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EXISTENCE THEOREM OF A FIXED POINT FOR ASYMPTOTICALLY NONEXPANSIVE NONSELF MAPPING IN CAT(0) SPACES

Kyung Soo Kim

Graduate School of Education, Mathematics Education
Kyungnam University, Changwon, Gyeongnam, 51767, Republic of Korea

e-mail: kksmj@kyungnam.ac.kr

Abstract. In this paper, we prove the existence of unique fixed point for asymptotically nonexpansive nonself mapping in $CAT(0)$ spaces.

1. Introduction

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a mapping c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x, c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a *geodesic* (or, *metric*) *segment* joining x and y. When it is unique, this geodesic is denoted by $[x, y]$. The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be *convex* if Y includes every geodesic segment joining any two of its points.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ is a geodesic metric space (X, d) consists of three points $x_1, x_2, x_3 \in X$ (the *vertices* of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for the geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\triangle}(x_1, x_2, x_3) = \triangle(\bar{x_1}, \bar{x_2}, \bar{x_3})$ in

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$\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x_i}, \bar{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists(see, [1], [18]).

A metric space X is a $CAT(0)$ space. This term is due to Gromov [9] and it is an acronym for Cartan, Aleksandrov and Toponogov. If it is geodesically connected, and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane(see, *e.g.*, [1], p.159). It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a $CAT(0)$ space. The precise definition is given below.

A geodesic metric space is said to be a $CAT(0)$ *space* if all geodesic triangles of appropriate size satisfy the following $CAT(0)$ comparison axiom.

> Let $\triangle$ be a geodesic triangle in X and let $\bar{\triangle} \subset \mathbb{R}^2$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ *inequality* if for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \bar{\triangle}$,
>
> $$d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y}).$$

If x, y_1, y_2 are points of a $CAT(0)$ space and if y_0 is the midpoint of the segment $[y_1, y_2]$, which we will denote by $\frac{y_1 \oplus y_2}{2}$, then the $CAT(0)$ inequality implies

$$d^2\left(x, \frac{y_1 \oplus y_2}{2}\right) = d^2(x, y_0) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2).$$

This inequality is the (CN) inequality of Bruhat and Tits [3]. In fact, a geodesic space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality(*cf.* [1], p.163). The above inequality has been extended by Khamsi and Kirk [11] as

$$\begin{aligned} &d^2(z, \alpha x \oplus (1-\alpha)y) \\ &\leq \alpha d^2(z, x) + (1-\alpha)d^2(z, y) - \alpha(1-\alpha)d^2(x, y), \end{aligned} \tag{CN*}$$

for any $\alpha \in [0, 1]$ and $x, y, z \in X$. The inequality (CN*) also appeared in [7].

In the recent years, $CAT(0)$ spaces have attracted many researchers as they played a very important role in different aspects of geometry and mathematics (see [1], [2], [4], [8], [14], [15]). Complete $CAT(0)$ spaces are often called *Hadamard spaces* (see [15]).

Let (X, d) be a metric space and C be a nonempty subset of X. Recall that C is called a *retract* of X if there exists a continuous mapping P from X onto C such that $Px = x$, for all $x \in C$. A mapping $P : X \to C$ is said to be *retraction* if $P^2 = P$. It follows that if a mapping P is retraction, then $Py = y$ for all y in the range of P.

Definition 1.1. Let C be a nonempty subset of a metric space (X, d). Let $P : X \to C$ be a nonexpansive retraction of X onto C.

(1) A nonself mapping $T : C \to X$ is said to be *nonexpansive*(cf. [12]) if

$$d(Tx, Ty) = d(T(PT)^0 x, T(PT)^0 y) \leq d(x, y),$$

for all $x, y \in C$.

(2) A nonself mapping $T : C \to X$ is said to be *asymptotically nonexpansive*([5]) if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$d(T(PT)^{n-1}x, T(PT)^{n-1}y) \leq k_n d(x, y), \quad \forall\, n \in \mathbb{N},$$

for all $x, y \in C$.

2. Preliminaries

Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers. Let C be a nonempty subset of a metric space (X, d). $\mathcal{F}(T) = \{x : Tx = x\}$ denotes the set of fixed points of T.

We write $(1-t)x \oplus ty$ for the unique point z in the geodesic segment joining from x to y such that

$$d(z, x) = td(x, y) \quad \text{and} \quad d(z, y) = (1-t)d(x, y).$$

We also denote by $[x, y]$ the geodesic segment joining from x to y, that is,

$$[x, y] = \{(1-t)x \oplus ty : t \in [0, 1]\}.$$

A subset C of a $CAT(0)$ space is convex if $[x, y] \subset C$ for all $x, y \in C$.

Now, we give the concept of $\triangle$-convergence and its some basic properties.

The concept of $\triangle$-convergence introduces by Lim [17] in 1976 was shown by Kirk and Panyanak [16] in $CAT(0)$ spaces to be very similar to the weak convergence in a Banach space setting.

Let X be a $CAT(0)$ space, and let $\{x_n\}$ be a bounded sequence in X. For $x \in X$, we let

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known that an asymptotic center $A(\{x_n\})$ consists of exactly one point in a complete $CAT(0)$ space(see, *e.g.*, [6], Proposition 7).

Definition 2.1. ([16]) A sequence $\{x_n\}$ in a complete $CAT(0)$ space X is said to be $\triangle$-*convergent to* $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we can write

$$x_n \xrightarrow{\triangle} x \quad \text{or} \quad \triangle - \lim_{n\to\infty} x_n = x$$

and call x the $\triangle$-limit of $\{x_n\}$.

Remark 2.2. In a $CAT(0)$ space, strong convergence in the metric implies $\triangle$-convergence(see, [10], [13]).

3. Existence theorem of a fixed point

Now, we shall prove the existence of a fixed point for asymptotically nonexpansive nonself mapping $T : C \to X$ in a complete $CAT(0)$ space.

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a complete $CAT(0)$ space X and let $T : C \to X$ be an asymptotically nonexpansive nonself mapping with a sequence $\{k_n\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$. Then T has a fixed point in C. Moreover, the set $\mathcal{F}(T)$ is closed and convex subset of X.*

Proof. For a given $x_0 \in C$, we define

$$\varphi(u) = \limsup_{n\to\infty} d(T(PT)^{n-1}x_0, u), \ \ \forall\, u \in C, \tag{3.1}$$

where P is a nonexpansive retraction of X onto C. Since T is an asymptotically nonexpansive nonself mapping, we have

$$\begin{aligned} d(T(PT)^{n+m-1}x_0, T(PT)^{m-1}u) &\le k_m d((PT)^n x_0, u) \\ &= k_m d(PT(PT)^{n-1}x_0, Pu) \\ &\le k_m d(T(PT)^{n-1}x_0, u) \end{aligned}$$

for any $n, m \in \mathbb{N}$. Taking $\limsup_{n\to\infty}$ in above inequality and using (3.1), we have that

$$\varphi(T(PT)^{m-1}u) \le k_m \varphi(u), \ \ \forall\, u \in C. \tag{3.2}$$

It is easy to see that the function $u \mapsto \varphi(u)$ is a lower semi-continuous function. Since C is closed and convex, there exists a point $w \in C$ such that

$$\varphi(w) = \inf_{u\in C} \varphi(u). \tag{3.3}$$

Letting $u = w$ in (3.2), for each $m \in \mathbb{N}$, we have

$$\varphi(T(PT)^{m-1}w) \le k_m \varphi(w). \tag{3.4}$$

By using (CN) inequality for any positive integers $n, m \in \mathbb{N}$, we obtain

$$\begin{aligned} & d^2\left(T(PT)^{n-1}x_0, \frac{T(PT)^{m-1}w \oplus T(PT)^{l-1}w}{2}\right) \\ & \leq \frac{1}{2}d^2(T(PT)^{n-1}x_0, T(PT)^{m-1}w) \\ & \quad + \frac{1}{2}d^2(T(PT)^{n-1}x_0, T(PT)^{l-1}w) \\ & \quad - \frac{1}{4}d^2(T(PT)^{m-1}w, T(PT)^{l-1}w). \end{aligned}$$

Taking $\limsup_{n\to\infty}$ on the both sides, from (3.3) and (3.4), we get

$$\begin{aligned} \varphi^2(w) \leq \varphi^2 & \left(\frac{T(PT)^{m-1}w \oplus T(PT)^{l-1}w}{2}\right) \\ & \leq \frac{1}{2}\varphi^2(T(PT)^{m-1}w) + \frac{1}{2}\varphi^2(T(PT)^{l-1}w) \\ & \quad - \frac{1}{4}d^2(T(PT)^{m-1}w, T(PT)^{l-1}w) \\ & \leq \frac{1}{2}(k_m^2 + k_l^2)\varphi^2(w) - \frac{1}{4}d^2(T(PT)^{m-1}w, T(PT)^{l-1}w). \end{aligned}$$

This implies that

$$d^2(T(PT)^{m-1}w, T(PT)^{l-1}w) \leq 2(k_m^2 + k_l^2 - 2)\varphi^2(w).$$

Taking $\limsup_{m,l\to\infty}$ on the both sides, we have

$$\limsup_{m,l\to\infty} d^2(T(PT)^{m-1}w, T(PT)^{l-1}w) \leq 0,$$

this means that

$$\limsup_{m,l\to\infty} d(T(PT)^{m-1}w, T(PT)^{l-1}w) \leq 0,$$

which implies that $\{T(PT)^{m-1}w\}$ is a Cauchy sequence in C. Since C is complete, it converges to some $v \in C$. Let

$$\triangle - \lim_{m\to\infty} T(PT)^{m-1}w = v.$$

From the continuity of TP, we have

$$\begin{aligned} v &= \triangle - \lim_{m\to\infty} T(PT)^m w \\ &= \triangle - \lim_{m\to\infty} TP(T(PT)^{m-1}w) \\ &= (TP)v = Tv. \end{aligned}$$

This means that T has a fixed point v.

Next, we have to prove that $\mathcal{F}(T)$ is closed and convex subset. Since T is continuous, $\mathcal{F}(T)$ is closed. To show that $\mathcal{F}(T)$ is convex, it is enough to show that

$$\frac{x \oplus y}{2} \in \mathcal{F}(T), \quad \forall\, x, y \in \mathcal{F}(T).$$

Let $p = \frac{x \oplus y}{2}$. Since C is convex, $p \in C$. By using (CN) inequality, we have

$$\begin{aligned} d^2(T(PT)^{n-1}p, p) &= d^2\left(T(PT)^{n-1}p, \frac{x \oplus y}{2}\right) \\ &\le \frac{1}{2} d^2(T(PT)^{n-1}p, x) + \frac{1}{2} d^2(T(PT)^{n-1}p, y) \\ &\quad - \frac{1}{4} d^2(x, y). \end{aligned} \tag{3.5}$$

Since $x \in \mathcal{F}(T)$, we obtain

$$\begin{aligned} d^2(T(PT)^{n-1}p, x) &= d^2(T(PT)^{n-1}p, T(PT)^{n-1}x) \\ &\le k_n^2 d^2(p, x) = k_n^2 d^2\left(\frac{x \oplus y}{2}, x\right) \\ &\le k_n^2 \left(\frac{1}{2} d(x, x) + \frac{1}{2} d(y, x)\right)^2 \\ &= \frac{1}{4} k_n^2 d^2(x, y). \end{aligned} \tag{3.6}$$

Similarly, since $y \in \mathcal{F}(T)$, we can get

$$d^2(T(PT)^{n-1}p, y) \le \frac{1}{4} k_n^2 d^2(x, y). \tag{3.7}$$

Substituting (3.6) and (3.7) into (3.5). we obtain

$$\begin{aligned} d^2(T(PT)^{n-1}p, p) &\le \frac{1}{8} k_n^2 d^2(x, y) + \frac{1}{8} k_n^2 d^2(x, y) - \frac{1}{4} d^2(x, y) \\ &= \frac{1}{4}(k_n^2 - 1) d^2(x, y). \end{aligned}$$

Hence, we have

$$\lim_{n\to\infty} d(T(PT)^{n-1}p, p) = 0,$$

that is

$$\triangle - \lim_{n\to\infty} T(PT)^{n-1}p = p.$$

From the continuity of TP and $p \in C$, we get

$$\begin{aligned} p &= \triangle - \lim_{n\to\infty} T(PT)^n p \\ &= \triangle - \lim_{n\to\infty} TP(T(PT)^{n-1}p) \\ &= (TP)p \\ &= Tp. \end{aligned}$$

It implies that

$$\frac{x \oplus y}{2} = p \in \mathcal{F}(T), \ \ \forall\, x, y \in \mathcal{F}(T).$$

This completes the proof. □

Corollary 3.2. *Let C be a nonempty, closed and convex subset of a complete $CAT(0)$ space X and let $T : C \to X$ be a nonexpansive mapping. Then T has a fixed point in C. Moreover, the set $\mathcal{F}(T)$ is closed and convex subset of X.*

4. Conclusion

In this paper, we show that the existence of fixed point and uniquess for asymptotically nonexpansive nonself mapping in $CAT(0)$ spaces. It is expected that this class will inspire and motivate further research in this area.

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