $\lim_{k\to\infty} d(x_{n_k}, x_{m_{k+1}}) = \epsilon$.

Motivated by the works of Wardowski [23], Piri et al. [16], Minak et al. [12], Cosentino et al. [6], Samet et al. [18] and Chandok et al. [5], Mebawondu et al. [11], we introduce the generalized Suzuki-(α,β)-F-contraction and the generalized (α,β)-F-contraction in the framework of a metric space and established some fixed point results. In addition, we establish the existence and uniqueness theorems of fixed points for such mappings in the framework of a complete metric space and we apply our fixed point results to establish the existence of a two-point boundary value problem of second-order differential equations.

2. Main results

In this section, we introduce the concept of the generalized Suzuki-(α,β)-F-contraction and the generalized (α,β)-F-contraction in the framework of a metric space and prove the existence and uniqueness theorems of fixed points for such mappings.

Definition 2.1. Let (X,d) be a metric space, $\alpha,\beta : X \times X \to [0,\infty)$ be two functions and T be a self map on X. The mapping T is said to be a generalized Suzuki-(α,β)-F-contraction mapping, if there exists $F \in \mathcal{F}, \tau > 0$ and $L \geq 0$ such that for all $x, y \in X$ with $Tx \neq Ty$ then

$$\frac{1}{2}d(x,Tx) \leq d(x,y)$$
$$\Rightarrow \quad \tau + F(\alpha(x,Tx)\beta(y,Ty)d(Tx,Ty)) \leq F(M(x,y) + LN(x,y)), \qquad (2.1)$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$$

and

$$N(x,y) = \min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}.$$

Definition 2.2. Let (X,d) be a metric space, $\alpha,\beta : X\times X \to [0,\infty)$ be two functions and T be a self map on X. The mapping T is said to be a generalized (α,β)-F-contraction mapping, if there exists $F\in\mathcal{F}, \tau>0$ and $L\geq 0$ such that for all $x,y\in X$ with $Tx\neq Ty$

$$2\tau + F(\alpha(x,Tx)\beta(y,Ty)d(Tx,Ty)) \leq F(M(x,y)), \tag{2.2}$$

where $M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$.

Remark 2.3. In Defintition 2.1, we note the following:

(1) If we take $\alpha(x,Tx)\beta(y,Ty) = 1$ and $L = 0$, we obtain

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \quad\Rightarrow\quad \tau + F(d(Tx,Ty)) \leq F(M(x,y)). \tag{2.3}$$

It is easy to see that (2.3) is a generalization of Definition 1.4, Definition 1.7 and inequality (1.5).

(2) If we suppose that $F(x) = \ln x$ in Definition 2.1. Thus (2.1) becomes

$$\begin{aligned}&\tau + In(\alpha(x,Tx)\beta(y,Ty)d(Tx,Ty)) \leq In(M(x,y)+LN(x,y))\\ &\Rightarrow\quad \alpha(x,Tx)\beta(y,Ty)d(Tx,Ty) \leq e^{-\tau}(M(x,y)+LN(x,y))\\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad = e^{-\tau}M(x,y)+e^{-\tau}LN(x,y))\\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad = \delta M(x,y)+L_1N(x,y)),\end{aligned}$$

we have

$$\begin{aligned}&\frac{1}{2}d(x,Tx) \leq d(x,y)\\ &\Rightarrow\quad \alpha(x,Tx)\beta(y,Ty)d(Tx,Ty) \leq \delta M(x,y)+L_1N(x,y)),\end{aligned} \tag{2.4}$$

where $\delta = e^{-\tau}\in(0,1)$ and $L_1 = e^{-\tau}L\geq 0$. Clearly, if

$$\alpha(x,Tx)\beta(y,Ty)d(Tx,Ty) = 1,$$

we obtain a generalization of Definition 1.1 and Definition 1.2.

(3) We also note that if, we take $\alpha = \beta = \gamma = \frac{1}{4}$ and $\delta = L = \frac{1}{8}$, clear $\alpha + \beta + \gamma + 2\delta = 1$. Then Definition 1.9, becomes

$$\begin{aligned}
&\tau + F(d(Tx, Ty)) \\
&\leq F\bigg(\frac{1}{4}\big[d(x,y) + d(x,Tx) + d(y,Ty) + \frac{d(x,Ty) + d(y,Tx)}{2}\big]\bigg) \\
&\leq F\big(\frac{1}{4}[4M(x,y)]\big) \\
&\leq F(M(x,y) + LN(x,y)\big).
\end{aligned}$$

Theorem 2.4. *Let (X, d) be a complete metric space and $T : X \to X$ be a generalized Suzuki-(α, β)-F-contraction mapping. Suppose the following conditions hold:*

(1) *T is an (α, β)-cyclic admissible mapping,*
(2) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$,*
(3) *T is continuous.*

Then T has a fixed point.

Proof. We define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. If we suppose that $x_{n+1} = x_n$, we obtain the desired result. Now, suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N} \cup \{0\}$. Since T is an (α, β)-cyclic admissible mapping and $\alpha(x_0, x_1) \geq 1$, we have $\beta(Tx_0, Tx_1) = \beta(x_1, x_2) \geq 1$ and this implies that $\alpha(x_2, x_3) = \alpha(Tx_1, Tx_2) \geq 1$, continuing the process, we have

$$\alpha(x_{2k}, x_{2k+1}) \geq 1 \text{ and } \beta(x_{2k+1}, x_{2k+2}) \geq 1, \ \forall\ k \in \mathbb{N} \cup \{0\}. \tag{2.5}$$

Using similar arguments, we have that

$$\beta(x_{2k}, x_{2k+1}) \geq 1 \text{ and } \alpha(x_{2k+1}, x_{2k+2}) \geq 1, \ \forall\ k \in \mathbb{N} \cup \{0\}. \tag{2.6}$$

It follows from (2.5) and (2.6) that $\alpha(x_n, x_{n+1}) \geq 1$, $\beta(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and

$$\frac{1}{2}d(x_n, Tx_n) = \frac{1}{2}d(x_n, x_{n+1}) < d(x_n, x_{n+1}).$$

Hence we obtain from (2.1)

$$\begin{aligned}
&\tau + F(d(x_{n+1}, x_{n+2})) \\
&\leq \tau + F(\alpha(x_n, x_{n+1})\beta(x_{n+1}, x_{n+2})d(Tx_n, Tx_{n+1})) \\
&\leq F(M(x_n, x_{n+1}) \\
&\quad + L\min\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_n, x_{n+2}), d(x_{n+1}, x_{n+1})\}) \\
&= F(M(x_n, x_{n+1}) + L.0) \\
&= F(M(x_n, x_{n+1})),
\end{aligned} \tag{2.7}$$

where

$$M(x_n, x_{n+1})) = \max\left\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})}{2}\right\}$$
$$= \max\left\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{d(x_n, x_{n+2})}{2}\right\}.$$

Since

$$\frac{d(x_n, x_{n+2})}{2} \leq \frac{1}{2}[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})]$$
$$\leq \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}.$$

we have that $M(x_n, x_{n+1}) = \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}$. If we take

$$M(x_n, x_{n+1}) = \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = d(x_{n+1}, x_{n+2}),$$

then we obtain a contradiction in (2.7) as such

$$M(x_n, x_{n+1}) = \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = d(x_n, x_{n+1}).$$

It therefore follows from (2.7) that

$$F(d(x_{n+1}, x_{n+2})) \leq F(d(x_n, x_{n+1})) - \tau.$$

Using a similar approach, it is easy to see that,

$$F(d(x_n, x_{n+1})) \leq F(d(x_{n-1}, x_n)) - \tau.$$

Thus inductively we obtain

$$F(d(x_n, x_{n+1})) \leq F(d(x_0, x_1)) - n\tau, \quad \forall\, n \in \mathbb{N} \cup \{0\}. \tag{2.8}$$

Since $F \in \mathcal{F}$, taking limit as $n \to \infty$ in (2.8), we have

$$\lim_{n\to\infty} F(d(x_n, x_{n+1})) = -\infty. \tag{2.9}$$

It follows from $(F_3^{'})$ and Lemma 1.11 that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.10}$$

We now show that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence, then by Lemma 1.20, there exists an $\epsilon > 0$ and sequences of positive integers $\{x_{n_k}\}$ and $\{x_{m_k}\}$ with $n_k > m_k \geq k$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$. For each $k > 0$, corresponding to m_k, we can choose n_k to be the smallest positive integer such that $d(x_{m_k}, x_{n_k}) \geq \epsilon, d(x_{m_k}, x_{n_{k-1}}) < \epsilon$ and $(1)-(4)$ of Lemma 1.20 hold. Since $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$, using Lemma 1.19, we obtain that $\alpha(x_{m_k}, x_{m_{k+1}})\beta(x_{n_k}, x_{n_{k+1}}) \geq 1$. Hence for all $k \geq n_0$, we have

$$
\begin{aligned}
&\tau + F(d(x_{m_{k+1}}, x_{n_{k+1}})) \\
&\leq \tau + F(\alpha(x_{m_k}, x_{m_{k+1}})\beta(x_{n_k}, x_{n_{k+1}})d(Tx_{m_k}, Tx_{n_k})) \\
&\leq F(M(x_{m_k}, x_{n_k}) \\
&\quad + L\min\{d(x_{m_k}, x_{m_{k+1}}), d(x_{n_k}, x_{n_{k+1}}), d(x_{m_k}, x_{n_{k+1}}), d(x_{n_k}, x_{m_{k+1}})\}) \\
&\leq \Bigg(\max\Bigg\{ d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{m_{k+1}}), d(x_{n_k}, x_{n_{k+1}}), \\
&\qquad\qquad \frac{d(x_{m_k}, x_{n_{k+1}}) + d(x_{n_k}, x_{m_{k+1}})}{2} \Bigg\} \\
&\quad + L\min\{d(x_{m_k}, x_{m_{k+1}}), d(x_{n_k}, x_{n_{k+1}}), d(x_{m_k}, x_{n_{k+1}}), d(x_{n_k}, x_{m_{k+1}})\}\Bigg).
\end{aligned}
\tag{2.11}
$$

Using Lemma 1.20, $(F_4^{'})$ and (2.10), we have that

$$
\begin{aligned}
&\tau + F(\epsilon) \\
&= \lim_{k\to\infty} [\tau + F(\alpha(x_{m_k}, x_{m_{k+1}})\beta(x_{n_k}, x_{n_{k+1}})d(Tx_{m_k}, Tx_{n_k}))] \\
&\leq \lim_{k\to\infty} \Bigg[F\Bigg(\max\Bigg\{ d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{m_{k+1}}), d(x_{n_k}, x_{n_{k+1}}), \\
&\qquad\qquad \frac{d(x_{m_k}, x_{n_{k+1}}) + d(x_{n_k}, x_{m_{k+1}})}{2} \Bigg\} \\
&\quad + L\min\{d(x_{m_k}, x_{m_{k+1}}), d(x_{n_k}, x_{n_{k+1}}), d(x_{m_k}, x_{n_{k+1}}), d(x_{n_k}, x_{m_{k+1}})\Bigg\}\Bigg)\Bigg] \\
&\leq F(\epsilon).
\end{aligned}
$$

That is,

$$\tau + F(\epsilon) \leq F(\epsilon),$$

which is a contradiction. We therefore have that $\{x_n\}$ is Cauchy. Since (X, d) is complete, it follows that there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Since T is continuous, we have that

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T\lim_{n\to\infty} x_n = Tx.$$

Thus T has a fixed point. □

Theorem 2.5. *Let (X, d) be a complete metric space and $T : X \to X$ be a generalized Suzuki-(α, β)-F-contraction mapping. Suppose the following conditions hold:*

(1) *T is an (α, β)-cyclic admissible mapping,*

(2) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$ *and* $\beta(x_0, Tx_0) \geq 1$,
(3) *if for any sequence* $\{x_n\}$ *in* X *such that* $x_n \to x$ *as* $n \to \infty$, *then* $\beta(x, Tx) \geq 1$ *and* $\alpha(x, Tx) \geq 1$.

Then T *has a fixed point.*

Proof. We define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. In Theorem 2.4, we have established that $\{x_n\}$ is Cauchy. Now suppose hypothesis (3) holds. Now, we claim that

$$d(x_n, x) < \frac{1}{2} d(x_n, x_{n+1})$$

and

$$d(x_{n+1}, x) < \frac{1}{2} d(x_{n+1}, x_{n+2}).$$

Indeed, by using the fact that $d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1})$, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(x_n, x) + d(x, x_{n+1}) \\ &< \frac{1}{2} d(x_n, x_{n+1}) + \frac{1}{2} d(x_{n+1}, x_{n+2}) \\ &= d(x_n, x_{n+1}). \end{aligned}$$

The above inequality is a contradiction, thus we must have that

$$d(x_n, x) \geq \frac{1}{2} d(x_n, x_{n+1}) \quad \text{or} \quad d(x_{n+1}, x) \geq \frac{1}{2} d(x_{n+1}, x_{n+2}).$$

Hence we have

$$\begin{aligned} &\tau + F(d(x_{n+1}, Tx)) \\ &\leq \tau + F(\alpha(x_n, x_{n+1}) \beta(x, Tx) d(Tx_n, Tx)) \\ &\leq F\Bigg(\max\left\{ d(x_n, x), d(x_n, x_{n+1}), d(x, Tx), \frac{d(x_n, Tx) + d(x, Tx_n)}{2} \right\} \\ &\quad + L \min\{d(x_n, x_{n+1}), d(x, Tx), d(x_n, Tx), d(x, Tx_n)\} \Bigg). \end{aligned}$$

Taking the limit $k \to \infty$ and using the fact that $F \in \mathcal{F}$, we have that

$$\tau + F(d(x, Tx)) \leq F(d(x, Tx)),$$

which is a contradiction. Then

$$x = Tx.$$

Hence T has a fixed point. □

Theorem 2.6. *Suppose that the hypothesis of Theorem* 2.5 *holds and in addition suppose* $\alpha(x, Tx) \geq 1$ *and* $\beta(y, Ty) \geq 1$ *for all* $x, y \in F(T)$. *Then* T *has a unique fixed point.*

Proof. Let $x, y \in F(T)$, that is $Tx = x$ and $Ty = y$ such that $x \neq y$. Since, $\alpha(x, Tx) \geq 1$ and $\beta(y, Ty) \geq 1$, we have $\alpha(x, Tx)\beta(y, Ty) \geq 1$ and $\frac{1}{2}d(x, Tx) = 0 \leq d(x, y)$, and consequently we obtain that

$$\begin{aligned}
F(d(x,y)) &= F(d(Tx,Ty)) < \tau + F(\alpha(x,Tx)\beta(y,Ty)d(Tx,Ty)) \\
&\leq F\Bigg(\max\left\{ d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2} \right\} \\
&\quad + L \min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\} \Bigg), \\
&= F(d(x,y)),
\end{aligned}$$

which implies that

$$F(d(x,y)) < F(d(x,y)).$$

Clearly, this is a contradiction. Thus T has a unique fixed point. □

Theorem 2.7. *Let (X, d) be a complete metric space and $T : X \to X$ be a generalized (α, β)-F-contraction mapping. Suppose the following conditions hold:*

(1) *T is an (α, β)-cyclic admissible mapping,*
(2) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$,*
(3) *T is continuous.*

Then T has a fixed point.

Proof. The proof follows a similar approach as of Theorem 2.4 and thus we omit it. □

Theorem 2.8. *Let (X, d) be a complete metric space and $T : X \to X$ be a generalized (α, β)-F-contraction mapping. Suppose the following conditions hold:*

(1) *T is an (α, β)-cyclic admissible mapping,*
(2) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$,*
(3) *if for any sequence $\{x_n\}$ in X such that $x_n \to x$ as $n \to \infty$, then $\beta(x, Tx) \geq 1$ and $\alpha(x, Tx) \geq 1$.*

Then T has a fixed point.

Proof. The proof follows a similar approach as of Theorem 2.5 and thus we omit it. □

Theorem 2.9. *Suppose that the hypothesis of Theorem 2.8 holds and in addition suppose $\alpha(x, Tx) \geq 1$ and $\beta(y, Ty) \geq 1$ for all $x, y \in F(T)$. Then T has a unique fixed point.*

Proof. The proof follows a similar approach as of Theorem 2.6 and thus we omit it. □

3. Application

In this section, we give an application of Theorem 2.8 and Theorem 2.9 to guarantee the existence and uniqueness of solutions for the following nonlinear Hammerstein integral equations

$$x(t) = g(t) + \int_0^1 G(t,s)K(s,x(s))ds \tag{3.1}$$

and the following two-point boundary value problem of the second-order differential equations:

$$\begin{cases} -\epsilon x'' = K(t,x) - c, \\ ax(0) - bx'(0) = 0, \\ dx(1) - ex'(1) = 0, \end{cases} \tag{3.2}$$

where $t \in (0,1)$, K is a continuous real valued function and the constants $\epsilon > 0, b, c, e \geq 0, a + b > 0, d + e > 0$ and $f := ad + ae + bd > 0$.

Let $X = C([0,1])$ be the space of all continuous real function defined on $I = [0,1]$. It is well-known that $C([0,1])$ with the metric $d(x,y) = \sup_{t\in I} |x(t) - y(t)|$ for all $x, y \in C(I)$ is a complete metric space. It is also well known that $x^* \in C([0,1]) \cap C^2([0,1])$ is a solution for (3.2) if and only if $x^* C([0,1])$ is a solution of the following nonlinear integral equation

$$x(t) = \frac{1}{\epsilon}\int_0^1 G(t,s)(K(s,x(s)) - c)ds, \quad t \in I \tag{3.3}$$

where $G(t,s)$ is the Green function defined by

$$G(t,s) = \frac{1}{f}\begin{cases} (b + as)(e + d(1-t)), & 0 \leq s \leq t \leq 1, \\ (b + at)(e + d(1-s)), & 0 \leq t \leq s \leq 1. \end{cases} \tag{3.4}$$

It is also well known that for $t \in I$,

$$\sup_{t\in I} \int_0^1 G(t,s)ds = \frac{1}{f^2}(4f(bd + 2be) + (ad + 2ae)^2) := N \neq 0.$$

It is worth mentioning that problem (3.2) is equivalent to the integral equation (3.3).

Theorem 3.1. *Let $X = C(I)$ and $T : X \to X$ be an operator defined by*

$$Tx(t) = \frac{1}{\epsilon}\int_0^1 G(t,s)(K(s,x(s)) - c)ds, \quad t \in I, x \in X.$$

Also, let $\alpha, \beta : X \times X \to [0, \infty)$ *be a given function. Suppose the following assertions hold:*

(1) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1, \beta(x_0, Tx_0) \geq 1$;

(2) *there exists* $\tau > 0$ *and* $K : I \times \mathbb{R} \to \mathbb{R}$ *such that*

$$|K(s, x(s)) - K(s, y(s))| \leq \frac{\epsilon |x(s) - y(s)| e^{-2\tau}}{N\alpha(x, Tx)\beta(y, Ty)}$$

for all $s \in I$ *and* $x, y \in X$;

(3) *for any sequence* $\{x_n\}$ *in* X *such that* $x_n \to x$ *in* X, *then* $\alpha(x, Tx) \geq 1, \beta(x, Tx) \geq 1$.

Then there exists a solution to the integral equation (3.3), *and hence there exists a solution of the problem* (3.2).

Proof. Define the function $\alpha, \beta : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \beta(x, y) = \begin{cases} 2, & \text{if} \quad \alpha(x, Tx) \geq 1, \beta(x, Tx) \geq 1, \\ 0, & \text{if} \quad \text{otherwise.} \end{cases}$$

Now observe that

$$\begin{aligned} |Tx(s) - Ty(s)| &= \frac{1}{\epsilon} \sup_{s \in I} \left| \int_0^1 G(t, s)(K(s, x(s)) - K(s, y(s)))ds \right| \\ &\leq \frac{1}{\epsilon} \sup_{s \in I} \int_0^1 G(t, s)|(K(s, x(s)) - K(s, y(s)))|ds \\ &\leq \frac{1}{\epsilon} \frac{\epsilon |x(s) - y(s)| e^{-2\tau}}{N\alpha(x, Tx)\beta(y, Ty)} \left(\sup_{s \in I} \int_0^1 G(t, s)ds \right) \qquad (3.5) \\ &\leq \frac{|x(s) - y(s)|}{\alpha(x, Tx)\beta(y, Ty)} e^{-2\tau}. \end{aligned}$$

Thus, we have that

$$\begin{aligned} &\alpha(x, Tx)\beta(y, Ty)d(Tx, Ty) \leq d(x, y)e^{-2\tau} \leq (M(x, y) + LN(x, y))e^{-2\tau} \\ &\Rightarrow \quad 2\tau + \ln(\alpha(x, Tx)\beta(y, Ty)d(Tx, Ty)) \leq \ln((M(x, y) + LN(x, y))), \end{aligned}$$

taking $F(t) = \ln t$, we have that

$$2\tau + F(\alpha(x, Tx)\beta(y, Ty)d(Tx, Ty)) \leq F((M(x, y) + LN(x, y))),$$

Clearly, all the conditions of Theorem 2.9 are satisfied, and so T has a fixed point. Hence, we have $x^*(t) = Tx^*(t) = \frac{1}{\epsilon} \int_0^1 G(t, s)(K(s, x^*(s)) - c)ds$ and consequently, x^* is a solution of the two-point boundary value problem (3.2). This completes the proof. □

Theorem 3.2. *Let $X = C(I)$ and $T : X \to X$ be the operator given by*

$$Tx(t) = g(t) + \int_0^1 G(t,s)K(t,x(s))ds$$

for all $t, s \in [0,1]$, where $G : [0,1] \times [0,1] \to \mathbb{R}^+$, $K : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $g : [0,1] \to \mathbb{R}$ are continuous functions. Furthermore, suppose the following conditions hold:

(1) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1, \beta(x_0, Tx_0) \geq 1$;*
(2) *there exists a continuous mapping $\mu : X \times X \to [0,\infty)$ such that*

$$|K(s,x(s)) - K(s,y(s))| \leq \mu(x,y)|x(s) - y(s)|$$

for all $s \in [a,b]$ and $x, y \in X$;
(3) *there exists $\tau > 0$ and $\alpha, \beta : X \to [0,\infty)$ such that for all $x \in X$, we have*

$$\int_0^1 G(t,s)\mu(x,y) \leq \frac{e^{-\tau}}{\alpha(x,Tx)\beta(y,Ty)};$$

(4) *for any sequence $\{x_n\}$ in X such that $x_n \to x$ in X, then $\alpha(x,Tx) \geq 1, \beta(x,Tx) \geq 1$.*

Then the integral equation (3.1) has a solution.

Proof. Define the function $\alpha, \beta : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \beta(x,y) = \begin{cases} 1, & \text{if} \quad \alpha(x,Tx) \geq 1, \beta(x,Tx) \geq 1, \\ 0, & \text{if} \quad \text{otherwise.} \end{cases}$$

Without a loss of generality, we suppose that $x \leq y$, so that

$$\sup\{|y(s) - x(s)| : s \in [0,1]\} \geq \sup\{|Tx(s) - x(s)| : s \in [0,1]\},$$

which implies that

$$d(y,x) \geq d(Tx,x) \geq \frac{1}{2}d(Tx,x).$$

Thus, we have that

$$\begin{aligned} |Ty(s) - Tx(s)| &\leq \int_0^1 |G(t,s)[K(t,y(s)) - K(t,x(s))]|ds \\ &\leq \int_0^1 G(t,s)\mu(x,y)|y(s) - x(s)|ds \\ &\leq \sup_{s\in[a,b]} |y(s) - x(s)| \int_a^b G(t,s)\mu(x,y)ds \\ &\leq d(y,x)\frac{e^{-\tau}}{\alpha(x,Tx)\beta(y,Ty)}. \end{aligned}$$

Therefore, we have

$$\alpha(x,Tx)\beta(y,Ty)d(Tx,Ty) \le d(x,y)e^{-\tau} \le (M(x,y)+LN(x,y))e^{-\tau}$$
$$\Rightarrow \quad \tau + In(\alpha(x,Tx)\beta(y,Ty)d(Tx,Ty)) \le \ln((M(x,y)+LN(x,y))).$$

Taking $F(t)=\ln t$, we have that

$$\frac{1}{2}d(x,Tx) \le d(x,y)$$
$$\Rightarrow \quad \tau + F(\alpha(x,Tx)\beta(y,Ty)d(Tx,Ty)) \le F((M(x,y)+LN(x,y))).$$

Clearly, all the conditions in Theorem 2.9 are satisfied, and so T has a fixed point. Hence, we have

$$x^*(t) = Tx^*(t) = g(t) + \int_0^1 G(t,s)K(t,x^*(s))ds$$

and consequently, x^* is a solution of the integral equation (3.1). □

References

[1] S. Banach, *Sur les oprations dans les ensembles abstraits et leur application aux quations intgrales,* Fund. Math., **3** (1922), 133–181.

[2] G.V.R. Babu and P.D. Sailaja, *A fixed point theorem of generalized weakly contractive maps in orbitally complete metric spaces,* Thai J. of Math., **9** (2011), 1–10.

[3] V. Berinde, *Approximating fixed points of weak contractions using the Picard iteration,* Nonlinear Anal. Forum, **9** (2004), 43–53.

[4] V. Berinde, *General constructive fixed point theorem for Ćirić-type almost contractions in metric spaces,* Carpath. J. Math., **24** (2008), 10–19.

[5] S. Chandok, K. Tas and A.H. Ansari, *Some fixed point results for TAC-type contractive mappings,* J. Function spaces, 2016, Art. ID 1907676, 6 pp.

[6] M. Cosentino and P. Vetro, *Fixed point results for F- contractive mappings of Hardy-Rogers type,* Filomat, **28** (2014), 715–722.

[7] N. Hussain, M.A. Kutbi and P. Salimi, *Fixed point theory in α-complete metric spaces with applications,* Abstr. Appl. Anal., (2014), Art. ID 280817, 11 pp.

[8] E. Karapinar, P. Kuman and P. Salimi, *On α-ψ-Meir-Keeler contractive mappings,* Fixed Point Theory Appl., **94** (2013), 1–12.

[9] E. Karapinar and K. Tas, *Generalized (C)-conditions and related fixed point theorems,* Comput. Math. Appl., **61** (2011), 3370–3380.

[10] M.S. Khan, M. Swaleh, and S. Sessa, *Fixed point theorems by altering distances between the points,* Bull. Austral. Math. Soc., **30** (1984), 1–9.

[11] A.A. Meabwondu and O.T. Mewomo, *Some fixed pont results for a modified F-contraction via a new type of (α,β)-cyclic admissible mappings in metric space,* Bulletin University. Transilvania Brasov, Series III: Mathematics, informatics, Physics, **12** (2019), 77–94.

[12] G. Minak, A. Helvaci and I. Altun, *Ćirić type generalized F-contraction on complete metric space and fixed point results,* Filomat, **28** (2014), 1143–1151.

[13] U. Mishra, H.K. Nashine and R. Kewat, *Fixed point theory for generalized* (ψ, ϕ)*-weak contractions involving* $f - g$ *reciprocally continuity,* Indian J. Math., **56** (2014), 153–168.
[14] P.P. Murthy, L.N. Mishra and U.D. Patel, *n-tupled fixed point theorems for weak-contraction in partially ordered complete G-metric spaces,* New Trends Math. Sci. **4** (2015), 50–75.
[15] A. Pansuwon, W. Sintunavarat, V. Parvaneh and Y.J. Cho, *Some fixed point theorems for* (α, θ, k)*-contractive multi-valued mappings with some applications,* Fixed Point Theory Appl., (2015), Art. ID 132.
[16] H. Piri and P. Kumam, *Some fixed point theorems concerning* F*-contraction in complete metric spaces,* Fixed Point Theory and Appl., (2014), Art. ID 210.
[17] P. Salimi and V. Pasquale, *A result of Suzuki type in partial G-metric spaces,* Acta Math. Sci. Ser. B (Engl. Ed.)., **34** (2014), 274–284.
[18] B. Samet, C. Vetro and P. Vetro, *Fixed point theorem for* α*-*ψ*-contractive type mappings,* Nonlinear Anal., **75** (2012), 2154–2165.
[19] N.A. Secelean, *Iterated function systems consisting of* F*-contractions,* Fixed Point Theory and Appl., (2013), Art. ID 277.
[20] W. Shatanawi and M. Postolache, *Some fixed-point results for a G-weak contraction in G-metric spaces,* Abstr. Appl. Anal., (2012), Art. ID 815870, 19 pp.
[21] T. Suzuki, *Fixed point theorems and convergence theorems for some generalized nonexpansive mappings,* J. Math. Anal. Appl., **340** (2008), 1088–1095.
[22] T. Suzuki, *A new type of fixed point theorem in metric spaces,* Nonlinear Anal., **71** (2009), 5313–5317.
[23] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces,* Fixed Point Theory and Appl., (2012), Art. ID 94.