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NONLOCAL MIXED PROBLEMS FOR SINGULAR PARABOLIC EQUATIONS

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Abstract. Mixed problems with nonlocal boundary condition or with nonlocal initial conditions were studied by many mathematicians lately [3, 4, 5, 7]. The importance of problems with integral condition has been pointed out by Samarskii [8]. Mathematical modelling by evolution problems with nonlocal constraint is encountered in heat transmission theory, thermoelasticity, chemical engineering, underground water flow, and plasma physics. In [1] the author derived a priori estimation of the solution for mixed problems with integral condition for singular parabolic equations, and in [2] it was proved that such problem is solvable. In this paper we prove a theorem about the existence and uniqueness of strong generalized solution of nonlocal mixed problems for singular parabolic equations.

1. Introduction

In the rectangle $Q = (0, L) \times (0, T)$, we consider the following mixed problem:

$$\mathcal{L}u = \frac{\partial u}{\partial t} - \frac{1}{x^m}\frac{\partial}{\partial x}\left(x^m \frac{\partial u}{\partial x}\right) = f(x,t),\ m \geq 0, \tag{1.1}$$

$$Lu \equiv u(x,0) = \varphi(x), \tag{1.2}$$

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$$\lim_{x\to 0} x^{\frac{m}{2}} \frac{\partial u\,(x,t)}{\partial x} = 0, \tag{1.3}$$

$$\ell^{\frac{m}{2}} u(\ell,t) - a^{\frac{m}{2}} u(a,t) = 0,\ 0 < a < \ell \tag{1.4}$$

2. A priori estimations

We denote by E_a Banach space, which is a completion by the norm

$$\begin{aligned} \| u \|_{E_a}^2 = & \sup_{0\le t\le T} \frac{1}{4} \int\limits_0^\ell \Psi_a(x) x^m u^2(x,t)dx \\ & + \int\limits_0^T \int\limits_0^\ell \Psi_a(x) x^m \left(\frac{\partial u}{\partial x}\right)^2 dxdt \\ & + \frac{m}{2\,(\ell-a)} \int\limits_0^T \int\limits_0^\ell x^{m-1} u^2\,(x,t)\,dxdt, \end{aligned} \tag{2.1}$$

set of sufficiently smooth functions $u(x,t)$, which satisfy the conditions (1.3), (1.4), and the function

$$\Psi_a(x) = \begin{cases} 1, & 0 \le x \le a \\ \frac{\ell - x}{\ell - a}, & a \le x \le \ell. \end{cases}$$

We will use this space for the solution $u(x,t)$ of the problem (1.1)-(1.4). For the right-hand side $f(x,t)$ of equation (1.1) and initial function $\varphi(x)$ of condition (1.2) we introduce the space F_a, which consist of vector-functions $\mathcal{F} = (f, \varphi)$ with the following norm

$$\| \mathcal{F} \|_{F_a}^2 = \int_Q \Psi_a(x) x^m \mid f(x,t) \mid^2 dxdt + \int\limits_a^\ell \Psi_a(x) x^m \varphi^2(x) dx. \tag{2.2}$$

Problem (1.1)-(1.4) generate the operator L with the domain $D(L)$, consisted of the functions $x^{\frac{m}{2}} u \in L_2(Q) : x^{\frac{m}{2}} \frac{\partial u}{\partial t} \in L_2(Q), x^{\frac{m}{2}} \frac{\partial u}{\partial x} \in L_2(Q), \frac{1}{x^{\frac{m}{2}}} \frac{\partial}{\partial x} \left(x^m \frac{\partial u}{\partial x}\right)$ $\in L_2(Q)$,which satisfy conditions (1.3) and (1.4). We note that for any functions $u \in D(L)$ at $m \ge 1$ condition (1.3) implied from the condition $|u\,(0,t)| < \infty$. In fact, we denote

$$\frac{1}{x^m} \frac{\partial}{\partial x} \left(x^m \frac{\partial u}{\partial x}\right) = h(x,t), \tag{2.3}$$

then

$$x^{\frac{m}{2}}\frac{\partial u\left(x,t\right)}{\partial x}=\frac{1}{x^{\frac{m}{2}}}\int\limits_{0}^{x}\xi^{m}h(\xi,t)d\xi+\frac{c_{1}}{x^{\frac{m}{2}}},\tag{2.4}$$

$$u\left(x,t\right)=\int\limits_{0}^{x}\frac{1}{\eta^{m}}\int\limits_{0}^{\eta}\xi^{m}h(\xi,t)d\xi d\eta+c_{1}\int\limits_{0}^{x}\frac{d\eta}{\eta^{m}}+c_{2}.\tag{2.5}$$

If $m\geq 1$ and $\left|u\left(0,t\right)\right|<\infty$, then from (2.5) implies that $c_1=0$, and then from (2.4) implies (1.3).

Theorem 2.1. *For any function $u\in D(L)$ the following inequality holds*

$$\left\|u\right\|_{E_a}^{2}\leq c\left\|\mathcal{F}\right\|_{F_a}^{2},\tag{2.6}$$

where $c=\max\left(\frac{1}{2},T\right)$.

Proof. Integrating by parts and from conditions (1.3) and (1.4), we get

$$\begin{aligned}\int\limits_{0}^{\tau}\int\limits_{0}^{\ell}\Psi_a(x)x^m\frac{\partial u}{\partial t}udxdt &=\frac{1}{2}\int\limits_{0}^{\ell}\Psi_a(x)x^m u^2(x,\tau)dx\\
&\quad-\frac{1}{2}\int\limits_{0}^{\ell}\Psi_a(x)x^m\varphi^2(x)dx\\
&\quad-\int\limits_{0}^{\tau}\int\limits_{0}^{\ell}\Psi_a(x)\frac{\partial}{\partial x}\left(x^m\frac{\partial u}{\partial x}\right)udxdt\\
&=\int\limits_{0}^{\tau}\int\limits_{0}^{\ell}\Psi_a(x)x^m\left(\frac{\partial u}{\partial x}\right)^2dxdt\\
&\quad-\frac{1}{\ell-a}\int\limits_{0}^{\tau}\int\limits_{a}^{\ell}x^m\frac{\partial u}{\partial x}udxdt\\
&\quad-\frac{1}{\ell-a}\int\limits_{0}^{\tau}\int\limits_{a}^{\ell}x^m\frac{\partial u}{\partial x}udxdt\end{aligned}$$

$$= \frac{m}{2(\ell-a)}\int_0^\tau\int_a^\ell x^{m-1}u^2(x,t)dxdt$$

$$- \frac{1}{\ell-a}\int_0^\tau[\ell^m u^2(\ell,t)-a^m u^2(a,t)]dt$$

$$= \frac{m}{2(\ell-a)}\int_0^\tau\int_a^\ell x^{m-1}u^2\left(x,t\right)dxdt. \tag{2.7}$$

Hence, from (2.7) we get the following equation

$$\frac{1}{2}\int_0^\ell \Psi_a(x)x^m u^2\left(x,\tau\right)dx + \int_0^\tau\int_0^\ell \Psi_a(x)x^m\left(\frac{\partial u}{\partial x}\right)^2 dxdt$$

$$+ \frac{m}{2(\ell-a)}\int_0^\tau\int_a^\ell x^{m-1}u^2\left(x,t\right)dxdt$$

$$= \frac{1}{2}\int_0^\ell \Psi_a(x)x^m\varphi^2(x)dx + \int_0^\tau\int_0^\ell \Psi_a(x)x^m f(x,t)u(x,t)dxdt. \tag{2.8}$$

We estimate the second term in the right-hand side of (2.8) as follows

$$\left|\int_0^\tau\int_0^\ell \Psi_a(x)x^m fudxdt\right| \le \sqrt{\int_0^\tau\int_0^\ell \Psi_a(x)x^m f^2(x,t)dxdt}$$

$$\times\sqrt{T\sup_{0\le t\le T}\int_0^\ell \Psi_a\left(x\right)x^m u^2\left(x,t\right)dx}$$

$$\le \frac{1}{4}\sup_{0\le t\le T}\int_0^\ell \Psi_a(x)x^m u^2\left(x,t\right)dx$$

$$+ T\int_0^T\int_0^\ell \Psi_a\left(x\right)x^m f^2\left(x,t\right)dxdt. \tag{2.9}$$

From (2.8) and (2.9) implies the following inequality

$$\frac{1}{2}\int_0^\ell \Psi_a(x)\, x^m u^2(x,\tau)dx + \int_0^\tau\int_0^\ell \Psi_a(x)\, x^m \left(\frac{\partial u}{\partial x}\right)^2 dxdt$$

$$+ \frac{m}{2(\ell-a)}\int_0^\tau\int_a^\ell x^{m-1}u^2 dxdt$$

$$\leq \frac{1}{2}\int_0^\ell \Psi_a(x)\, x^m \varphi^2(x)\, dx + T\int_Q \Psi_a(x)\, x^m f^2(x,t)dxdt$$

$$+ \frac{1}{4}\sup_{0\leq t\leq T}\int_0^\ell \Psi_a(x)\, x^m u^2(x,t)\, dx. \tag{2.10}$$

The right-hand side of (2.10) does not depend on T. Then we take the sup in the left-hand side of (2.10) by T and we get (2.6). This completes the proof. □

3. Existence of generalized solution

We consider the operator L, which maps E_a into F_a with the domain $D(L)$. In a standard way (see [6]) it is proved that the operator L admit the closure which we denote by $\overline{L}$ with the domain $D\left(\overline{L}\right)$.

Definition 3.1. Solution of the equation $\bar{L}u = \mathcal{F}$ is called strong generalized solution of the problem (1.1)-(1.4). In other words, the function u, is called strong generalized solution of (1.1)-(1.4), if there exist a sequence of functions $u_n \in D(L)$, such that $\|u_n - u\|_{E_a} \to 0$ and $\|u_n - \mathcal{F}\|_{F_a} \to 0$ as $n \to \infty$.

Theorem 3.2. *For any* $\mathcal{F} = (f,\ell) \in F_a$, *there exists a unique strong generalized solution of the problem* (1.1)-(1.4).

Proof. For $u_n \in D(L)$, the following inequality holds

$$\|u_n\|_{F_a} \leq \|u_n\|_{E_a}^2, \tag{3.1}$$

which implies from Theorem 2.1. In (3.1) passing on to the limit as $n \to \infty$, we get the inequality

$$\|u\|_{E_a}^2 \leq c\,\|Lu\|_{F_a}^2, \quad u \in D\left(\overline{L}\right). \tag{3.2}$$

From (3.2), we know that the strong generalized solution of (1.1)-(1.4) is unique, the range $R\left(\overline{L}\right)$ of the operator $\overline{L}$ has a closure in F_a and $R\left(\overline{L}\right) = \overline{R\left(L\right)}$. Hence for the proof of the existence of strong generalized solution of (1.1)-(1.4) we need to prove that the range $R(L)$ of the operator L is dense in F_a. Since the range of track operator is dense in the space with norm $\left(\int\limits_0^\ell \Psi_a\left(x\right)x^m\varphi^2\left(x\right)dx\right)^{\frac{1}{2}}$, then it is sufficient to prove that from the equality

$$\int_Q \Psi_a\left(x\right)x^m\mathcal{L}ug\left(x,t\right)dxdt = 0, \tag{3.3}$$

where

$$u \in D_0\left(L\right) = \left\{u \in D\left(L\right) : u\left(x,0\right) = 0\right\}$$

and

$$\int_Q \Psi_a\left(x\right)x^m g^2\left(x,t\right)dxdt < \infty,$$

implies that $g = 0$. In (3.3) we set

$$u\left(x,t\right) = \int\limits_0^t \{\int\limits_0^x \frac{1}{\eta^m}\int\limits_0^\eta \xi^m g\left(\xi,\tau\right)d\xi + \frac{a^{\frac{m}{2}}}{\ell^{\frac{m}{2}} - a^{\frac{m}{2}}}\int\limits_0^a \frac{1}{\eta^m}\int\limits_0^\eta \xi^m g\left(\xi,\tau\right)d\xi$$
$$+ \frac{\ell^{\frac{m}{2}}}{a^{\frac{m}{2}} - \ell^{\frac{m}{2}}}\int\limits_0^\ell \frac{1}{\eta^m}\int\limits_0^\eta \xi^m g\left(\xi,\tau\right)d\xi\}d\tau.$$

It is not hard to see that $u \in D_0\left(L\right)$ and $g\left(x,t\right) = \frac{1}{x^m}\frac{\partial}{\partial x}\left(x^m\frac{\partial u}{\partial x}\right)$. From (3.3), we get

$$\int_Q \Psi_a\left(x\right)\frac{\partial u}{\partial t}\frac{\partial}{\partial x}\left(x^m\frac{\partial^2 u}{\partial x\partial t}\right)dxdt$$
$$-\int_Q \Psi_a\left(x\right)\frac{1}{x^m}\frac{\partial}{\partial x}\left(x^m\frac{\partial u}{\partial x}\right)\frac{\partial}{\partial x}\left(x^m\frac{\partial^2 u}{\partial x\partial t}\right)dxdt = 0. \tag{3.4}$$

Also as in (2.7), we have

$$\int_Q \Psi_a\left(x\right)\frac{\partial u}{\partial t}\frac{\partial}{\partial x}\left(x^m\frac{\partial^2 u}{\partial x\partial t}\right)dxdt = -\int_Q \Psi_a\left(x\right)x^m\left(\frac{\partial^2 u}{\partial x\partial t}\right)^2 dxdt$$
$$-\frac{m}{2(\ell - a)}\int\limits_0^T\int\limits_a^\ell x^{m-1}\left(\frac{\partial u}{\partial t}\right)^2 dxdt. \tag{3.5}$$

and

$$-\int_{Q}\Psi_a\left(x\right)\frac{1}{x^m}\frac{\partial}{\partial x}\left(x^m\frac{\partial u}{\partial x}\right)\frac{\partial}{\partial x}\left(x^m\frac{\partial^2 u}{\partial x\partial t}\right)dxdt$$
$$=-\frac{1}{2}\int_{0}^{\ell}\Psi_a\left(x\right)\frac{1}{x^m}\left(\frac{\partial}{\partial x}\left(x^m\frac{\partial u\left(x,T\right)}{\partial x}\right)\right)^2 dx. \tag{3.6}$$

From (3.4)-(3.6) implies that u is constant, but since $u \in D_0(L)$, that is, $u\left(x,0\right)=0$, then $u\equiv 0$. Consequently $g\equiv 0$. This completes the proof. □

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