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EXISTENCE OF SOCIAL EQUILIBRIA IN GENERALIZED NASH GAMES WITH ADDITIVELY COUPLED PAYOFFS

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Abstract. In this paper, we introduce a new model of generalized Nash game with additively coupled payoff functions which generalizes Balder's model in [2], and next give two social equilibrium existence theorems for general strategic games which are comparable with the previous results due to Arrow and Debreu, Balder, Debreu, and Park in several aspects.

1. Introduction

In 1951, Nash [7] established the pioneering result on the existence of equilibrium for abstract economies, and next, in 1952, Debreu [5] established the existence of social equilibrium existence theorem using constraint correspondences. Since then, there are many generalizations and applications of these two theorems as basic references for the existence of Nash equilibrium for generalized games, e.g., see [3,6,8–11] and references therein.

Among them, we met an economic condition which presents a psychologic behavior of payoff functions. For a generalized game $\Gamma = (X_i; T_i, f_i)_{i \in I}$, we encountered an individual and independent payoff function f_i on the constraint correspondence $T_i : X \to 2^{X_i}$ which satisfies the optimal inequality with respect to the utility function f_i. That is, if the strategy x_j is feasible in the game Γ, the j-th player can choose the strategy x_j restricted on the constraint set $T_j(x)$ so that we want to find the optimal value of utility function f_j. This is a kind of natural economic sense in the real strategic game situation, and in

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[2], Balder introduced a basic optimization problem where the each player is faced with an individual and independent payoff so that an existence of Nash equilibrium is obtained.

In this paper, we first introduce a new model of strategic Nash game with additively coupled payoff functions which generalizes Balder's model in [2] using constraint correspondences. Next, we establish two new social equilibrium existence theorems for general strategic games by using Bauer's maximum theorem and Berge's maximum theorem, respectively, which are comparable with the previous results due to Arrow and Debreu [1], Balder [2], Debreu [5], Ding-Kim-Tan [6], Park [8], and others in several aspects. Finally, we give an example which is suitable for our theorems but the previous equilibrium existence theorems can not be applied.

2. Preliminaries

We begin with some notions and terminologies in generalized Nash equilibrium for non-cooperative pure strategic games. Let the set I of players be possibly countable. Then, a *generalized Nash game of normal form* (or *social system*) is the system of ordered triples $\Gamma = (X_i; T_i, f_i)_{i\in I}$, where for each player $i \in I$, the nonempty set X_i is a player's pure strategy space, $T_i : X \to 2^{X_i}$ is a player's constraint correspondence, and $f_i : X \to \mathbb{R}$ is a player's payoff (or utility) function. The set X, *joint strategy space*, is the Cartesian product of the individual strategy spaces, and the element of X_i is called a *strategy.* When I is any set of players, we shall use the notation as

$$X_{-i} := \prod_{j\in I; j\neq i} X_j;$$

and hence we write a typical strategy profile $x = (x_i, x_{-i}) \in X = \Pi_{i\in I} X_i = X_i \times X_{-i}$. Then, a strategy profile $\bar{x} = (\bar{x}_i, \bar{x}_{-i}) \in X$ is called the *social equilibrium* (or *generalized Nash equilibrium*) for the generalized Nash game Γ if the following system of inequalities holds: for each $i \in I$,

$$\bar{x}_i \in T_i(\bar{x}), \quad \text{and} \quad f_i(\bar{x}_i, \bar{x}_{-i}) \leq f_i(x_i, \bar{x}_{-i}) \quad \text{for each } x_i \in T_i(\bar{x}).$$

Next, we now introduce an economic condition which presents a kind of psychologic behavior as follows: Let $\Gamma = (X_i; T_i, f_i)_{i\in I}$ be a generalized Nash game with the set I of players which is finite (possibly countable). Then we can consider an individual and independent payoff for the constraint correspondence $T_i : X \to 2^{X_i}$ which satisfies the following optimal inequality with respect to the utility function f_i. That is, we want to find the optimal strategy $x = (x_i)_{i\in I} \in X$ such that all of individual players might imagine and guess that there might be a better strategy $y \in T_i(x)$ satisfying that

$f_i(x_i, x_{-i}) \leq f_i(y, x_{-i})$, i.e., there might be a strategy having least payoff value. This is a kind of psychologic and natural economic sense in the real strategic game situation.

We first introduce a generalized additively coupled condition of payoff functions for a generalized Nash game $\Gamma = (X_i; T_i, f_i)_{i\in I}$ as follows:

Definition 2.1. Let $\Gamma = (X_i; T_i, f_i)_{i\in I}$ be a generalized Nash game with a finite (possibly countable) set I of players. Then the payoff functions $\{f_i\}_{i\in I}$ are called *additively coupled* if for each $i \in I$, there exists component functions $f_{i,j} : X_j \to \mathbb{R}$, $j \in I$, such that

$$f_i(x_1, x_2, \cdots, x_n) \;=\; \sum_{j=1}^{n} f_{i,j}(x_j)$$

for each $x = (x_1, x_2, \cdots, x_n) \in X = \Pi_{j\in I} X_j$, and $x_j \in T_j(x)$.

When $T_i(x) := X_i$, for each $i \in I$ in Definition 2.1, this coincides with Balder's definition in [2], and he introduced an optimization problem where the each player is faced with an individual and independent payoff so that an existence of Nash equilibrium is obtained.

In most equilibrium existence theorems, payoff functions should be satisfied uniform kind of convex assumptions, e.g. [3, 6, 9–11]. However, in some games, payoff functions can not be satisfied with uniform convex assumptions. For example, let $\Gamma = (X_i; T_i, f_i)_{i\in I}$ be a generalized Nash game with the each player i is faced with an individual and independent payoff function as

$$f_i(x_1, x_2, \cdots, x_n) \;=\; \sum_{j=1}^{n} f_{i,j}(x_j),$$

where $f_{i,j}(x_j) := \frac{1}{j}(-1)^{i+j}x_j^2$ for each $x = (x_1, x_2, \cdots, x_n) \in X$, and $x_j \in T_j(x)$. Then, when $i + j$ is even, $f_{i,j}$ is convex, and when $i + j$ is odd, $f_{i,j}$ is concave. Therefore, we can not apply the previous existence theorems of Nash equilibria as in [1, 5–11] for this game Γ; however we can apply the next Theorems 3.1 or 3.3 for the existence of social equilibrium in this game Γ.

Next, we shall need the following which is a basic tool for proving the existence of social equilibrium for a generalized Nash game:

Lemma 2.2. *Let $\Gamma = (X_i; T_i, f_i)_{i\in I}$ be the generalized Nash game with $I = \{1, \ldots, n\}$ the (possibly countable) set of players, whose payoff functions $\{f_i\}_{i\in I}$ are additively coupled. Then we have*

$\bar{x} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) \in X$ *is a social equilibrium for* Γ, *if and only if, for each $i \in I$,*

$$\bar{x}_i \in T_i(\bar{x}) \quad \textit{and} \quad f_{i,i}(\bar{x}_i) = \inf_{x_i \in T_i(\bar{x})} f_{i,i}(x_i).$$

Proof. For the necessity, we let $\bar{x} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) \in X$ be a social equilibrium for Γ. Then, for each $i \in I$, $f_i(\bar{x}_i, \bar{x}_{-i}) \leq f_i(x_i, \bar{x}_{-i})$ for each $x_i \in T_i(\bar{x})$ so that $f_{i,i}(\bar{x}_i) \leq f_{i,i}(x_i)$. Indeed, since the payoff function f_i is additively coupled, for each $x_i \in T_i(\bar{x})$, we have

$$f_i(\bar{x}_i, \bar{x}_{-i}) = f_{i,1}(\bar{x}_1) + f_{i,2}(\bar{x}_2) + \cdots + f_{i,i}(\bar{x}_i) + \cdots + f_{i,n}(\bar{x}_n);$$
$$f_i(x_i, \bar{x}_{-i}) = f_{i,1}(\bar{x}_1) + f_{i,2}(\bar{x}_2) + \cdots + f_{i,i}(x_i) + \cdots + f_{i,n}(\bar{x}_n);$$

so that $f_{i,i}(\bar{x}_i) \leq f_{i,i}(x_i)$, and hence

$$f_{i,i}(\bar{x}_i) \leq \inf_{x_i \in T_i(\bar{x})} f_{i,i}(x_i).$$

Since $\bar{x}_i \in T_i(\bar{x})$, $\inf_{x_i \in T_i(\bar{x})} f_{i,i}(x_i) \leq f_{i,i}(\bar{x}_i)$, thus $f_{i,i}(\bar{x}_i) = \inf_{x_i \in T_i(\bar{x})} f_{i,i}(x_i)$.

For the sufficiency, if $f_{i,i}(\bar{x}_i) = \inf_{x_i \in T_i(\bar{x})} f_{i,i}(x_i)$ for each $i \in I$, then

$$\begin{aligned} f_i(\bar{x}_i, \bar{x}_{-i}) &= f_{i,1}(\bar{x}_1) + f_{i,2}(\bar{x}_2) + \cdots + f_{i,i}(\bar{x}_i) + \cdots + f_{i,n}(\bar{x}_n) \\ &\leq f_{i,1}(\bar{x}_1) + f_{i,2}(\bar{x}_2) + \cdots + f_{i,i}(x_i) + \cdots + f_{i,n}(\bar{x}_n) \\ &= f_i(x_i, \bar{x}_{-i}); \end{aligned}$$

for all $x_i \in T_i(\bar{x})$. Hence, $\bar{x} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) \in X$ is a social equilibrium for Γ. □

Remark 2.3. Lemma 2.2 further generalizes Proposition 1.1 in [2] to a generalized Nash game $\Gamma = (X_i; T_i, f_i)_{i \in I}$ in the following aspects:

(a) for each $x \in X$, $T_i(x)$ need not be a constant multimap;
(b) the set of players I need not be finite.

Let X be a convex subset of a vector space E. Recall that a point $x \in X$ is said to be *an extreme point* of X if, whenever elements y and z of X satisfy $x = \lambda y + (1 - \lambda) z$, where $\lambda \in [0, 1]$, then either $\lambda = 0$ or $\lambda = 1$.

Let A and B be two convex subsets of a vector space E. Also, recall that a function $f : A \to B$ is called an *affine map* if for every family $\{(a_i, \lambda_i)\}_{i \in I}$ of weighted points in A such that $\sum_{i \in I} \lambda_i = 1$, we have

$$f\left(\sum_{i \in I} \lambda_i a_i\right) = \sum_{i \in I} \lambda_i f(a_i).$$

For the existence of extreme social equilibrium for a generalized Nash game, we shall need the following minimum version of Bauer's maximum theorem for extreme points of compact convex sets:

Lemma 2.4. (Theorem 25.9, [4])*Let X be a nonempty compact convex subset of a locally convex Hausdorff topological vector space E. Suppose that $f : X \to \mathbb{R}$ is a concave and lower semicontinuous function on X. Then there exists an extreme point of X where f assumes its minimum value.*

We also need Berge's theorem for continuous multimaps which is well known for the existence of fixed points in nonlinear analysis:

Lemma 2.5. ([11]) *Let X and Y be topological spaces, $f : X \times Y \to \mathbb{R}$ a real function, $T : X \to 2^Y$ a multimap, and*

$$\hat{f}(x) := \inf_{y \in T(x)} f(x, y), \quad S(x) := \{y \in T(x) \mid f(x, y) = \hat{f}(x)\}$$

for each $x \in X$. Then we have

(1) *If f is l.s.c. and T is u.s.c. with compact values, then $\hat{f}$ is l.s.c.;*
(2) *If f is u.s.c. and T is l.s.c., then $\hat{f}$ is u.s.c.;*
(3) *If f is continuous and T is continuous with compact values, then $\hat{f}$ is continuous and S is u.s.c.*

From now on, let X_i be a nonempty compact convex subset of a locally convex Hausdorff topological vector space E. For the other standard notations and terminologies, we shall refer to Border [3], Ding-Kim-Tan [6], Yuan-Tarafdar [11], and the references therein.

3. Existence of social equilibria with additively coupled payoffs

Using Lemma 2.2, we begin with the existence of social equilibria in generalized Nash games with additively coupled payoffs as follows:

Theorem 3.1. *Let $\Gamma = (X_i; T_i, f_i)_{i \in I}$ be a generalized Nash game with $I = \{1, \ldots, n\}$ the (possibly countable) set of players, where for each player $i \in I$, X_i is a nonempty compact convex subset of a locally convex Hausdorff topological vector space E, and $X := \Pi_{i \in I} X_i = X_i \times X_{-i}$. Suppose that for each $i \in I$,*

(1) *$T_i : X \to 2^{X_i}$ is a continuous constraint multimap such that $T_i(x)$ is a nonempty compact and convex subset of X_i for each $x \in X$;*
(2) *$f_i : X \to \mathbb{R}$ is an additively coupled payoff function such that $f_{i,i} : X_i \to \mathbb{R}$ is continuous;*
(3) *$f_{i,i} : X_i \to \mathbb{R}$ is a quasi-convex component function on $T_i(x)$.*

Then there exists a social equilibrium $\bar{x} = (\bar{x}_i)_{i \in I} \in X$ for Γ, i.e., for each $i \in I$,

$$\bar{x}_i \in T_i(\bar{x}), \text{ and } f_i(\bar{x}_i, \bar{x}_{-i}) \leq f_i(x_i, \bar{x}_{-i}) \quad \text{for each } x_i \in T_i(\bar{x}).$$

Proof. For each $i \in I$, we first define a multimap $\Phi_i : X \to 2^{X_i}$ by

$$\Phi_i(x) := \Big\{ z \in T_i(x) \mid f_{i,i}(z) = \inf_{y \in T_i(x)} f_{i,i}(y) \Big\}$$

for each $x = (x_i, x_{-i}) \in X$.

By the assumptions (1) and (2), each $f_{i,i}$ is a continuous function, and T_i is a continuous multimap with compact values. Therefore, by Lemma 2.5, Φ_i is an upper semicontinuous multimap with nonempty values. Since $f_{i,i}$ is continuous and T_i is continuous, it is easy to see that $\Phi_i(x)$ is nonempty closed for each $x \in X$. For the convexity of $\Phi_i(x)$, we let $z_1, z_2 \in T_i(x)$ such that $f_{i,i}(z_k) = \inf_{y \in T_i(x)} f_{i,i}(y)$ for each $k = 1, 2$. For any $\lambda \in [0, 1]$, we shall show that $\hat{z} = \lambda z_1 + (1 - \lambda) z_2 \in T_i(x)$ such that $f_{i,i}(\hat{z}) = \inf_{y \in T_i(x)} f_{i,i}(y)$. Since $T_i(x)$ is convex, and $z_1, z_2 \in T_i(x)$, it is clear that $\hat{z} \in T_i(x)$. If we let

$$A := \Big\{ z \in T_i(x) \mid f_{i,i}(z) \le \inf_{y \in T_i(x)} f_{i,i}(y) \Big\},$$

then we have $z_1, z_2 \in A$. By the assumption (3), since $f_{i,i} : X \to \mathbb{R}$ is a quasi-convex function on $T_i(x)$, A is convex so that $\hat{z} = \lambda z_1 + (1 - \lambda) z_2 \in A$ for all $\lambda \in [0, 1]$, that is, $f_{i,i}(\hat{z}) \le \inf_{y \in T_i(x)} f_{i,i}(y)$. Therefore, $\Phi_i(x)$ is convex for each $x \in X$.

Next, we define a multimap $\Phi : X \to 2^X$ by

$$\Phi(x) := \Pi_{i \in I} \Phi_i(x) \text{ for each } x = (x_i, x_{-i}) \in X.$$

Then, Φ is an upper semicontinuous multimap such that $\Phi(x)$ is a nonempty compact convex subset of X for each $x \in X$. By the Fan-Glicksberg fixed point theorem, there exists a fixed point $\bar{x} \in \Phi(\bar{x})$, that is, for each $i \in I$,

$$\bar{x}_i \in T_i(\bar{x}) \text{ and } f_{i,i}(\bar{x}_i) = \inf_{x_i \in T_i(\bar{x})} f_{i,i}(x_i).$$

Therefore, by Lemma 2.2, the fixed point $\bar{x}$ for Φ is exactly a social equilibrium for the game $\Gamma = (X_i; T_i, f_i)_{i \in I}$ which completes the proof. □

Remark 3.2. Theorem 3.1 asserts an existence of a social equilibrium in generalized Nash games $\Gamma = (X_i; T_i, f_i)_{i \in I}$ with the (possibly countable) set of players. However, we shall need the additional continuity and quasi-convex assumption on the component function $f_{i,i}$ for each $i \in I$ so that Theorem 3.1 is comparable with Corollary 1.2 [2] in special case of $T_i(x) = X_i$.

Next, using Lemma 2.4, we shall prove an existence theorem of an extreme social equilibrium as follows:

Theorem 3.3. *Let* $\Gamma = (X_i; T_i, f_i)_{i \in I}$ *be a generalized Nash game with* $I = \{1, \ldots, n\}$ *the (possibly countable) set of players, where for each player* $i \in I$,

X_i is a nonempty compact convex subset of a locally convex Hausdorff topological vector space E. Suppose that for each $i \in I$,

(1) *$T_i : X \to 2^{X_i}$ is a continuous constraint multimap such that $T_i(x)$ is a nonempty compact and convex subset of X_i for each $x \in X$;*
(2) *$f_i : X \to \mathbb{R}$ is an additively coupled payoff function such that $f_{i,i} : X_i \to \mathbb{R}$ is a continuous function;*
(3) *$f_{i,i} : X_i \to \mathbb{R}$ is a quasi-convex component function on $T_i(x)$;*
(4) *the mapping $x \mapsto \inf_{y \in T_i(x)} f_{i,i}(y)$ is concave on X.*

Then there exists a social equilibrium $\bar{x} = (\bar{x}_i)_{i \in I} \in X$ for the game Γ such that $\bar{x}_i$ is an extreme point of $T_i(\bar{x})$ for each $i \in I$.

Proof. By the same proof of Theorem 3.1, we can obtain a social equilibrium $\bar{x} = (\bar{x}_i)_{i \in I} \in X$ for Γ. It remains to show that $\bar{x}_i$ is an extreme point of $T_i(\bar{x})$ for each $i \in I$.

For each $i \in I$, by the assumptions (1) and (2), each $f_{i,i}$ is a continuous function, and T_i is a continuous multimap with compact values so that by Lemma 2.5(1), the function $\phi_i : X \to \mathbb{R}$ defined by

$$\phi_i(x) := \inf_{y \in T_i(x)} f_{i,i}(y) \text{ for each } x \in X,$$

is a lower semicontinuous function on X. By the assumption (4), ϕ_i is a concave mapping on X. Therefore, since each $T_i(x)$ is nonempty compact convex, by Lemma 2.4, we obtain that ϕ_i has an extreme point of $T_i(x)$ where ϕ_i assumes its minimum value on $T_i(x)$, i.e., the set

$$\Big\{ z \in T_i(x) \mid f_{i,i}(z) = \inf_{y \in T_i(x)} f_{i,i}(y) \Big\}$$

has values of extreme points of $T_i(x)$ for each $x \in X$. Therefore, we can obtain a social equilibrium $\bar{x} = (\bar{x}_i)_{i \in I} \in X$ for Γ where $\bar{x}_i$ is an extreme point of $T_i(\bar{x})$ for each $i \in I$. This completes the proof. □

Remark 3.4. (1) In Theorem 3.3, if we assume the following condition instead of the assumption (4), then we can obtain the same conclusion:

(4′) for each $x_1, x_2 \in X$, and any $\lambda \in (0,1)$,

$$T_i\big(\lambda x_1 + (1-\lambda)x_2\big) \subseteq T_i(x_k) \quad \text{for each} \quad k = 1, 2.$$

In fact, we will show that ϕ_i is concave function. Indeed, for any $x_1, x_2 \in X$ such that $\phi_i(x_1) = \inf_{y \in T_i(x_1)} f_{i,i}(y)$, and $\phi_i(x_2) = \inf_{y \in T_i(x_2)} f_{i,i}(y)$, we shall show that for any $\lambda \in (0,1)$,

$$\phi_i\big(\lambda x_1 + (1-\lambda)x_2\big) \geq \lambda \inf_{y \in T_i(x_1)} f_{i,i}(y) + (1-\lambda) \inf_{y \in T_i(x_2)} f_{i,i}(y).$$

Indeed, by the assumption (4′), for any $\lambda \in (0,1)$,

$$\inf_{y\in T_i\big(\lambda x_1+(1-\lambda)x_2\big)} f_{i,i}(y) \;\geq\; \lambda \inf_{y\in T_i(x_1)} f_{i,i}(y) + (1-\lambda) \inf_{y\in T_i(x_2)} f_{i,i}(y);$$

thus

$$\phi_i\big(\lambda x_1 + (1-\lambda)x_2\big) \geq \lambda\phi_i(x_1) + (1-\lambda)\phi_i(x_2)$$

so that ϕ_i is a concave function on X, and we obtain the conclusion.

(2) When $T_i(x) := X_i$, for each $x \in X$, then the condition (4′) is automatically satisfied so that the assumptions (1) and (4) in Theorem 3.3 are clearly satisfied. In this case, Theorem 3.3 is comparable with Corollary 1.3 [2] in the following aspects:

(a) the component function $f_{i,i}$ need not be convex but quasi-convex; however we shall need lower semicontinuity of $f_{i,i}$ for each $i \in I$;
(b) the set of players I need not be finite.

Note that the following existence theorem is contained in Theorem 3.3, and it shows that the additive coupledness is an inherent aspect of affine condition as follows:

Theorem 3.5. *Let $\Gamma = (X_i; T_i, f_i)_{i\in I}$ be a generalized Nash game with $I = \{1,\ldots,n\}$ the (possibly countable) set of players, where for each player $i \in I$, X_i is a nonempty compact convex subset of a locally convex Hausdorff topological vector space E. For each $i \in I$, suppose that*

(1) *$T_i : X \to 2^{X_i}$ is a continuous constraint multimap such that $T_i(x)$ is a nonempty compact and convex subset of X_i for each $x \in X$;*
(2) *$f_i : X \to \mathbb{R}$ is a continuous and affine payoff function on X;*
(3) *for any $\lambda \in (0,1)$, $T_i\big(\lambda x_1 + (1-\lambda)x_2\big) \subseteq T_i(x_k)$ for each $k = 1,2$.*

Then there exists a social equilibrium $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ for the game Γ, i.e., for each $i \in I$,

$$\bar{x}_i \in T_i(\bar{x}), \text{ and } f_i(\bar{x}_i, \bar{x}_{-i}) \leq f_i(x_i, \bar{x}_{-i}) \text{ for each } x_i \in T_i(\bar{x});$$

and also $\bar{x}_i$ is an extreme point of $T_i(\bar{x})$ for each $i \in I$.

Proof. By applying Theorem 3.3, we shall show that f_i is an additively coupled function on X. For this, we let $\hat{x} = (\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n) \in X = \Pi_{i\in I} X_i$ be fixed. For each $j \in I$, we now define a component function $f_{i,j} : X_j \to \mathbb{R}$ by

$$f_{i,j}(x_j) := f_i(x_j, \hat{x}_{-j}) - \left(1 - \frac{1}{n}\right) f_i(\hat{x}_j, \hat{x}_{-j}) \text{ for each } x_j \in X_j.$$

Then, by the affine assumption of f_i on X, for each $x = (x_1, x_2, \cdots, x_n) \in X$, we have

$$\begin{aligned}
\sum_{j=1}^{n} \frac{1}{n} f_{i,j}(x_j) &= \sum_{j=1}^{n} \frac{1}{n} \big[f_i(x_j, \hat{x}_{-j}) - (1 - \frac{1}{n}) f_i(\hat{x}) \big] \\
&= \sum_{j=1}^{n} f_i(\frac{1}{n}\, x_j, \frac{1}{n}\, \hat{x}_{-j}) \,-\, (1 - \frac{1}{n}) f_i(\hat{x}) \\
&= f_i\big(\frac{1}{n}[(x_1, \hat{x}_{-1}) + \cdots + (x_n, \hat{x}_{-n})]\big) - (1 - \frac{1}{n}) f_i(\hat{x}) \\
&= f_i\big(\frac{1}{n}\, x + (1 - \frac{1}{n})\, \hat{x}\big) - (1 - \frac{1}{n}) f_i(\hat{x}) \\
&= \frac{1}{n}\, f_i(x) + (1 - \frac{1}{n})\, f_i(\hat{x}) - (1 - \frac{1}{n}) f_i(\hat{x}) \\
&= \frac{1}{n}\, f_i(x),
\end{aligned}$$

so that we have $f_i(x) = \sum_{j=1}^{n} f_{i,j}(x_j)$ for each $x \in X$; thus f_i is additively coupled by $\{f_{i,j}\}$ for each $i \in I$. It remains to show that for each $i \in I$, $f_{i,i} : X_i \to \mathbb{R}$ is a quasi-convex component function on $T_i(x)$. Indeed, since

$$f_{i,i}(x_i) = f_i(x_i, \hat{x}_{-i}) - (1 - \frac{1}{n}) f_i(\hat{x}_i, \hat{x}_{-i})$$

and $f_i : X \to \mathbb{R}$ is affine on X_i, $f_{i,i}$ is clearly an affine map on X_i and hence $f_{i,i}$ is a quasi-convex component function on $T_i(x)$. Therefore, all the hypotheses of Theorem 3.3 are satisfied so that we can obtain a desired conclusion. This completes the proof. □

Finally, we give an example of a generalized Nash game which is suitable for Theorem 3.1 or Theorem 3.3, but the previous equilibrium existence theorems in Border [3], Ding-Kim-Tan [6], and Park [8] for compact games can not be applied:

Example 3.6. Let $\Gamma = (X_i; T_i, f_i)_{i \in I}$ be a generalized Nash game where for each player $i \in I = \{1, 2, \cdots, n\}$, and for each $x = (x_1, x_2, \cdots, x_n) \in X$, suppose that

(1) $X_i := [0, 1]$ is a nonempty compact convex strategy set, and $X = \Pi_{i \in I} X_i$;
(2) $T_i : X \to 2^{X_i}$ is a continuous constraint multimap such that $T_i(x) := [0, 1]$ for each $x \in X$;

(3) $f_i : X \to \mathbb{R}$ is an additively coupled continuous payoff function on X such that

$$f_i(x_1, x_2, \cdots, x_n) = \sum_{j=1}^{n} f_{i,j}(x_j),$$

where $f_{i,j}(x_j) := \frac{1}{j}(-1)^{i+j}x_j^2$ for each $x = (x_1, x_2, \cdots, x_n) \in X = \Pi_{j\in I}X_j$.

Then, for each $i \in I$, $f_{i,i}$ is clearly a (quasi-)convex component function on $T_i(x)$, and for any $\lambda \in (0,1)$, $T_i\big(\lambda x_1 + (1-\lambda)x_2\big) \subseteq T_i(x_k)$ for each $k = 1, 2$. Therefore, all hypotheses of Theorem 3.3 are satisfied so that there exists an extreme social equilibrium $\bar{x} = (0, 0, \cdots, 0) \in X$ for the game Γ. Indeed, for each $i \in I$, $0 \in T_i(\bar{x})$, and

$$\begin{aligned} 0 = f_i(0, 0, \cdots, 0) \;&\le\; f_i(x_i, \bar{x}_{-i}) \\ &= f_{i,1}(0) + f_{i,2}(0) + \cdots + f_{i,i}(x_i) + \cdots + f_{i,n}(0) \\ &= f_{i,i}(x_i) = \frac{1}{i}x_i^2 \end{aligned}$$

for each $x_i \in T_i(\bar{x}) = [0,1]$, and also 0 is an extreme point of $T_i(\bar{x}) = [0,1]$.

However, since $f_i(\cdot, x_i)$ is neither (quasi-)convex nor (quasi-)concave for each $i \in I$, we can not apply the previous equilibrium existence theorems in Border [3], Ding-Kim-Tan [6], and Park [8] for this compact game Γ.

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