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α-ADMISSIBLE PREŠIĆ TYPE F-CONTRACTION

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Abstract. In this paper, we introduce α-admissible mappings on product spaces and obtain F-contraction results for α-admissible Prešić type operators. Our results extend, unify and generalize some known results of the literature. We illustrate an example for support our results.

1. Introduction

The Banach contraction principle [2] is one of the most important analytical results and considered as the main source of metric fixed point theory. It is the most widely applied fixed point result in many branches of mathematics. This result has been generalized in many different directions. Subsequently, in 2012, Wordowski [16] introduced the concept of F-contraction which generalized the Banach contraction principal in many ways.

In 2016, Shuklaa and Shahzad [14] introduced α-admissible mappings on product spaces and obtain fixed point results for α-admissible Prešić type operators. Abbas et al. [1] introduced the convergence of the Prešić type k-step iterative process for a class of operators $f : X^k \to X$ satisfying Prešić type F-contractive condition in the setting of metric spaces. Gopal et al. [4] introduced new concepts of α-type F-contractive mappings which are essentially

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weaker than the class of F-contractive mappings given in [17]. Padcharoen et al. [6] introduced the concept of α-type F-contraction in the setting of modular metric spaces and establish fixed point and periodic point results for such a contraction. In [5], Jain et al. presented a new approach to study the existence of fixed points for multivalued F-contraction in the setting of modular metric spaces. Moreover, In [15], Sumalai, modified and prove some common fixed point theorems by using the $(CLRg)$-property along with the weakly compatible mapping. Padcharoen et al. [7] introduced some results on the existence of coincidence and periodic point of F-contractive mappings in the framework of modular metric spaces endowed with a graph.

Following this direction of research, in this paper, we define α-admissible mappings on product spaces and obtain F-contraction results for α-admissible Prešìc type operators. Some examples are provided, which illustrate the results proved herein and show the applicability of results.

2. Preliminaries

We give some definitions and their properties for our main results.

Theorem 2.1. ([8, 9]) *Let (X, d) be a complete metric space and let k be a positive integer. If $f : X^k \to X$ satisfies the following contractive condition*:

$$d(f(x_1, x_2 \dots, x_k), f(x_2, x_3 \dots, x_{k+1})) \le \alpha_1 d(x_1, x_2) + \dots + \alpha_k d(x_k, x_{k+1}),$$

for all $x_1, x_2 \dots, x_{k+1} \in X$, $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}^+$ and $\alpha_1 + \alpha_2 + \dots + \alpha_k = \alpha < 1$. Then we have the followings:

- (i) *f has a unique fixed point $x^* \in X$ such that $f(x^*, x^*, \dots, x^*) = x^*$;*
- (ii) *if $x_1, x_2, \dots, x_k$ are arbitrary points in X and if for all $n \in \mathbb{N}$, $x_{n+k} = f(x_n, x_{n+1}, \dots, x_{n+k-1})$, then the sequence $\{x_n\}$ is convergent and $\lim x_n = f(\lim x_n, \lim x_n, \dots, \lim x_n)$.*

Theorem 2.1 was proved in Rus [11], see also [12], for operators f fulfilling the more general condition:

$$d(f(x_1, x_2, \dots, x_k), f(x_2, x_3, \dots, x_{k+1})) \le \varphi(\alpha_1 d(x_1, x_2), \dots, \alpha_k d(x_k, x_{k+1})),$$

for all $x_1, x_2, \dots, x_{k+1} \in X$, where $\varphi : \mathbb{R}^{+k} \to \mathbb{R}^+$ satisfies certain conditions.

Another important generalization of Prešić Theorem 2.1 result was recently obtained by Cirić and Prešić in [3].

Theorem 2.2. ([3]) *Let (X, d) be a complete metric space and let k be a positive integer. If $f : X^k \to X$ satisfies the following contractive condition*:

$$d(f(x_1, x_2, \dots, x_k), f(x_2, x_3, \dots, x_{k+1})) \le \lambda \max\{\alpha_1 d(x_1, x_2), \dots, \alpha_k d(x_k, x_{k+1})\},$$

where $\lambda \in (0, 1)$ is constant and $x_1, x_2, \dots, x_{k+1}$ are arbitrary elements in X. Then there exists a point x^* in X such that $f(x^*, x^*, \dots, x^*) = x^*$. Moreover, *if* $x_1, x_2, \ldots, x_k$ *are arbitrary points in* X *and for* $n \in \mathbb{N}$, $x_{n+k} = f(x_n, x_{n+1}, \ldots, x_{n+k-1})$ *then the sequence* $\{x_n\}$ *is convergent and* $\lim x_n = f(\lim x_n, \lim x_n, \ldots, \lim x_n)$.

Other general Prešić type fixed point results have been very recently obtained by the first author in [14]. The main result in [10] is the following fixed point theorem.

Theorem 2.3. ([10]) *Let* (X, d) *be a complete metric space,* k *be a positive integer,* $a \in \mathbb{R}$ *be a constant such that* $0 \leq ak(k+1) < 1$ *and* $f : X^k \to X$ *be an operator satisfying the following condition*:

$$d(f(x_1, x_2, \ldots, x_k), f(x_2, x_3, \ldots, x_{k+1})) \leq a \sum_{i=1}^{k+1} d(x_i, f(x_i, x_i, \ldots, x_i)), \quad (2.1)$$

for all $x_1, x_2, \ldots, x_{k+1} \in X$*. Then we have the followings*:

(i) f *has a unique fixed point* $x^* \in X$ *such that* $f(x^*, x^*, \ldots, x^*) = x^*$;

(ii) *the sequence* $\{y_n\}_{n \geq 0}$ *defined by* $y_{n+1} = f(y_n, y_n, \ldots, y_n)$, $n \geq 0$, *converges to* x^*;

(iii) *the sequence* $\{x_n\}_{n \geq 0}$ *with* $x_0, x_1, \ldots, x_{k-1} \in X$ *and* $x_n = f(x_{n-k}, x_{n-k+1}, \ldots, x_{n-1})$, $n \geq k$, *also converges to* x^*, *with a rate estimated by*

$$d(x_{n+1}, x^*) \leq M\theta^n, \; n \geq 0,$$

for a positive constant M *and a certain* $\theta \in (0, 1)$.

In [13], Samet et al. presented the concept of α-admissible mappings as the following:

Definition 2.4. ([13]) Let $f : X \to X$ and $\alpha : X \times X \to [0, \alpha)$. Then f is called α-admissible if for all $x, y \in X$ with $\alpha(x, y) \geq 1$ implies $\alpha(fx, fy) \geq 1$.

Definition 2.5. ([13]) Let $f : X \to X$ and $\alpha : X \times X \to [0, \infty)$. Then f is called a triangular α-admissible mapping if

(i) f is α-admissible;

(ii) $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$ imply $\alpha(x, y) \geq 1$.

Wardowski [16] introduce a new type of contractions which is called F-contraction and proved new fixed point theorems concerning F-contraction (see [1, 4, 6, 7, 15]).

Definition 2.6. ([16]) Let $\mathcal{F}$ be the collection of all mappings $F : \mathbb{R}^+ \to \mathbb{R}$ that satisfy the following conditions:

(i) F is strictly increasing on $\mathbb{R}^+$;

(ii) For every sequence $\{\alpha_n\}$ of positive real numbers, $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} F(\alpha_n) = -\infty$ are equivalent;

(iii) There exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Definition 2.7. ([16]) Let (X,d) be a metric space. A mapping $f : X \to X$ is called an F-contraction on X if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$ with $d(fx, fy) > 0$, we have

$$\tau + F(d(fx, fy)) \leq F(d(x,y)).$$

Note that every F-contraction is continuous (see [16]).

Theorem 2.8. ([10]) *Let (X,d) be a complete metric space and $f : X \to X$ be an F-contraction. Then there exists a unique $x \in X$ such that $x = f(x)$. Moreover, for any $x_0 \in X$, the iterative sequence $\{x_n\}$ defined by $x_{n+1} = f(x_n)$ converges to x.*

Let $f : X^k \to X$, where $k \geq 1$ is a positive integer. A point $x^* \in X$ is called a fixed point of f if $x^* = f(x^*, x^*, \ldots, x^*)$.

Consider the k-th order nonlinear difference equation:

$$x_{n+k} = f(x_n, x_{n+1}, \ldots, x_{n+k-1}), \quad n = 1, 2, \ldots. \tag{2.2}$$

with the initial values $x_1, \ldots, x_k \in X$. Equation (2.2) can be studied by means of fixed point theory in view of the fact that x^* in X is a solution of (2.2) if and only if x^* is a fixed point of mapping $T : X \to X$ given by

$$T(x^*) = f(x^*, x^*, \ldots, x^*), \quad \text{for all} \quad x^* \in X.$$

Păcurar [10] derived a convergence result for Prešić-Kannan operators as follows:

Theorem 2.9. ([10]) *Let (X,d) be a complete metric space, k be a positive integer, $a \in \mathbb{R}$ be a constant such that $0 \leq ak(k+1) < 1$ and $f : X^k \to X$ be an operator satisfying the following condition:*

$$d(f(x_1, x_2, \ldots, x_k), f(x_2, x_3, \ldots, x_{k+1})) \leq a \sum_{i=1}^{k+1} d(x_i, f(x_i, x_i, \ldots, x_i)),$$

for any $(x_1, x_2, \ldots, x_{k+1}) \in X^{k+1}$ where $a_1, a_2, \ldots, a_k$ are nonnegative constants such that $a_1 + a_2 + \cdots + a_k < 1$. Then,

(i) *f has a unique fixed point x^*, that is, there exists a unique $x^* \in X$ such that $f(x^*, x^*, \ldots, x^*) = x^*$;*

(ii) *the sequence $\{y_n\}_{n\geq 0}$,*

$$y_{n+1} = f(y_n, y_n, \ldots, y_n), \qquad n \geq 0,$$

converges to x^*;

(iii) *the sequence* $\{x_n\}_{n\geq 0}$ *with* $x_1, x_2, \ldots, x_k \in X$ *and* $x_{n+k} = f(x_1, y_2, \ldots, y_{n+k-1})$, $n \in \mathbb{N}$, *also converges to* x^*.

Definition 2.10. ([10]) Let (X, d) be a metric space and $F \in \mathcal{F}$. A mapping $f : X^k \to X$ is said to be α-admissible Prešić type F-contraction if there exists $\tau > 0$ and $\alpha : X \times X \to [0, +\infty)$ such that

$$d(f(x_1, x_2, \ldots, x_k), f(x_2, x_3, \ldots, x_{k+1})) > 0$$

implies that

$$\begin{aligned} &\tau + \min\{\alpha(x_i, x_{i+1})\} F(d(f(x_1, x_2, \ldots, x_k), f(x_2, x_3, \ldots, x_{k+1}))) \\ &\leq F(\max\{d(x_i, x_{i+1})\}), \quad 1 \leq i \leq k, \end{aligned} \tag{2.3}$$

for all $(x_1, x_2, \ldots, x_{k+1}) \in X^{k+1}$.

3. Main results

Theorem 3.1. *Let* (X, d) *be a complete metric space,* k *be a positive integer and* $f : X^k \to X$ *be an* α*-Prešić type* F*-contraction. Suppose, the following conditions are satisfied:*

(i) f *is an* α*-admissible operator* ;

(ii) *there exist* $x_1, x_2, \ldots, x^k \in X$ *such that*

$$\min\{\alpha(x_i, x_{i+1}), \alpha(x_k, f(x_1, x_2, \ldots, x_k)) : 1 \leq i \leq k-1\} \geq 1;$$

(iii) f *is diagonally* α*-continuous.*

Then f *has a fixed point in* X.

Proof. Let $x_1, x_2, \ldots, x^k \in X$ such that

$$\min\{\alpha(x_i, x_{i+1}), \alpha(x_k, f(x_1, x_2, \ldots, x_k)) : 1 \leq i \leq k\} \geq 1.$$

We define a sequence $\{x_n\}$ in X by

$$x_{n+k} = f(x_n, x_{n+1} \ldots, x_{n+k-1}), \quad \text{for all} \quad n \geq 1. \tag{3.1}$$

Obviously, if there exists $n_0 \in \{1, 2, 3, \ldots, k\}$ for which $x_{n_0+1} = x_{n_0}$, then

$$x_{n_0+k} = f(x_{n_0}, x_{n_0+1}, \ldots, x_{n_0+k-1}) = f(x_{n_0+k}, x_{n_0+k}, \ldots, x_{n_0+k}),$$

that is, x_{n_0+k} is a fixed point of f and the proof is finished. Thus, we suppose that $x_{n+k} \neq x_{n+k+1}$ for every $n \geq 1$. We will show that $\{x_n\}$ is a termwise α-sequence in X. From (ii) and (i), we get that

$$\alpha(f(x_1, x_2, \ldots, x_k), f(x_2, x_3, \ldots, x_{k+1})) \geq 1,$$

that is, $\alpha(x_{k+1}, x_{k+2}) \geq 1$. So, $\min\{\alpha(x_i, x_{i+1}) : 2 \leq i \leq k+1\} \geq 1$. Again, as f is an α-admissible operator, we have

$$\alpha(f(x_2, x_3, \ldots, x_{k+1}), f(x_3, x_4 \ldots, x_{k+2})) \geq 1,$$

that is, $\alpha(x_{k+2}, x_{k+3}) \geq 1$. Following a similar process, we obtain $\alpha(x_i, x_{i+1}) \geq 1$ for all $n \geq 1$. Thus, $\{x_n\}$ is a termwise α-sequence.

Denote $\delta_{n+k} = d(x_{n+k}, x_{n+k+1})$, $n \geq 1$, and

$$\gamma = \max\{d(x_1, x_2), d(x_2, x_3), \ldots, d(x_k, x_{k+1})\},$$

then we have $\delta_{n+k} > 0$ for all $n \geq 1$ and $\gamma > 0$. Now, for $n \leq k$, we have the following inequalities:

$$\begin{aligned} F(\delta_{k+1}) &= F(d(x_{k+1}, x_{k+2})) \\ &= F(d(f(x_1, x_2, \ldots, x_k), f(x_2, x_3, \ldots, x_{k+1}))) \\ &\leq \min\{\alpha(x_i, x_{i+1}) : 1 \leq i \leq k\} \\ &\quad \times F(d(f(x_1, x_2, \ldots, x_k), f(x_2, x_3, \ldots, x_{k+1}))) \\ &\leq F(\max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\}) - \tau \\ &= F(\gamma) - \tau, \end{aligned} \tag{3.2}$$

$$\begin{aligned} F(\delta_{k+2}) &= F(d(x_{k+2}, x_{k+3})) \\ &= F(d(f(x_2, x_3, \ldots, x_{k+1}), f(x_3, x_4, \ldots, x_{k+2}))) \\ &\leq \min\{\alpha(x_i, x_{i+1}) : 2 \leq i \leq k+1\} \\ &\quad \times F(d(f(x_2, x_3, \ldots, x_{k+1}), f(x_3, x_4, \ldots, x_{k+3}))) \\ &\leq F(\max\{d(x_i, x_{i+1}) : 2 \leq i \leq k+1\}) - \tau \\ &= F(\gamma) - 2\tau. \end{aligned} \tag{3.3}$$

Continuing in this fashion, for $n \geq 1$ we have

$$\begin{aligned} F(\delta_{k+n}) &= F(d(x_{n+k}, x_{n+k+1})) \\ &= F(d(f(x_n, x_{n+1}, \ldots, x_{n+k-1}), f(x_{n+1}, x_{n+2}, \ldots, x_{n+k}))) \\ &\leq \min\{\alpha(x_i, x_{i+1}) : n \leq i \leq n+k-1\} \\ &\quad \times F(d(f(x_n, x_{n+1}, \ldots, x_{n+k-1}), f(x_{n+1}, x_{n+2}, \ldots, x_{n+k}))) \\ &\leq F(\max\{d(x_i, x_{i+1}) : 1 \leq i \leq n+k-1\}) - \tau \\ &= F(\gamma) - n\tau. \end{aligned} \tag{3.4}$$

On taking limit as $n \to \infty$, we obtain that $\lim_{n\to\infty} F(\delta_{k+n}) = -\infty$ and hence $\lim_{n\to\infty} \delta_{k+n} = 0$ from Definition 2.7 (ii).

Now, from Definition 2.7 (iii). there exists $\sigma \in (0, 1)$ such that

$$\lim_{n\to\infty} \delta_{k+n}^{\sigma} F(\delta_{k+n}) = 0.$$

By (3.4), we have

$$\delta_{k+n}^{\sigma} F(\delta_{k+n}) - \delta_{k+n}^{\sigma} F(\gamma) \leq \delta_{k+n}^{\sigma} (F(\gamma) - n\tau) - \delta_{k+n}^{\sigma} F(\gamma) \leq -\delta_{k+n}^{\sigma} n\tau.$$

On taking limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} n\delta_{k+n}^{\sigma} = 0.$$

Then there exists $n_0 \in \mathbb{N}$ such that $n\delta_{k+n}^{\sigma} \leq 1$ for all $n \geq n_0$, that is,

$$\delta_{k+n} \leq \frac{1}{n^{1/\sigma}} \quad \text{for all} \quad n \geq n_0.$$

For any $n, m \in \mathbb{N}$ with $m \geq n \geq n_0$, we have

$$\begin{aligned} d(x_{k+n}, x_{k+m}) &\leq d(x_{n+k}, x_{n+k+1}) + d(x_{n+k+1}, x_{n+k+2}) \\ &\quad + \cdots + d(x_{m+k-1}, x_{m+k}) \\ &= \delta_{n+k} + \delta_{n+k+1} + \cdots + \delta_{m+k-1} \\ &< \sum_{i=n}^{\infty} \delta_{i+k} \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/\sigma}} \to 0. \end{aligned} \tag{3.5}$$

This means that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, there exists $x^* \in X$ such that

$$\lim_{m,n\to\infty} d(x_n, x_m) = \lim_{n\to\infty} d(x_n, x^*) = 0.$$

Finally, the continuity of f yields

$$\begin{aligned} x^* &= \lim_{n\to\infty} x_{n+k} \\ &= \lim_{n\to\infty} f(x_n, x_{n+1}, \ldots, x_{n+k-1}) \\ &= f(\lim_{n\to\infty} x_n, \lim_{n\to\infty} x_{n+1}, \ldots, \lim_{n\to\infty} x_{n+k-1}) \\ &= f(x^*, x^*, \ldots, x^*), \end{aligned} \tag{3.6}$$

that is, x^* is a fixed point of f. This completes the proof. □

Example 3.2. Let $X = [0, 1]$ and d be a usual metric of X. Let k be a positive integer and $f : X^k \to X$ be the mapping defined by

$$f(x_1, x_2, \ldots, x_k) = \frac{x_1 + x_k}{2k} \quad \text{for all} \quad x_1, x_2 \ldots, x_k \in X,$$

and

$$\alpha(x_i, x_{i+1}) = \begin{cases} 1, & \text{if} \quad x_i, x_{i+1} \in X,\ 1 \leq i \leq k, \\ 0, & \text{otherwise.} \end{cases}$$

Define $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(\beta) = \beta + \ln(\beta)$. Note that $F \in \mathcal{F}$ ([16]). Also, we know that, for $\tau = \ln(2k) > 0$ and $x_1, x_2, \ldots, x_{k+1} \in X$, with

$$d(f(x_1, x_2, \ldots, x_k), f(x_2, x_3, \ldots, x_{k+1})) > 0,$$

we have

$$\begin{aligned}
&\tau+\min\{\alpha(x_i,x_{i+1}):1\le i\le k\}F(d(f(x_1,x_2,\dots,x_k),f(x_2,x_3,\dots,x_{k+1})))\\
&=\ln(2k)+F\Big(\big|\frac{x_1+x_k}{2k}-\frac{x_2+x_{k+1}}{2k}\big|\Big)\\
&=\ln(2k)+F\Big(\frac{1}{2k}\big|(x_1-x_2)+(x_k-x_{k+1})\big|\Big)\\
&=\ln(2k)+\frac{1}{2k}\big|(x_1-x_2)+(x_k-x_{k+1})\big|\\
&\quad+\ln\Big(\frac{1}{2k}\big|(x_1-x_2)+(x_k-x_{k+1})\big|\Big)\\
&=\ln(2k)+\frac{1}{2k}\max\big\{d(x_1,x_2),d(x_k,x_{k+1})\big\}\\
&\quad+\ln\Big(\frac{1}{2k}\max\big\{d(x_1,x_2),d(x_k,x_{k+1})\big\}\Big)\\
&=\frac{1}{2k}\max\big\{d(x_1,x_2),d(x_k,x_{k+1})\big\}+\ln\big(\max\big\{d(x_1,x_2),d(x_k,x_{k+1})\big\}\big)\\
&\le\max\big\{d(x_i,x_{i+1}):1\le i\le k\big\}+\ln\big(\max\big\{d(x_i,x_{i+1}):1\le i\le k\big\}\big)\\
&\le F\big(\max\big\{d(x_i,x_{i+1}):1\le i\le k\big\}\big).
\end{aligned}$$

Thus, all the required hypotheses of Theorem 3.1 are satisfied. Moreover, for any arbitrary points $x_1,x_2,\dots,x_k\in X$, the sequence $\{x_n\}$ defined by (3.1) converges to $x^*=0$, which is the unique fixed point of f.

4. Application to matrix difference equations

In this section, we consider matrix difference equations:

$$X_{n+k}=Q+\frac{1}{k}\sum_{i=0}^{k-1}A^*\varphi(X_{n+i})A,\quad i=1,2,\dots,\tag{4.1}$$

where Q is an $N\times N$ Hermitian positive semidefinite matrix, k is a positive integer, A is an $N\times N$ nonsingular matrix, A^* is the conjugate transpose of A and $\varphi:P(N)\to P(N)$, $P(N)$ is the set of $N\times N$ Hermitian positive definite matrix.

Lemma 4.1. ([1]) *For any $A,B,C,D\in P(N)$,*

$$d(A+B,C+D)\le\max\{d(A,C),d(B,D)\}.$$

Furthermore, for all positive semidefinite A and $B,C\in P(N)$,

$$d(A+B,A+C)\le d(B,C).$$

Let φ is an an α-Prešić type F-contraction mapping with respect to the Thompson metric d. Our main result in this section is the following.

Theorem 4.2. *Equation* (4.1) *has a unique equilibrium point* $X \in P(N)$.

Proof. Define the mapping $f : P(N)^k \to P(N)$ by

$$f(V_1, V_2, \ldots, V_k) = Q + \frac{1}{k}[A^*\varphi(V_1)A + A^*\varphi(V_2)A + \cdots + A^*\varphi(V_k)A],$$

for all $V_1, V_2, \ldots, V_k \in P(N)$. Let $V_1, V_2, \ldots, V_{k+1} \in P(N)$. Then by using Lemma 4.1, we have

$$\begin{aligned}
&d(f(V_1, V_2, \ldots, V_k),\, f(V_2, V_3, \ldots, V_{k+1})) \\
&= d\Big(Q + \frac{1}{k}\sum_{i=1}^{k} A^*\varphi(V_i)A,\, Q + \frac{1}{k}\sum_{j=2}^{k+1} A^*\varphi(V_j)A\Big) \\
&\le d\Big(\frac{1}{k}\sum_{i=1}^{k} A^*\varphi(V_i)A,\, \frac{1}{k}\sum_{j=2}^{k+1} A^*\varphi(V_j)A\Big) \\
&= d\Big(\sum_{i=1}^{k}(\frac{1}{\sqrt{k}}A)^*\varphi(V_i)(\frac{1}{\sqrt{k}}A),\, \sum_{j=2}^{k+1}(\frac{1}{\sqrt{k}}A)^*\varphi(V_j)(\frac{1}{\sqrt{k}}A)\Big).
\end{aligned}$$

Denote $U = \dfrac{1}{\sqrt{k}}A$. Then, using again Lemma 4.1, we have

$$\begin{aligned}
&d(f(V_1, V_2, \ldots, V_k),\, f(V_2, V_3, \ldots, V_{k+1})) \\
&\le d\Big(\sum_{i=1}^{k} U^*\varphi(V_i)U,\, \sum_{j=2}^{k+1} U^*\varphi(V_j)U\Big) \\
&= d(U^*\varphi(V_1)U + U^*\varphi(V_2)U + \cdots + U^*\varphi(V_k)U, \\
&\qquad U^*\varphi(V_2)U + U^*\varphi(V_3)U + \cdots + U^*\varphi(V_{k+1})U) \\
&\le \max\{d(U^*\varphi(V_1)U,\, U^*\varphi(V_2)U),\, \ldots,\, d(U^*\varphi(V_k)U,\, U^*\varphi(V_{k+1})U)\} \\
&= \max\{d(U^*\varphi(V_i)U,\, U^*\varphi(V_{i+1})U) : i = 1, 2, \ldots, k\}.
\end{aligned}$$

Since A is nonsingular, the matrix U is also nonsingular. For all $i = 1, 2, \ldots, k$, we have

$$d(U^*\varphi(V_i)U,\, U^*\varphi(V_{i+1})U) = d(\varphi(V_i), \varphi(V_{i+1})).$$

But φ is an α-Prešić type F-contraction. Then, for all $i = 1, 2, \ldots, k$, we have

$$\tau + \min\{\alpha(V_i, V_{i+1})\}F(d(U^*\varphi(V_i)U,\, U^*\varphi(V_{i+1})U)) \le F(d(V_i, V_{i+1})).$$

Hence, we have

$$\begin{aligned}
&\tau + \min\{\alpha(V_i, V_{i+1})\}F(d(f(V_1, V_2, \ldots, V_k),\, f(V_2, V_3, \ldots, V_{k+1}))) \\
&\le \max\{d(V_i, V_{i+1})\},
\end{aligned}$$

for all $V_1, V_2, \ldots, V_{k+1} \in P(N)$. Now, Applying Theorem 3.1, we obtain the existence of a global attractor equilibrium point $X \in P(N)$. □

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References

[1] M. Abbas, M. Berzig, T. Nazir and E. Karapnar, *Iterative approximation of fixed points for Prešić type F-contraction operator*, U.P.B. Sci. Bull., Series A, **78** (2016), 147–160.

[2] S. Banach, *Sur les opérations dans les ensembles abstraits et leur applications aux S. équations intégrales*, Fund. Math., **3** (1922), 133–181.

[3] L.B. Ćirić and S.B. Prešić, *On Prešić type generalization of the Banach contraction mapping principle*, Acta Math. Univ. Comenian, **76** (2007), 143–147.

[4] D. Gopal, M. Abbas, D.K. Patel and C. Vetro, *Fixed points of α-type F-contractive mappings with an application to nonlinear fractional differential equation*, Acta Math. Sci., **36B**(3) (2016), 1–14.

[5] D. Jain, A. Padcharoen, P. Kumam and D. Gopal, *A new approach to study fixed point of multivalued mappings in modular metric spaces and applications*, Mathematics, **4** (2016), 51.

[6] A. Padcharoen, D. Gopal, P. Chaipunya and P. Kumam, *Fixed point and periodic point results for α-type F-contractions in modular metric spaces*, Fixed Point Theory Appl., **2016**, 2016:39 .

[7] A. Padcharoen, P. Kumam and D. Gopal, Coincidence and periodic point results in a modular metric space endowed with a graph and applications, Creat. Math. Inform., **26** (2017), 95–104.

[8] S.B. Prešić, *Sur la convergence des suites*, Comptes Rendus de l'Acade mie des Sciences, vol. **260** (1965), 3828–3830.

[9] S.B. Prešić, *Sur une classe dinéquations aux différences finies et sur la convergence de certaines suites*, Publ. Inst. Math., **5(19)** (1965), 75–78.

[10] M. Păcurar, A multi-step iterative method for approximating fixed points of Prešić-kannan operators, Acta Math. Univ. Comenianae. **79** (2010), 77–88.

[11] I.A. Rus, *An iterative method for the solution of the equation $x = f(x, ..., x)$*, Rev. Anal. Numer. Theor. Approx., **10** (1981), 95–100.

[12] I.A. Rus, *An abstract point of view in the nonlinear difference equations*, Conf. on An., Functional Equations, Appl. and Convexity, Cluj-Napoca, October 15-16, (1999), 272–276.

[13] B. Samet, C. Vetro, and P. Vetro, *Fixed point theorems for α-ψ-contractive type mappings*, Nonlinear Analysis, **75** (2012) 2154–2165.

[14] S. Shukla and N. Shahzad, *α-admissible Prešić type operators and fixed points*, Nonlinear Analysis: Modelling and Control, **21** (2016), 424–436.

[15] P. Sumalai, P. Kumam, Y.J. Cho and A. Padcharoen, *The (CLRg)-property for coincidence point theorems and Fredholm integral equations in modular metric spaces*, European J. Pure Appl. Math., **10** (2017), 238–254.

[16] D. Wardowski, *Fixed points of new type of contractive mappings in complete metric space* Fixed Point Theory Appl., **2012**, 2012: 94.

[17] D. Wardowski and N. Van Dung, *Fixed points of F -weak contractions on complete metric space*, Demonstratio Math., **47** (2014), 146–155.