Nonlinear Functional Analysis and Applications
Vol. 15, No. 3 (2010), pp. 487-502

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

UNIFORM APPROXIMATION IN STATISTICAL SENSE BY DOUBLE GAUSS-WEIERSTRASS SINGULAR INTEGRAL OPERATORS

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Abstract. In this paper, we study the statistical approximation properties of a sequence of double smooth Gauss-Weierstrass singular integral operators which are not positive in general. We also show that our statistical approximation results are stronger than the classical uniform approximations.

1. Introduction

In the approximation theory, it is a quite difficult problem to approximate a function by linear operators that do not need to be positive. The uniform and L_p-approximation properties of some non-positive operators may be found in the papers [1, 2, 3, 6, 7, 8, 9, 16].

A similar problem also occurs in the statistical approximation theory. In this paper, using the concept of statistical convergence from the summability theory, we study the statistical approximation properties of the double Gauss-Weierstrass singular integral operators which are not positive in general.

In recent years, the statistical convergence has been used in the Korovkin-type approximation theory which deals with the problem of approximation of a function by means of a sequence of positive linear operators. Recall that

[0]Received July 12, 2009. Revised September 28, 2009.
[0]2000 Mathematics Subject Classification: 41A36, 62L20.
[0]Keywords: A-statistical convergence, statistical approximation, Gauss-Weierstrass singular integral operators.

it is possible to approximate (in statistical sense) a function by means of a sequence of positive linear operators although the limit of the sequence fails (see, e.g., [4, 10, 11, 12, 13]).

Let $A := [a_{jn}]$, $j, n = 1, 2, ...$, be an infinite summability matrix and assume that, for a given sequence $x = (x_n)_{n\in\mathbb{N}}$, the series $\sum_{n=1}^{\infty} a_{jn}x_n$ converges for every $j \in \mathbb{N}$. Then, by the A-transform of x, we mean the sequence $Ax = ((Ax)_j)_{j\in\mathbb{N}}$ such that, for every $j \in \mathbb{N}$, $(Ax)_j := \sum_{n=1}^{\infty} a_{jn}x_n$. A summability matrix A is said to be regular (see [17]) if for every $x = (x_n)_{n\in\mathbb{N}}$ for which $\lim_{n\to\infty} x_n = L$ we get $\lim_{j\to\infty} (Ax)_j = L$. Now, fix a non-negative regular summability matrix A. Then, a given sequence $x = (x_n)_{n\in\mathbb{N}}$ is said to be A-statistically convergent to L if, for every $\varepsilon > 0$, $\lim_{j\to\infty} \sum_{n\,:\,|x_n-L|\geq\varepsilon} a_{nj} = 0$. This limit is denoted by $st_A - \lim_n x_n = L$ (see [15]). It is easy to check that if $A = C_1 = [c_{jn}]$, the Cesáro matrix of order one defined to be $c_{jn} = 1/j$ if $1 \leq n \leq j$, and $c_{jn} = 0$ otherwise, then C_1-statistical convergence coincides with the concept of statistical convergence, which was first introduced by Fast [14]. In this case, we use the notation $st - \lim$ instead of $st_{C_1} - \lim$. Every convergent sequence is A-statistically convergent, however, its converse is not always true. Not all properties of convergent sequences hold true for A-statistical convergence (or statistical convergence). For instance, although it is well-known that a subsequence of a convergent sequence is convergent, this is not always true for A-statistical convergence. Another example is that every convergent sequence must be bounded, however it does not need to be bounded of an A-statistically convergent sequence.

2. Construction of the operators

In this section we introduce a sequence of double smooth Gauss-Weierstrass singular integral operators. We first give some notation used in the paper. Let

$$\alpha_{j,r}^{[m]} := \begin{cases} (-1)^{r-j}\binom{r}{j} j^{-m} & \text{if } j = 1, 2, ..., r, \\ 1 - \sum\limits_{j=1}^{r} (-1)^{r-j}\binom{r}{j} j^{-m} & \text{if } j = 0. \end{cases} \tag{2.1}$$

and

$$\delta_{k,r}^{[m]} := \sum_{j=1}^{r} \alpha_{j,r}^{[m]} j^k, \quad k = 1, 2, ..., m \in \mathbb{N}. \tag{2.2}$$

Then it is clear that $\sum_{j=0}^{r} \alpha_{j,r}^{[m]} = 1$ and $-\sum_{j=1}^{r} (-1)^{r-j}\binom{r}{j} = (-1)^r \binom{r}{0}$ hold. We also consider the set

$$\mathbb{D} := \left\{(s,t) \in \mathbb{R}^2 : s^2 + t^2 \leq \pi^2\right\}.$$

Assume now that $(\xi_n)_{n\in\mathbb{N}}$ is a sequence of positive real numbers. Setting

$$\lambda_n := \frac{1}{\pi\left(1-e^{-\pi^2/\xi_n^2}\right)}, \tag{2.3}$$

we define the double smooth Gauss-Weierstrass singular integral operators as follows:

$$W_{r,n}^{[m]}(f;x,y) = \frac{\lambda_n}{\xi_n^2}\sum_{j=0}^{r}\alpha_{j,r}^{[m]}\left(\iint\limits_{\mathbb{D}} f\left(x+sj,y+tj\right)e^{-(s^2+t^2)/\xi_n^2}dsdt\right), \tag{2.4}$$

where $(x,y)\in\mathbb{D}$, $n,r\in\mathbb{N}$, $m\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}$, and also $f:\mathbb{D}\to\mathbb{R}$ is a Lebesgue measurable function. In this case, we observe that our operators $W_{r,n}^{[m]}$ are not positive in general. For example, if we take $\varphi(u,v)=u^2+v^2$ and also take $r=2$, $m=3$, $x=y=0$, then we get

$$\begin{aligned}
W_{2,n}^{[3]}(\varphi;0,0) &= \frac{\lambda_n}{\xi_n^2}\left(\sum_{j=1}^{2}j^2\alpha_{j,2}^{[3]}\right)\iint\limits_{\mathbb{D}}\left(s^2+t^2\right)e^{-(s^2+t^2)/\xi_n^2}dsdt \\
&= \frac{\lambda_n}{\xi_n^2}\left(\alpha_{1,2}^{[3]}+4\alpha_{2,2}^{[3]}\right)\int_{-\pi}^{\pi}\int_{0}^{\pi}\rho^3e^{-\rho^2/\xi_n^2}d\rho d\theta \\
&= \frac{2\pi\lambda_n}{\xi_n^2}\left(-2+\frac{1}{2}\right)\int_{0}^{\pi}\rho^3e^{-\rho^2/\xi_n^2}d\rho \\
&= -\frac{3\pi\lambda_n}{\xi_n^2}\left(-\frac{\pi^2\xi_n^2e^{-\pi^2/\xi_n^2}}{2}+\frac{\left(1-e^{-\pi^2/\xi_n^2}\right)\xi_n^4}{2}\right) \\
&= -\frac{3\xi_n^2}{2}+\frac{3\pi^2e^{-\pi^2/\xi_n^2}}{2\left(1-e^{-\pi^2/\xi_n^2}\right)}<0,
\end{aligned}$$

by the fact that

$$1+u\le e^u \quad \text{for all } u\ge 0.$$

We observe that the operators $W_{r,n}^{[m]}$ given by (2.4) preserve the constant functions in two variables. Indeed, for the constant function $f(x,y)=C$, by (2.1), (2.3) and (2.4), we get, for every $r, n \in \mathbb{N}$ and $m \in \mathbb{N}_0$, that

$$\begin{aligned} W_{r,n}^{[m]}(C;x,y) &= \frac{C\lambda_n}{\xi_n^2} \iint_{\mathbb{D}} e^{-(s^2+t^2)/\xi_n^2} dsdt \\ &= \frac{C\lambda_n}{\xi_n^2} \int_{-\pi}^{\pi} \int_0^{\pi} e^{-\rho^2/\xi_n^2} \rho d\rho d\theta \\ &= C. \end{aligned}$$

We also need the following lemma.

Lemma 2.1. *Let $k \in \mathbb{N}$. Then, it holds, for each $\ell = 0, 1, ..., k$ and for every $n \in \mathbb{N}$, that*

$$\iint_{\mathbb{D}} s^{k-\ell} t^{\ell} e^{-(s^2+t^2)/\xi_n^2} dsdt = \begin{cases} 0 & \text{if } k \text{ is odd} \\ 2\gamma_{n,k} B\left(\frac{k-\ell+1}{2}, \frac{\ell+1}{2}\right) & \text{if } k \text{ is even} \end{cases}$$

where $B(a,b)$ denotes the Beta function, and

$$\gamma_{n,k} := \int_0^{\pi} \rho^{k+1} e^{-\rho^2/\xi_n^2} d\rho = \frac{\xi_n^{k+2}}{2} \left\{ \Gamma\left(1 + \frac{k}{2}\right) - \Gamma\left(1 + \frac{k}{2}, \left(\frac{\pi}{\xi_n}\right)^2\right) \right\}, \quad (2.5)$$

where $\Gamma(\alpha, z) = \int_z^{\infty} t^{\alpha-1} e^{-t} dt$ is the incomplete gamma function and Γ is the gamma function.

Proof. It is clear that if k is odd, then the integrand is a odd function with respect to s and t; and hence the above integral is zero. Also, if k is even, then the integrand is a even function with respect to s and t. If we define

$$\mathbb{D}_1 := \left\{ (s,t) \in \mathbb{R}^2 : 0 \le s \le \pi \text{ and } 0 \le t \le \sqrt{\pi^2 - s^2} \right\}, \quad (2.6)$$

then we may write that

$$\begin{aligned} \iint_{\mathbb{D}} s^{k-\ell} t^{\ell} e^{-(s^2+t^2)/\xi_n^2} dsdt &= 4 \iint_{\mathbb{D}_1} s^{k-\ell} t^{\ell} e^{-(s^2+t^2)/\xi_n^2} dsdt \\ &= 4 \int_0^{\pi/2} \int_0^{\pi} (\cos\theta)^{k-\ell} (\sin\theta)^{\ell} e^{-\rho^2/\xi_n^2} \rho^{k+1} d\rho d\theta \\ &= 4\gamma_{n,k} \int_0^{\pi/2} (\cos\theta)^{k-\ell} (\sin\theta)^{\ell} d\theta \\ &= 2\gamma_{n,k} B\left(\frac{k-\ell+1}{2}, \frac{\ell+1}{2}\right) \end{aligned}$$

whence the result. $\square$

3. Estimates for the operators (2.4)

Let $f \in C_{\pi}(\mathbb{D})$, the space of all continuous functions on $\mathbb{D}$, 2π-periodic per coordinate. Then, the rth (double) modulus of smoothness of f is given by (see, e.g., [5])

$$\omega_r(f;h) := \sup_{\sqrt{u^2+v^2}\leq h;\ (u,v)\in\mathbb{D}} \left\|\Delta^r_{u,v}(f)\right\| < \infty, \quad h>0, \tag{3.1}$$

where $\|\cdot\|$ is the sup-norm and

$$\Delta^r_{u,v}\left(f(x,y)\right) = \sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j} f(x+ju, y+jv). \tag{3.2}$$

Let $m \in \mathbb{N}_0$. By $C^{(m)}_{\pi}(\mathbb{D})$ we mean the space of functions 2π-periodic per coordinate, having m times continuous partial derivatives with respect to the variables x and y. Observe that if $f \in C^{(m)}_{\pi}(\mathbb{D})$, then we see that

$$\left\|\frac{\partial^m f(\cdot,\cdot)}{\partial^{m-\ell}x\partial^{\ell}y}\right\| := \sup_{(x,y)\in\mathbb{D}}\left|\frac{\partial^m f(x,y)}{\partial^{m-\ell}x\partial^{\ell}y}\right| < \infty, \tag{3.3}$$

for every $\ell = 0,1,...,m$.

3.1. Estimates in the case of $m \in \mathbb{N}$.

Now we consider the case of $m \in \mathbb{N}$. Then, define the function

$$\begin{aligned} G^{[m]}_{x,y}(s,t) :=\ & \frac{1}{(m-1)!}\sum_{j=0}^{r}\binom{r}{j}\int_0^1 (1-w)^{m-1} \\ & \times\left\{\sum_{\ell=0}^{m}\binom{m}{m-\ell}\left|\frac{\partial^m f(x+jsw, y+jtw)}{\partial^{m-\ell}x\partial^{\ell}y}\right|\right\} dw \end{aligned} \tag{3.4}$$

for $m \in \mathbb{N}$ and $(x,y),(s,t) \in \mathbb{D}$. Notice that $G^{[m]}_{x,y}(s,t)$ is well-defined for each fixed $m \in \mathbb{N}$ when $f \in C^{(m)}_{\pi}(\mathbb{D})$ due to the condition (3.3).

Theorem 3.1. *Let $m \in \mathbb{N}$ and $f \in C^{(m)}_{\pi}(\mathbb{D})$. Then, for the operators $W^{[m]}_{r,n}$, we have*

$$\begin{aligned} &\left|W^{[m]}_{r,n}(f;x,y) - f(x,y) - I_m(x,y)\right| \\ &\qquad \leq \frac{\lambda_n}{\xi_n^2}\iint_{\mathbb{D}} G^{[m]}_{x,y}(s,t)\left(|s|^m + |t|^m\right) e^{-(s^2+t^2)/\xi_n^2}dsdt, \end{aligned} \tag{3.5}$$

where λ_n *is given by* (2.3) *and*

$$I_m(x,y) := \frac{2\lambda_n}{\xi_n^2}\sum_{i=1}^{[m/2]}\frac{\gamma_{n,2i}\delta_{2i,r}^{[m]}}{(2i)!} \times\left\{\sum_{\ell=0}^{2i}B\left(\frac{2i-\ell+1}{2},\frac{2i+1}{2}\right)\binom{2i}{2i-\ell}\frac{\partial^{2i}f(x,y)}{\partial^{2i-\ell}x\partial^{\ell}y}\right\}. \tag{3.6}$$

The sum in (3.6) *collapses when* $m=1$.

Proof. Let $(x,y)\in\mathbb{D}$ be fixed. For every $f\in C_{\pi}(\mathbb{D})$ we may write that

$$\begin{aligned}&\sum_{j=0}^{r}\alpha_{j,r}^{[m]}\left(f(x+js,y+jt)-f(x,y)\right)\\ =\;&\sum_{k=1}^{m}\frac{\delta_{k,r}^{[m]}}{k!}\sum_{\ell=0}^{k}\binom{k}{k-\ell}s^{k-\ell}t^{\ell}\frac{\partial^{k}f(x,y)}{\partial^{k-\ell}x\partial^{\ell}y}\\ &+\frac{1}{(m-1)!}\int_0^1(1-w)^{m-1}\varphi_{x,y}^{[m]}(w;s,t)dw,\end{aligned}$$

where

$$\begin{aligned}\varphi_{x,y}^{[m]}(w;s,t)\;:\;&=\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}\\ &\times\left\{\sum_{\ell=0}^{m}\binom{m}{m-\ell}s^{m-\ell}t^{\ell}\frac{\partial^{m}f(x+jsw,y+jtw)}{\partial^{m-\ell}x\partial^{\ell}y}\right\}.\end{aligned}$$

Hence, using the definition (2.4), one can get

$$\begin{aligned}W_{r,n}^{[m]}(f;x,y)-f(x,y)\;=\;&\frac{\lambda_n}{\xi_n^2}\sum_{k=1}^{m}\frac{\delta_{k,r}^{[m]}}{k!}\sum_{\ell=0}^{k}\binom{k}{k-\ell}\frac{\partial^{k}f(x,y)}{\partial^{k-\ell}x\partial^{\ell}y}\\ &\times\left(\iint_{\mathbb{D}}s^{k-\ell}t^{\ell}e^{-(s^2+t^2)/\xi_n^2}dsdt\right)\\ &+R_n^{[m]}(x,y),\end{aligned}$$

where

$$\begin{aligned}R_n^{[m]}(x,y)\;:\;&=\frac{\lambda_n}{\xi_n^2(m-1)!}\iint_{\mathbb{D}}\left(\int_0^1(1-w)^{m-1}\varphi_{x,y}^{[m]}(w;s,t)dw\right)\\ &\times e^{-(s^2+t^2)/\xi_n^2}dsdt.\end{aligned}$$

Also, using Lemma 2.1, we obtain that

$$W_{r,n}^{[m]}(f;x,y)-f(x,y)-I_m(x,y)=R_n^{[m]}(x,y), \tag{3.7}$$

where $I_m(x,y)$ is given by (3.6). Since

$$\begin{aligned}\left|\varphi_{x,y}^{[m]}(w;s,t)\right| &\le (|s|^m+|t|^m)\sum_{j=0}^{r}\binom{r}{j}\\ &\quad\times\left\{\sum_{\ell=0}^{m}\binom{m}{m-\ell}\left|\frac{\partial^m f(x+jsw,y+jtw)}{\partial^{m-\ell}x\partial^{\ell}y}\right|\right\},\end{aligned}$$

it is clear that

$$\left|R_n^{[m]}(x,y)\right|\le\frac{\lambda_n}{\xi_n^2}\iint_{\mathbb{D}}G_{x,y}^{[m]}(s,t)\left(|s|^m+|t|^m\right)e^{-(s^2+t^2)/\xi_n^2}dsdt. \tag{3.8}$$

Therefore, combining (3.7) and (3.8) the proof is completed. □

Corollary 3.2. *Let $m\in\mathbb{N}$ and $f\in C_{\pi}^{(m)}(\mathbb{D})$. Then, for the operators $W_{r,n}^{[m]}$, we have*

$$\left\|W_{r,n}^{[m]}(f)-f\right\|\le\frac{C_{r,m}\lambda_n}{\xi_n^2}\left(\gamma_{n,m}+\sum_{i=1}^{[m/2]}\gamma_{n,2i}\right) \tag{3.9}$$

for some positive constant $C_{r,m}$ depending on r and m, where $\gamma_{n,k}$ is given by (2.5)*. Also, the sums in* (3.9) *collapse when $m=1$.*

Proof. From (3.5) and (3.6), we may write that

$$\left\|W_{r,n}^{[m]}(f)-f\right\|\le\|I_m\|+\frac{\lambda_n}{\xi_n^2}\iint_{\mathbb{D}}\left\|G_{x,y}^{[m]}(s,t)\right\|\left(|s|^m+|t|^m\right)e^{-(s^2+t^2)/\xi_n^2}dsdt.$$

We first estimate $\|I_m\|$. It is easy to see that

$$\begin{aligned}\|I_m\| &\le \frac{2\lambda_n}{\xi_n^2}\sum_{i=1}^{[m/2]}\frac{\gamma_{n,2i}\delta_{2i,r}^{[m]}}{(2i)!}\\ &\quad\times\left\{\sum_{\ell=0}^{2i}B\left(\frac{2i-\ell+1}{2},\frac{2i+1}{2}\right)\binom{2i}{2i-\ell}\left\|\frac{\partial^m f(\cdot,\cdot)}{\partial^{m-\ell}x\partial^{\ell}y}\right\|\right\}\\ &\le \frac{K_{r,m}\lambda_n}{\xi_n^2}\sum_{i=1}^{[m/2]}\gamma_{n,2i},\end{aligned}$$

where

$$
\begin{aligned}
&K_{r,m} : \\
&= \max_{1\leq i\leq [m/2]} \left\{ \frac{2\delta_{2i,r}^{[m]}}{(2i)!} \left(\sum_{\ell=0}^{2i} B\left(\frac{2i-\ell+1}{2}, \frac{2i+1}{2} \right) \binom{2i}{2i-\ell} \left\| \frac{\partial^m f(\cdot,\cdot)}{\partial^{m-\ell}x\partial^\ell y} \right\| \right) \right\}.
\end{aligned}
$$

On the other hand, observe that

$$
\left\| G_{x,y}^{[m]}(s,t) \right\| \leq \frac{2^r}{m!} \sum_{\ell=0}^{m} \binom{m}{m-\ell} \left\| \frac{\partial^m f(\cdot,\cdot)}{\partial^{m-\ell}x\partial^\ell y} \right\| := L_{r,m}.
$$

Then, combining these results we observe that

$$
\begin{aligned}
\left\| W_{r,n}^{[m]}(f) - f \right\| &\leq \frac{K_{r,m}\lambda_n}{\xi_n^2} \sum_{i=1}^{[m/2]} \gamma_{n,2i} \\
&\quad + \frac{L_{r,m}\lambda_n}{\xi_n^2} \iint_{\mathbb{D}} \left(|s|^m + |t|^m \right) e^{-(s^2+t^2)/\xi_n^2} ds dt \\
&= \frac{K_{r,m}\lambda_n}{\xi_n^2} \sum_{i=1}^{[m/2]} \gamma_{n,2i} \\
&\quad + \frac{4L_{r,m}\lambda_n}{\xi_n^2} \iint_{\mathbb{D}_1} \left(s^m + t^m \right) e^{-(s^2+t^2)/\xi_n^2} ds dt \\
&= \frac{K_{r,m}\lambda_n}{\xi_n^2} \sum_{i=1}^{[m/2]} \gamma_{n,2i} \\
&\quad + \frac{4L_{r,m}\lambda_n}{\xi_n^2} \int_0^{\pi/2} \int_0^{\pi} \rho^{m+1} (\cos^m\theta + \sin^m\theta) e^{-\rho^2/\xi_n^2} d\rho d\theta \\
&= \frac{K_{r,m}\lambda_n}{\xi_n^2} \sum_{i=1}^{[m/2]} \gamma_{n,2i} + \frac{4\lambda_n L_{r,m}}{\xi_n^2} B\left(\frac{m+1}{2}, \frac{1}{2} \right) \gamma_{n,m},
\end{aligned}
$$

which yields

$$
\left\| W_{r,n}^{[m]}(f) - f \right\| \leq \frac{C_{r,m}\lambda_n}{\xi_n^2} \left(\gamma_{n,m} + \sum_{i=1}^{[m/2]} \gamma_{n,2i} \right),
$$

where

$$
C_{r,m} := \max \left\{ K_{r,m},\ 4L_{r,m} B\left(\frac{m+1}{2}, \frac{1}{2} \right) \right\}.
$$

So, the proof is completed. $\square$

3.2. **Estimates in the case of** $m = 0$**.**

Now we only consider the case of $m = 0$. Then, we first get the following result.

Theorem 3.3. *Let* $f \in C_\pi(\mathbb{D})$. *Then, we have*

$$\left| W_{r,n}^{[0]}(f; x, y) - f(x, y) \right| \leq \frac{4\lambda_n}{\xi_n^2} \iint_{\mathbb{D}_1} \omega_r \left(f; \sqrt{s^2 + t^2} \right) e^{-(s^2+t^2)/\xi_n^2} ds dt, \quad (3.10)$$

where λ_n *and* $\mathbb{D}_1$ *are given by* (2.3) *and* (2.6), *respectively.*

Proof. Let $(x, y) \in \mathbb{D}$. Taking $m = 0$ in (2.1) we observe that

$$\begin{aligned} W_{r,n}^{[0]}(f; x, y) - f(x, y) &= \frac{\lambda_n}{\xi_n^2} \iint_{\mathbb{D}} \left\{ \sum_{j=1}^{r} \alpha_{j,r}^{[0]} \left(f(x + sj, y + tj) - f(x, y) \right) \right\} \\ &\quad \times e^{-(s^2+t^2)/\xi_n^2} ds dt \\ &= \frac{\lambda_n}{\xi_n^2} \iint_{\mathbb{D}} \left\{ \left(\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} f(x + sj, y + tj) \right) \right. \\ &\quad \left. - \left(\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} f(x, y) \right) \right\} e^{-(s^2+t^2)/\xi_n^2} ds dt. \end{aligned}$$

Then, we have

$$\begin{aligned} W_{r,n}^{[0]}(f; x, y) - f(x, y) &= \frac{\lambda_n}{\xi_n^2} \iint_{\mathbb{D}} \left\{ \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f(x + sj, y + tj) \right\} \\ &\quad \times e^{-(s^2+t^2)/\xi_n^2} ds dt \end{aligned}$$

and hence

$$W_{r,n}^{[0]}(f; x, y) - f(x, y) = \frac{\lambda_n}{\xi_n^2} \iint_{\mathbb{D}} \Delta_{s,t}^r (f(x, y)) e^{-(s^2+t^2)/\xi_n^2} ds dt.$$

Therefore, we obtain that

$$\begin{aligned} \left| W_{r,n}^{[0]}(f; x, y) - f(x, y) \right| &\leq \frac{\lambda_n}{\xi_n^2} \iint_{\mathbb{D}} \left| \Delta_{s,t}^r (f(x, y)) \right| e^{-(s^2+t^2)/\xi_n^2} ds dt \\ &\leq \frac{\lambda_n}{\xi_n^2} \iint_{\mathbb{D}} \omega_r \left(f; \sqrt{s^2 + t^2} \right) e^{-(s^2+t^2)/\xi_n^2} ds dt, \end{aligned}$$

which completes the proof. $\square$

Corollary 3.4. *Let $f \in C_{\pi}(\mathbb{D})$. Then, we have*

$$\left\| W_{r,n}^{[0]}(f) - f \right\| \leq S_r \lambda_n \omega_r(f;\xi_n) \tag{3.11}$$

for some positive constant S_r depending on r.

Proof. Using (3.10) and also considering the fact that

$$\omega_r(f;\lambda u) \leq (1+\lambda)^r \omega_r(f;u), \lambda, u > 0,$$

we may write that

$$\begin{aligned}
\left\| W_{r,n}^{[0]}(f) - f \right\| &\leq \frac{4\lambda_n}{\xi_n^2} \iint\limits_{\mathbb{D}_1} \omega_r\left(f;\sqrt{s^2+t^2}\right) e^{-(s^2+t^2)/\xi_n^2} ds dt \\
&\leq \frac{4\lambda_n \omega_r(f;\xi_n)}{\xi_n^2} \iint\limits_{\mathbb{D}_1} \left(1 + \frac{\sqrt{s^2+t^2}}{\xi_n}\right)^r e^{-(s^2+t^2)/\xi_n^2} ds dt \\
&= \frac{4\lambda_n \omega_r(f;\xi_n)}{\xi_n^2} \int_0^{\pi/2}\int_0^{\pi} \left(1+\frac{\rho}{\xi_n}\right)^r \rho e^{-\rho^2/\xi_n^2} d\rho d\theta \\
&= \frac{2\pi\lambda_n \omega_r(f;\xi_n)}{\xi_n^2} \int_0^{\pi} \left(1+\frac{\rho}{\xi_n}\right)^r \rho e^{-\rho^2/\xi_n^2} d\rho.
\end{aligned}$$

Now setting $u = \frac{\rho}{\xi_n}$, we get

$$\begin{aligned}
\left\| W_{r,n}^{[0]}(f) - f \right\| &\leq 2\pi\lambda_n \omega_r(f;\xi_n) \int_0^{\pi/\xi_n} (1+u)^r u e^{-u^2} du \\
&\leq 2\pi\lambda_n \omega_r(f;\xi_n) \int_0^{\infty} \frac{(1+u)^{r+1}}{e^{u^2}} du \\
&=: S_r \lambda_n \omega_r(f;\xi_n)
\end{aligned}$$

where

$$S_r := 2\pi \int_0^{\infty} \frac{(1+u)^{r+1}}{e^{u^2}} du < \infty.$$

Therefore, the proof is completed. ∎

4. Statistical approximation by the operators (2.4)

4.1. **Statistical approximation in the case of $m \in \mathbb{N}$.**

We need the following lemma.

Lemma 4.1. *Let $A = [a_{jn}]$ be a non-negative regular summability matrix, and let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers for which*

$$st_A - \lim_n \xi_n = 0. \tag{4.1}$$

Then, for each fixed $k = 1, 2, ..., m \in \mathbb{N}$, we have

$$st_A - \lim_n \frac{\gamma_{n,k}\lambda_n}{\xi_n^2} = 0,$$

where λ_n and $\gamma_{n,k}$ are given by (2.3) *and* (2.5)*, respectively.*

Proof. Let $k = 1, 2, ..., m$ be fixed. Then, by (2.5), we get

$$\begin{aligned}
\frac{\gamma_{n,k}\lambda_n}{\xi_n^2} &= \frac{\lambda_n}{\xi_n^2}\int_0^\pi \rho^{k+1} e^{-\rho^2/\xi_n^2} d\rho \\
&= \frac{\lambda_n}{\xi_n^2}\int_0^\pi \rho^{k-2}\rho^2 \left(\rho e^{-\rho^2/\xi_n^2}\right) d\rho \\
&\leq \frac{\pi^{k-2}\lambda_n}{\xi_n^2}\int_0^\pi \rho^2 \left(\rho e^{-\rho^2/\xi_n^2}\right) d\rho \\
&\quad \text{(by change of variable and integration by parts)} \\
&= \frac{\pi^{k-2}\lambda_n}{\xi_n^2}\left\{\frac{\pi^2\xi_n^2 e^{-\pi^2/\xi_n^2}}{2} + \frac{\xi_n^4\left(1 - e^{-\pi^2/\xi_n^2}\right)}{2}\right\}
\end{aligned}$$

Now using (2.3), we obtain that

$$\frac{\gamma_{n,k}\lambda_n}{\xi_n^2} \leq \frac{\pi^{k-1} e^{-\pi^2/\xi_n^2}}{2\left(1 - e^{-\pi^2/\xi_n^2}\right)} + \frac{\pi^{k-3}\xi_n^2}{2},$$

which gives

$$0 < \frac{\gamma_{n,k}\lambda_n}{\xi_n^2} \leq m_k \left(\frac{1}{e^{\pi^2/\xi_n^2} - 1} + \frac{\xi_n^2}{\pi^2}\right), \tag{4.2}$$

where

$$m_k := \frac{\pi^{k-1}}{2}.$$

On the other hand, the hypothesis (4.1) implies that

$$st_A - \lim_n \frac{1}{e^{\pi^2/\xi_n^2} - 1} = 0 \quad \text{and} \quad st_A - \lim_n \xi_n^2 = 0. \tag{4.3}$$

Now, for a given $\varepsilon > 0$, consider the following sets:

$$\begin{aligned} D &: = \left\{ n \in \mathbb{N} : \frac{\gamma_{n,k}\lambda_n}{\xi_n^2} \geq \varepsilon \right\}, \\ D_1 &: = \left\{ n \in \mathbb{N} : \frac{1}{e^{\pi^2/\xi_n^2} - 1} \geq \frac{\varepsilon}{2m_k} \right\}, \\ D_2 &: = \left\{ n \in \mathbb{N} : \xi_n^2 \geq \frac{\varepsilon\pi^2}{2m_k} \right\}. \end{aligned}$$

Then, from (4.2), we easily see that

$$D \subseteq D_1 \cup D_2,$$

which yields that, for each $j \in \mathbb{N}$,

$$\sum_{j \in D} a_{jn} \leq \sum_{j \in D_1} a_{jn} + \sum_{j \in D_2} a_{jn}. \tag{4.4}$$

Letting $j \to \infty$ in (4.4) and also using (4.3) we get

$$\lim_j \sum_{j \in D} a_{jn} = 0,$$

which completes the proof. □

Now, we are ready to give our first statistical approximation theorem for the operators (2.4) in the case of $m \in \mathbb{N}$.

Theorem 4.2. *Let $A = [a_{jn}]$ be a non-negative regular summability matrix, and let $(\xi_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers for which (4.1) holds. Then, for each fixed $m \in \mathbb{N}$ and for all $f \in C_\pi^{(m)}(\mathbb{D})$, we have*

$$st_A - \lim_n \left\| W_{r,n}^{[m]}(f) - f \right\| = 0.$$

Proof. Let $m \in \mathbb{N}$ be fixed. Then, by (3.9), the inequality

$$\left\| W_{r,n}^{[m]}(f) - f \right\| \leq C_{r,m} \left(\frac{\gamma_{n,m}\lambda_n}{\xi_n^2} + \sum_{i=1}^{[m/2]} \frac{\gamma_{n,2i}\lambda_n}{\xi_n^2} \right) \tag{4.5}$$

holds for some positive constant where $C_{r,m}$. Now, for a given $\varepsilon > 0$, define the following sets:

$$\begin{aligned} E &: = \left\{ n \in \mathbb{N} : \left\| W_{r,n}^{[m]}(f) - f \right\| \geq \varepsilon \right\}, \\ E_i &: = \left\{ n \in \mathbb{N} : \frac{\gamma_{n,2i}\lambda_n}{\xi_n^2} \geq \frac{\varepsilon}{(1 + [m/2])\, C_{r,m}} \right\},\ i = 1, ..., \left[\frac{m}{2}\right], \\ E_{1+[\frac{m}{2}]} &: = \left\{ n \in \mathbb{N} : \frac{\gamma_{n,m}\lambda_n}{\xi_n^2} \geq \frac{\varepsilon}{(1 + [m/2])\, C_{r,m}} \right\}. \end{aligned}$$

Then, the inequality (4.5) implies that

$$E \subseteq \bigcup_{i=1}^{1+[\frac{m}{2}]} E_i,$$

and hence, for every $j \in \mathbb{N}$,

$$\sum_{n \in E} a_{jn} \leq \sum_{i=1}^{1+[\frac{m}{2}]} \sum_{n \in E_i} a_{jn}.$$

Now taking limit as $j \to \infty$ in the both sides of the above inequality and using Lemma 4.1 we obtain that

$$\lim_j \sum_{n \in E} a_{jn} = 0,$$

which is the desired result. $\square$

4.2. **Statistical approximation in the case of $m = 0$.**

We now investigate the statistical approximation properties of the operators (2.4) when $m = 0$. We need the following result.

Lemma 4.3. *Let $A = [a_{jn}]$ be a non-negative regular summability matrix, and let $(\xi_n)_{n\in\mathbb{N}}$ be a bounded sequence of positive real numbers for which* (4.1) *holds. Then, for every $f \in C_\pi(\mathbb{D})$, we have*

$$st_A - \lim_n \lambda_n \omega_r (f; \xi_n) = 0.$$

Proof. It follows from (4.1) and (2.3) that

$$st_A - \lim_n \lambda_n = \frac{1}{\pi}.$$

Also, using the right-continuity of $\omega_r(f; \cdot)$ at zero, it is not hard to see that

$$st_A - \lim_n \omega_r (f; \xi_n) = 0.$$

Combining these results, the proof is completed. $\square$

Then, we get the next statistical approximation theorem.

Theorem 4.4. *Let $A = [a_{jn}]$ be a non-negative regular summability matrix, and let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers for which* (4.1) *holds. Then, for all $f \in C_\pi(\mathbb{D})$, we have*

$$st_A - \lim_n \left\| W_{r,n}^{[0]}(f) - f \right\| = 0.$$

Proof. By (3.11), the inequality

$$\left\| W_{r,n}^{[0]}(f) - f \right\| \leq S_r \lambda_n \omega_r (f; \xi_n)$$

holds for some positive constant S_r. Then, for a given $\varepsilon > 0$, we can write that

$$\left\{ n \in \mathbb{N} : \left\| W_{r,n}^{[0]}(f) - f \right\| \geq \varepsilon \right\} \subseteq \left\{ n \in \mathbb{N} : \lambda_n \omega_r (f; \xi_n) \geq \frac{\varepsilon}{S_r} \right\},$$

which gives, for every $j \in \mathbb{N}$, that

$$\sum_{n : \left\| W_{r,n}^{[0]}(f) - f \right\| \geq \varepsilon} a_{jn} \leq \sum_{n : \lambda_n \omega_r (f; \xi_n) \geq \frac{\varepsilon}{S_r}} a_{jn}.$$

Now, taking limit as $j \to \infty$ in the both sides of the last inequality and also using Lemma 4.3, we obtain that

$$\lim_j \sum_{n : \left\| W_{r,n}^{[0]}(f) - f \right\| \geq \varepsilon} a_{jn} = 0,$$

whence the result. □

5. Concluding remarks

Taking $A = C_1$, the Cesáro matrix of order one, and also combining Theorems 4.2 and 4.4, we immediately get the following result.

Corollary 5.1. *Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers for which $st - \lim_n \xi_n = 0$ holds. Then, for each fixed $m \in \mathbb{N}_0$ and for all $f \in C_\pi^{(m)}(\mathbb{D})$, we have $st - \lim_n \left\| W_{r,n}^{[m]}(f) - f \right\| = 0$.*

Furthermore, choosing $A = I$, the identity matrix, in Theorems 4.2 and 4.4, we have the next approximation theorems with the usual convergence.

Corollary 5.2. *Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers for which $\lim_n \xi_n = 0$ holds. Then, for each fixed $m \in \mathbb{N}_0$ and for all $f \in C_\pi^{(m)}(\mathbb{D})$, the sequence $\left(W_{r,n}^{[m]}(f) \right)_{n\in\mathbb{N}}$ is uniformly convergent to f on $\mathbb{D}$.*

Now define a sequence $(\xi_n)_{n\in\mathbb{N}}$ by

$$\xi_n := \begin{cases} \sqrt{n}, & \text{if } n = k^2,\ k = 1, 2, ... \\ \frac{1}{n}, & \text{otherwise.} \end{cases} \tag{5.1}$$

Then, observe that $st - \lim_n \xi_n = 0$ although it is unbounded above. In this case, taking $A = C_1$, we obtain from Corollary 5.1 (or, Theorems 4.2 and 4.4) that

$$st - \lim_n \left\| W_{r,n}^{[m]}(f) - f \right\| = 0$$

holds for each $m \in \mathbb{N}_0$ and for all $f \in C_\pi^{(m)}(\mathbb{D})$. However, since the sequence $(\xi_n)_{n\in\mathbb{N}}$ given by (5.1) is non-convergent, the (classical) uniform approximation to a function f by the sequence $\left(W_{r,n}^{[m]}(f)\right)_{n\in\mathbb{N}}$ does not hold, i.e., Corollary 5.2 fails for the operators $W_{r,n}^{[m]}(f)$ obtained from the sequence $(\xi_n)_{n\in\mathbb{N}}$ defined by (5.1).

As a result, we can say that our statistical approximation results obtained in this paper can be still valid although the operators $W_{r,n}^{[m]}$ are not positive in general and also the sequence $(\xi_n)_{n\in\mathbb{N}}$ is non-convergent or unbounded.

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