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COMMON COUPLED FIXED POINT IN A PARTIALLY ORDERED b-METRIC SPACE

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Abstract. A common coupled fixed point theorem supported with an illustrative example, and a related problem of existence of solution of system of Fredhlom type integral equations, are presented for two mappings, which satisfy mixed weakly monotone property in a partially ordered b-metric space.

1. Introduction

Let Y be a nonempty set and $\rho_s : Y \times Y \to \mathbb{R}$ such that

(bs1) $\rho_s(x, y) \geq 0$ for all $x, y \in X$ with $\rho_s(x, y) = 0$ iff $x = y$,

(bs2) $\rho_s(x, y) = \rho_s(y, x)$ for all $x, y \in X$,

(bs3) $\rho_s(x, y) \leq s[\rho_s(x, y) + \rho_s(y, z)]$ for all $x, y \in X$ and $s \geq 1$.

Bakthin [1] introduced ρ_s as a b-metric on Y, and the pair (Y, ρ_s) denotes a b-metric space with constant s. If $(Y, \leq)$ is a partially ordered set, the triad $(Y, \rho_s, \leq)$ gives a partially ordered b-metric space. The completeness and convergence in a partially ordered b-metric space is similar to those in a b-metric space [7] . In establishing the existence and uniqueness of solution of periodic

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boundary value problems, Bhaskar and Lakshmikantham [2] introduced *mixed monotone mappings* in partially ordered metric space as given below:

Definition 1.1. Let $(Y, \leq)$ be a partially ordered set. A a mapping $S : Y \times Y \to Y$ is said to have mixed monotone property, if

(a) $x_1, x_2 \in Y, x_1 \leq x_2 \Rightarrow S(x_1, y) \leq S(x_2, y)$,
(b) $y_1, y_2 \in Y, y_1 \leq y_2 \Rightarrow S(x, y_1) \geq S(x, y_2)$ for all $x, y \in Y$.

Definition 1.2. Let $(Y, \leq)$ be a partially ordered set. A pair (S, T) of mappings $S : Y \times Y \to Y$ and $T : Y \times Y \to Y$ is said to have *mixed weakly monotone* property, if for all $x, y \in Y$,

(a) $x \leq S(x, y), S(y, x) \leq y \Rightarrow S(x, y) \leq T(S(x, y), S(y, x)), S(y, x) \geq T(S(y, x), S(x, y))$,
(b) $x \leq T(x, y), T(y, x) \leq y \Rightarrow T(x, y) \leq S(T(x, y), T(y, x)), T(y, x) \geq S(T(y, x), T(x, y))$.

Let (Y, d) be a metric space. An element $(x, y) \in Y \times Y$ is said to be a *coupled fixed point* [4, 7] of a mapping $f : Y \times Y \to Y$, if $f(x, y) = x$ and $f(y, x) = y$. While, $(x, y) \in Y \times Y$ is a *common coupled fixed point* [3, 4, 5] of mappings $f : Y \times Y \to Y$ and $g : Y \times Y \to Y$, if $x = g(x, y) = f(x, y)$ and $y = g(y, x) = f(y, x)$.

In this paper, we prove a common coupled fixed point theorem, supported by an illustrative example, and present a problem of existence of solution of system of Fredhlom type integral equations, for a pair of mappings, which satisfy mixed weakly monotone property in partially ordered b-metric space.

2. Main results

Lemma 2.1. ([6]) *Let (Y, ρ_s) be a b-metric space. Then $Y \times Y$ is a b-metric space endowed with b-metric ρ_d as follows:*

$$\rho_d((x, y), (u, v)) = \rho_s(x, y) + \rho_s(u, v). \tag{2.1}$$

Theorem 2.2. *Let $(Y, \rho_s, \leq)$ be a partially ordered complete b-metric space with constant $s \geq 1$ and mappings $S, T : Y \times Y \to Y$ have weakly mixed monotone property on Y. Suppose that there exists $k \in [0, 1/4s)$ such that*

$$\begin{aligned}\rho_s(S(x, y), T(u, v)) \leq k \max\Big\{ & \frac{1 + \rho_d((x, y), (S(x, y), S(y, x)))}{1 + \rho_d((x, y), (u, v))}, \\ & \rho_d((u, v), (T(u, v), T(v, u))), \rho_d((x, y), (u, v)), \\ & \rho_d((x, y), (S(x, y), S(y, x))) + \rho_d((u, v), (T(u, v), T(v, u))), \\ & \rho_d((u, v), (S(x, y), S(y, x))) + \rho_d((x, y), (T(u, v), T(v, u)))\Big\}\end{aligned} \tag{2.2}$$

for all $x, y, u, v \in Y$ *with* $x \leq u$, $y \geq v$ *and* ρ_d *is given by* (2.1). *Also, suppose that* $x_0, y_0 \in Y$ *such that*

$$x_0 \leq S(x_0, y_0) \text{ and } y_0 \geq S(y_0, x_0)$$

or

$$x_0 \leq T(x_0, y_0) \text{ and } y_0 \geq T(y_0, x_0).$$

If S *or* T *is continuous, then* S *and* T *have a common coupled fixed point.*

Proof. Let $(x_0, y_0) \in Y \times Y$. Define $\langle x_n \rangle_{n=1}^{\infty}, \langle y_n \rangle_{n=1}^{\infty} \subset Y$ by:

$$\begin{aligned}
&x_{2n+1} = S(x_{2n}, y_{2n}), \quad y_{2n+1} = S(x_{2n}, y_{2n}),\\
&x_{2n+2} = T(x_{2n+1}, y_{2n+1}), \quad y_{2n+2} = T(x_{2n+1}, y_{2n+1}) \text{ for } n \geq 1.
\end{aligned}$$

Since S and T have weakly mixed monotone property, we have

$$\begin{aligned}
x_1 = S(x_0, y_0) \leq T(S(x_0, y_0), S(y_0, x_0)) = T(x_1, y_1) = x_2 \Rightarrow x_1 \leq x_2,\\
x_2 = T(x_1, y_1) \leq S(T(x_1, y_1), T(y_1, x_1)) = S(x_2, y_2) = x_3 \Rightarrow x_2 \leq x_3.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
y_1 = S(y_0, x_0) \geq T(S(y_0, x_0), S(x_0, y_0)) = T(y_1, x_1) = y_2 \Rightarrow y_1 \geq y_2,\\
y_2 = T(y_1, x_1) \geq S(T(y_1, x_1), T(x_1, y_1)) = S(y_2, x_2) = x_3 \Rightarrow y_2 \geq y_3.
\end{aligned}$$

By induction, $x_0 \leq x_1 \leq x_2 \leq x_3 \leq \cdots$ and $y_0 \geq y_1 \geq y_2 \geq y_3 \geq \cdots$. That is, $\langle x_n \rangle_{n=1}^{\infty}$ is increasing, while $\langle y_n \rangle_{n=1}^{\infty}$ is decreasing.

Now using (2.2), we get

$$\begin{aligned}
\rho_s(x_{2n+1}, x_{2n+2}) &= \rho_s(S(x_{2n}, y_{2n}), T(x_{2n+1}, y_{2n+1}))\\
&\leq k \max \Bigg\{ \frac{1 + \rho_d((x_{2n}, y_{2n}), (S(x_{2n}, y_{2n}), S(y_{2n}, x_{2n})))}{1 + \rho_d((x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1}))},\\
&\qquad \rho_d((x_{2n+1}, y_{2n+1}), (T(x_{2n+1}, y_{2n+1}), T(y_{2n+1}, x_{2n+1}))),\\
&\qquad \rho_d((x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})),\\
&\qquad \rho_d((x_{2n}, y_{2n}), (S(x_{2n}, y_{2n}), S(y_{2n}, x_{2n})))\\
&\qquad + \rho_d((x_{2n+1}, y_{2n+1}), (T(x_{2n+1}, y_{2n+1}), T(y_{2n+1}, x_{2n+1}))),\\
&\qquad \rho_d((x_{2n+1}, y_{2n+1}), (S(x_{2n}, y_{2n}), S(y_{2n}, x_{2n})))\\
&\qquad + \rho_d((x_{2n}, y_{2n}), (T(x_{2n+1}, y_{2n+1}), T(y_{2n+1}, x_{2n+1}))) \Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&\le k\max\Big\{\rho_d((x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2}))+\rho_d((x_{2n},y_{2n}),(x_{2n+1},y_{2n+1})),\\
&\qquad \rho_d((x_{2n},y_{2n}),(x_{2n+1},y_{2n+1}))+\rho_d((x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2})),\\
&\qquad s[\rho_d((x_{2n},y_{2n}),(x_{2n+1},y_{2n+1}))+\rho_d((x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2}))]\Big\}\\
&\le ks[\rho_d((x_{2n},y_{2n}),(x_{2n+1},y_{2n+1}))+\rho_d((x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2}))]
\end{aligned}
$$

or

$$
\begin{aligned}
\rho_s(x_{2n+1},x_{2n+2}) \le ks[&d(x_{2n},x_{2n+1})+d(y_{2n},y_{2n+1})\\
&+d(x_{2n+1},x_{2n+2})+d(y_{2n+1},y_{2n+2})]. \qquad (2.3)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\rho_s(y_{2n+1},y_{2n+2}) &= \rho_s(S(y_{2n},x_{2n}),T(y_{2n+1},x_{2n+1}))\\
&\le k\max\Big\{\frac{1+\rho_d((y_{2n},x_{2n}),(y_{2n+1},x_{2n+1}))}{1+\rho_d((y_{2n},x_{2n}),(y_{2n+1},x_{2n+1}))},\\
&\qquad \rho_d((y_{2n+1},x_{2n+1}),(y_{2n+2},x_{2n+2})),\\
&\qquad \rho_d((y_{2n},x_{2n}),(y_{2n+1},x_{2n+1})),\\
&\qquad \rho_d((y_{2n},x_{2n}),(y_{2n+1},x_{2n+1}))\\
&\qquad\quad +\rho_d((y_{2n+1},x_{2n+1}),(y_{2n+2},x_{2n+2})),\\
&\qquad \rho_d((y_{2n+1},x_{2n+1}),(y_{2n+1},x_{2n+1}))\\
&\qquad\quad +\rho_d((y_{2n},x_{2n}),(y_{2n+2},x_{2n+2}))\Big\}\\
&\le ks[\rho_d((y_{2n},x_{2n}),(y_{2n+1},x_{2n+1}))\\
&\quad +\rho_d((y_{2n+1},x_{2n+1}),(y_{2n+2},x_{2n+2}))]
\end{aligned}
$$

or

$$
\begin{aligned}
\rho_s(y_{2n+1},y_{2n+2}) \le ks[&\rho_s(x_{2n},x_{2n+1})+\rho_s(y_{2n},y_{2n+1})\\
&+\rho_s(x_{2n+1},x_{2n+2})+\rho_s(y_{2n+1},y_{2n+2})]. \qquad (2.4)
\end{aligned}
$$

Adding (2.3) and (2.4) and then simplifying, we get

$$
\rho_s(x_{2n+1},x_{2n+2})+\rho_s(y_{2n+1},y_{2n+2}) \le c[\rho_s(x_{2n},x_{2n+1})+\rho_s(y_{2n},y_{2n+1})], \quad (2.5)
$$

where $c = 2ks/(1-2ks)$. The choice of k implies that $0 \le c < 1$. Again by (2.2),

$$\begin{aligned}
&\rho_s(x_{2n+2}, x_{2n+3})\\
&= \rho_s(T(x_{2n+1}, y_{2n+1}), S(x_{2n+2}, y_{2n+2}))\\
&\le k \max\left\{\frac{1+\rho_d((x_{2n+1}, y_{2n+1}), (T(x_{2n+1}, y_{2n+1}), T(y_{2n+1}, x_{2n+1})))}{1+\rho_d((x_{2n+1}, y_{2n+1})(x_{2n+2}, y_{2n+2}))},\right.\\
&\qquad \rho_d((x_{2n+2}, y_{2n+2}), (S(x_{2n+2}, y_{2n+2}), S(y_{2n+2}, x_{2n+2}))),\\
&\qquad \rho_d((x_{2n+1}, y_{2n+1}), (x_{2n+2}, y_{2n+2})),\\
&\qquad \rho_d((x_{2n+1}, y_{2n+1}), (T(x_{2n+1}, y_{2n+1}), T(y_{2n+1}, x_{2n+1})))\\
&\qquad\ + \rho_d((x_{2n+2}, y_{2n+2}), (S(x_{2n+2}, y_{2n+2}), S(y_{2n+2}, x_{2n+2}))),\\
&\qquad \rho_d((x_{2n+1}, y_{2n+1}), (S(x_{2n+2}, y_{2n+2}), S(y_{2n+2}, x_{2n+1})))\\
&\qquad\ \left. + \rho_d((x_{2n+2}, y_{2n+2}), (T(x_{2n+1}, y_{2n+1}), T(y_{2n+1}, x_{2n+1})))\right\}\\
&= k \max\left\{\frac{1+\rho_d((x_{2n+1}, y_{2n+1}), (x_{2n+2}, y_{2n+2}))}{1+\rho_d((x_{2n+1}, y_{2n+1}), (x_{2n+2}, y_{2n+2}))},\right.\\
&\qquad \rho_d((x_{2n+2}, y_{2n+2}), (x_{2n+3}, y_{2n+3})),\\
&\qquad \rho_d((x_{2n+1}, y_{2n+1}), (x_{2n+2}, y_{2n+2})),\\
&\qquad \rho_d((x_{2n+1}, y_{2n+1}), (x_{2n+2}, y_{2n+2}))\\
&\qquad\ + \rho_d((x_{2n+2}, y_{2n+2}), (x_{2n+3}, y_{2n+3})),\\
&\qquad \rho_d((x_{2n+1}, y_{2n+1}), (x_{2n+3}, y_{2n+3}))\\
&\qquad\ \left. + \rho_d((x_{2n+2}, y_{2n+2}), (x_{2n+2}, y_{2n+2}))\right\}\\
&\le k \max\left\{\rho_d((x_{2n+1}, y_{2n+1}), (x_{2n+2}, y_{2n+2}))\right.\\
&\qquad\ + \rho_d((x_{2n+2}, y_{2n+2}), (x_{2n+3}, y_{2n+3})),\\
&\qquad \rho_d((x_{2n+1}, y_{2n+1}), (x_{2n+2}, y_{2n+2}))\\
&\qquad\ + \rho_d((x_{2n+2}, y_{2n+2}), (x_{2n+3}, y_{2n+3})),\\
&\qquad s[\rho_d((x_{2n+1}, y_{2n+1}), (x_{2n+2}, y_{2n+2}))\\
&\qquad\ \left. + \rho_d((x_{2n+2}, y_{2n+2}), (x_{2n+3}, y_{2n+3}))]\right\}\\
&\le ks[\rho_d((x_{2n+1}, y_{2n+1}), (x_{2n+2}, y_{2n+2})) + \rho_d((x_{2n+2}, y_{2n+2}), (x_{2n+3}, y_{2n+3}))]
\end{aligned}$$

or

$$\begin{aligned}
\rho_s(x_{2n+2}, x_{2n+3}) \le ks[&\rho_s(x_{2n+1}, x_{2n+2}) + \rho_s(y_{2n+1}, y_{2n+2})\\
&+ \rho_s(x_{2n+2}, x_{2n+3}) + \rho_s(y_{2n+2}, y_{2n+3})]. \qquad (2.6)
\end{aligned}$$

Similarly,

$$\rho_s(y_{2n+2}, y_{2n+3}) \leq ks[\rho_s(x_{2n+1}, x_{2n+2}) + \rho_s(y_{2n+1}, y_{2n+2}) + \rho_s(x_{2n+2}, x_{2n+3}) + \rho_s(y_{2n+2}, y_{2n+3})]. \tag{2.7}$$

Adding (2.6) and (2.7),

$$\rho_s(x_{2n+2}, x_{2n+3}) + \rho_s(y_{2n+2}, y_{2n+3}) \leq c[\rho_s(x_{2n+1}, x_{2n+2}) + \rho_s(y_{2n+1}, y_{2n+2})]. \tag{2.8}$$

From (2.5) and (2.8),

$$\rho_s(x_{2n+2}, x_{2n+3}) + \rho_s(y_{2n+2}, y_{2n+3}) \leq c^2[\rho_s(x_{2n}, x_{2n+1}) + \rho_s(y_{2n}, y_{2n+1})]. \tag{2.9}$$

Continuning in this process, we obtain

$$\begin{aligned}\rho_s(x_{2n+1}, x_{2n+2}) + \rho_s(y_{2n+1}, y_{2n+2}) &\leq c[\rho_s(x_{2n}, x_{2n+1}) + \rho_s(y_{2n}, y_{2n+1})]\\ &\leq c^3[\rho_s(x_{2n-2}, x_{2n-1}) + \rho_s(y_{2n-2}, y_{2n-1})]\\ &\leq c^5[\rho_s(x_{2n-4}, x_{2n-3}) + \rho_s(y_{2n-4}, y_{2n-3})]\\ &\;\;\vdots\\ &\leq c^{2n+1}.[\rho_s(x_0, x_1) + \rho_s(y_0, y_1)]\end{aligned}$$

and

$$\begin{aligned}\rho_s(x_{2n+2}, x_{2n+3}) + \rho_s(y_{2n+2}, y_{2n+3}) &\leq c[\rho_s(x_{2n+1}, x_{2n+2}) + \rho_s(y_{2n+1}, y_{2n+2})]\\ &\leq c^{2n+2}[\rho_s(x_0, x_1) + \rho_s(y_0, y_1)]\end{aligned}$$

for $n \geq 1$.

Now, for all $m, n \geq 1$ with $n \leq m$, we have

$$\begin{aligned}&\rho_s(x_{2n+1}, x_{2m+1}) + \rho_s(y_{2n+1}, y_{2m+1})\\ &\leq s[\rho_s(x_{2n+1}, x_{2n+2}) + \rho_s(x_{2n+2}, y_{2m+1})]\\ &\quad + s[\rho_s(y_{2n+1}, y_{2n+2}) + \rho_s(y_{2n+2}, y_{2m+1})]\\ &\leq s[\rho_s(x_{2n+1}, x_{2n+2}) + \rho_s(y_{2n+1}, y_{2n+2})]\\ &\quad + s^2[\rho_s(x_{2n+2}, x_{2n+3}) + \rho_s(y_{2n+2}, y_{2n+3})]\\ &\quad + s^2[\rho_s(x_{2n+3}, x_{2m+1}) + \rho_s(y_{2n+3}, y_{2m+1})]\\ &\leq sc^{2n+1}[\rho_s(x_0, x_1) + \rho_s(y_0, y_1)]\\ &\quad + s^2c^{2n+2}[\rho_s(x_0, x_1) + \rho_s(y_0, y_1)]\\ &\quad + \cdots + s^{2(m-n)}.c^{2m}[\rho_s(x_0, x_1) + \rho_s(y_0, y_1)]\end{aligned}$$

$$\leq sc^{2n+1}[1 + (cs) + (cs)^2 + (cs)^3 + ... + (cs)^{2m-2n-1}]$$
$$\times [\rho_s(x_0, x_1) + \rho_s(y_0, y_1)]$$
$$< \frac{sc^{2n+1}}{1 - cs} \; [\rho_s(x_0, x_1) + \rho_s(y_0, y_1)].$$

Similarly,

$$\rho_s(x_{2n}, x_{2m+1}) + \rho_s(y_{2n}, y_{2m+1}) \leq \frac{sc^{2n}}{1 - cs} \; [\rho_s(x_0, x_1) + \rho_s(y_0, y_1)],$$
$$\rho_s(x_{2n}, x_{2m}) + \rho_s(y_{2n}, y_{2m}) \leq \frac{sc^{2n}}{1 - cs} \; [\rho_s(x_0, x_1) + \rho_s(y_0, y_1)],$$
$$\rho_s(x_{2n+1}, x_{2m}) + \rho_s(y_{2n+1}, y_{2m}) \leq \frac{sc^{2n+1}}{1 - cs} \; [\rho_s(x_0, x_1) + \rho_s(y_0, y_1)].$$

Hence for all $m, n \geq 1$ with $n \leq m$, we see that

$$\rho_s(x_n, x_m) + \rho_s(y_n, y_m) \leq \frac{sc^n}{1 - cs} \; [\rho_s(x_0, x_1) + \rho_s(y_0, y_1)].$$

Since $0 \leq c < 1$, $\rho_s(x_n, x_m) + \rho_s(y_n, y_m) \to 0$ as $n \to \infty$, which implies that $\rho_s(x_n, x_m) \to 0$ and $\rho_s(y_n, y_m) \to 0$ as $m, n \to \infty$. This means that $\langle x_n \rangle_{n=1}^{\infty}$ and $\langle y_n \rangle_{n=1}^{\infty}$ are Cauchy sequences in complete Y, so there exist $x, y \in Y$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

First, suppose that S is continuous. Then

$$x = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} S(x_{2n}, y_{2n}) = S(x, y)$$

and

$$y = \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} S(y_{2n}, x_{2n}) = S(y, x),$$

which imply that (x, y) is a coupled fixed point of S.

Using (2.2) with $u = x$ and $v = y$, we have

$$\begin{aligned}
&\rho_s(S(x, y), T(x, y)) \\
&\leq k \max\Bigg\{ \frac{1 + \rho_d((x, y), (S(x, y), S(y, x)))}{1 + \rho_d((x, y), (x, y))} \times \rho_d((x, y), (T(x, y), T(y, x))), \\
&\quad \rho_d((x, y), (x, y)), [\rho_d((x, y)(S(x, y), S(y, x))) \\
&\quad + \rho_d((x, y)(T(x, y), T(y, x)))], \\
&\quad [\rho_d((x, y)(S(x, y), S(y, x))) + \rho_d((x, y)(T(x, y), T(y, x)))] \Bigg\}
\end{aligned}$$

$$= k\max\left\{\frac{1+\rho_d((x,y),(x,y))}{1+\rho_d((x,y),(x,y))}\times\rho_d((x,y),(T(x,y),T(y,x))),\right.$$
$$\rho_d((x,y),(x,y))+\rho_d((x,y),(T(x,y),T(y,x))),$$
$$\left.[\rho_d((x,y),(x,y))+\rho_d((x,y)(T(x,y),T(y,x)))]\right\}$$

or

$$\rho_s(x,T(x,y))\leq k\ \rho_d((x,y),(T(x,y),T(y,x))). \tag{2.10}$$

Similarly, we can get

$$\rho_s(y,T(y,x))\leq k\,\rho_d((y,x),(T(y,x),T(x,y))). \tag{2.11}$$

From (2.10) and (2.11)

$$\begin{aligned}&\rho_s(x,T(x,y))+\rho_s(x,T(x,y))\\&\leq k\ [\rho_d((x,y),(T(x,y),T(y,x)))+\rho_d((x,y),(T(x,y),T(y,x)))]\\&=2k[\rho_s(x,T(x,y))+\rho_s(y,T(y,x))],\end{aligned}$$

which implies that $\rho_s(x,T(x,y))=0$ and $\rho_s(y,T(y,x))=0$, since $k<1/2$. That is, (x,y) is a coupled fixed point of T, and henceit is a common coupled fixed point of S and T. □

The following example illustrates Theorem 2.2.

Example 2.3. Let $Y=\mathbb{R}$. Define $\rho_s: Y\times Y\to[0,\infty)$ by $\rho_s(x,y)=|x-y|^2$, where $s=2$. Clearly, $(Y,\rho_s,\leq)$ is a partially orderded complete b-metric space. Set $S(x,y)=\frac{6x-3y+33}{36}$ and $T(x,y)=\frac{8x-4y+44}{48}$. Then the pair (S,T) satisfies mixed weakly monotone property. Now

$$\begin{aligned}&\rho_s(S(x,y),T(u,v)))\\&=|S(x,y)-T(u,v)|^2=\left|\tfrac{6x-3y+33}{36}-\tfrac{8u-4v+44}{48}\right|^2\\&\leq\left(\tfrac{1}{6}\,|x-u|+\tfrac{1}{8}\,|y-v|\,\right)^2\leq\left(\tfrac{1}{6}(|x-u|+|y-v|)\right)^2\\&\leq\tfrac{1}{18}\left(\,|x-u|^2+|y-v|^2\,\right)=\tfrac{1}{18}[\rho_s(x,u)+\rho_s(y,v)]\\&=\tfrac{1}{18}\ \ \rho_d((x,y),(u,v))\\&\leq k\max\left\{\tfrac{1+\rho_d((x,y)(S(x,y),S(y,x)))\rho_d((u,v),(T(u,v),T(v,u))}{1+\rho_d((x,y)(u,v))},\rho_d((x,y)(u,v)),\right.\\&\qquad[\rho_d((x,y)(S(x,y),S(y,x)))+\rho_d((u,v)(T(u,v),T(v,u)))],\\&\qquad\left.[\rho_d((u,v)(S(x,y),S(y,x)))+\rho_d((x,y)(T(u,v),T(v,u)))]\right\}.\end{aligned}$$

where $k = 1/18$. Note that $0 \leq k < 1/4s$ for $s = 2$. Thus all the conditions of Theorem 2.2 are satisfied. Therefore, S and T have a common coupled fixed point, namely $(1, 1)$.

Remark 2.4. If Y is a totally ordered set, then common coupled fixed point of S and T in Theorem 2.2 is unique. In fact, suppose that (p, q) is another common coupled fixed point of S and T. That is, $S(p, q) = p, S(q, p) = q$ and $T(p, q) = p, T(q, p) = q$. Now, using (2.2), we get

$$\begin{aligned}
&\rho_s(x,p) + \rho_s(y,q)\\
&= \rho_s(S(x,y),T(p,q)) + \rho_s(S(y,x),T(q,p))\\
&\leq k\max\Big\{\tfrac{1+\rho_d((x,y)(S(x,y),S(y,x)))\rho_d((p,q),(T(p,q),T(q,p))}{1+\rho_d((x,y)(p,q))}, \rho_d((x,y),(p,q)),\\
&\qquad [\rho_d((x,y),(S(x,y),S(y,x))) + \rho_d((p,q),(T(p,q),T(q,p))],\\
&\qquad [\rho_d((p,q),(S(x,y),S(y,x))) + \rho_d((x,y),(T(p,q),T(q,p))]\Big\}\\
&\quad + k\max\Big\{\tfrac{1+\rho_d((y,x),(S(y,x),S(x,y)))\rho_d((q,p),(T(q,p),T(p,q))}{1+\rho_d((y,x)(q,p))}, \rho_d((y,x),(q,p)),\\
&\qquad [\rho_d((y,x)(S(y,x),S(x,y))) + \rho_d((q,p),(T(q,p),T(q,p))],\\
&\qquad [\rho_d((q,p),(S(y,x),S(x,y))) + \rho_d((y,x),(T(q,p),T(p,q))]\Big\}\\
&= 2k[\rho_d((p,q),(x,y)) + \rho_d((x,y),(p,q))]\\
&= [d(x,p) + d(y,q)]
\end{aligned}$$

or

$$(1 - 4k)(\rho_s(x,p) + \rho_s(y,q)) \leq 0.$$

Since $k < 1/4$ for $s \geq 1$, it follows that $\rho_s(x,p) + \rho_s(y,q) = 0$, which in turn implies that $x = p$ and $y = q$. Thus common coupled fixed point of S and T is unique.

Taking $S = T$ and $s = 1$ in the Theorem 2.2, we get

Corollary 2.5. *Suppose that $(Y, \rho_s, \leq)$ is a partially ordered complete b-metric space with constant $s = 1$ and $T : Y \times Y \to Y$ is a mapping which has a mixed monotone property on Y and there exists $k \in [0, 1/4)$ such that*

$$\begin{aligned}
&\rho_s(T(x,y),T(u,v))\\
&\leq k\max\Big\{\tfrac{1+\rho_d((x,y),(T(x,y),T(y,x)))\rho_d((u,v),(T(u,v),T(v,u))}{1+\rho_d((x,y)(u,v))}, \rho_d((x,y),(u,v)),\\
&\qquad [\rho_d((x,y),(T(x,y),T(y,x))) + \rho_d((u,v),(T(u,v),T(v,u)))],\\
&\qquad [\rho_d((u,v),(T(x,y),T(y,x))) + \rho_d((x,y),(T(u,v),T(v,u)))]\Big\}
\end{aligned} \tag{2.12}$$

for all $x, y, u, v \in Y$ with $x \leq u$, $y \geq v$, and $\rho_d((x,y),(u,v)) = \rho_s(x,y) + \rho_s(u,v)$. Let x_0 and x_0 be any two elements in Y such that $x_0 \leq T(x_0,y_0)$ and $y_0 \geq T(y_0,x_0)$. If T is continuous, then T has a coupled fixed point.

3. An application to system of Fredolom type integral equations

Consider the following system of Fredholom type integral equations:

$$f(w) = q(w) + \int_a^b \lambda(w,t)[T_1(t,f(t)) + T_2(t,g(t))]dt, \tag{3.1}$$
$$g(w) = q(w) + \int_a^b \lambda(w,t)[T_1(t,g(t)) + T_2(t,f(t))]dt.$$

Let $Y = C([a,b],\mathbb{R})$ be the class of all real valued continuous functions on $[a,b]$. Define $\rho_s(f,g) = \max\{|f(w) - g(w)| / w \in [a,b]\}$ and the partial ordered relation on Y as

$$f \leq g \Leftrightarrow f(w) \leq g(w) \text{ for all } f, g \in Y \text{ and } w \in [a,b]. \tag{3.2}$$

Then $(Y, \rho_s, \leq)$ is a partially orderded complete metric space. We make the the following assumptions:

(a) The mappings $T_1 : [a,b] \times \mathbb{R} \to \mathbb{R}$, $T_2 : [a,b] \times \mathbb{R} \to \mathbb{R}$,$q : [a,b] \to \mathbb{R}$ and $\lambda : [a,b] \times \mathbb{R} \to [0,\infty)$ are continuous,

(b) There exists $c > 0$ and $k \in [0, 1/4)$ such that

$$0 \leq T_1(w,y) - T_1(w,x) \leq ck(y-x),$$
$$0 \leq T_2(w,x) - T_2(w,y) \leq ck(y-x)$$

for all $x, y \in \mathbb{R}$ with $y \geq x$ and $w \in [a,b]$,

(c) $c \max\{\int_a^b \lambda(w,t)dt : w \in [a,b]\} \leq 1$,

(d) There exists u_0 and v_0 in Y such that

$$u_0(w) \geq q(w) + \int_a^b \lambda(w,t)[T_1(t,u_0(t)) + T_2(t,v_0(t))]dt,$$
$$v_0(w) \leq q(w) + \int_a^b \lambda(w,t)[T_1(t,v_0(t)) + T_2(t,u_0(t))]dt.$$

Then the system (3.1) has a solution in $Y \times Y$.

To achieve this, define $T : Y \times Y \to Y$ as

$$T(f,g)(w) = q(w) + \int_a^b \lambda(w,t)[T_1(t,f(t)) + T_2(t,g(t))]dt$$

for all $f, g \in Y$ and $w \in [a,b]$. Then, using condition (b), it can be shown that T has mixed monotone property.

Now for $x, y, u, v \in Y$ with $x \geq u$ and $y \leq v$,

$$
\begin{aligned}
&\rho_s(T(x,y),T(u,v))\\
&=\max\{|T(x,y)(w)-T(u,v)(w)|\,/w\in[a,b]\}\\
&=\max\Big\{|\int_a^b \lambda(w,t)[T_1(t,x(t))+T_2(t,y(t))]dt\\
&\qquad -\int_a^b \lambda(w,t)[T_1(t,u(t))+T_2(t,v(t))]dt|/w\in[a,b]\Big\}\\
&\le ck\max\Big\{\int_a^b |x(t)-u(t)|\,|\lambda(w,t)|\,dt\\
&\qquad +\int_a^b |y(t)-v(t)|\,|\lambda(w,t)|\,dt/w\in[a,b]\Big\}\\
&\le k\max\Big\{|x(w)-u(w)|\,/w\in[a,b]\}+\max\{|y(w)-v(w)|\,/w\in[a,b]\Big\}\\
&\quad -c\max\Big\{\int_a^b |\lambda(w,t)|\,dt/w\in[a,b]\Big\}\\
&\le k\max\{|x(w)-u(w)|\,/w\in[a,b]\}+\max\{|y(w)-v(w)|\,/w\in[a,b]\}\\
&=k[\rho_s((x,u)+(y,v))]\\
&=k[\rho_d((x,y),(u,v))]
\end{aligned}
$$

or

$$
\begin{aligned}
&\rho_s(T(x,y),T(u,v))\\
&\le k\max\Big\{\frac{1+\rho_d((x,y),(T(x,y),T(y,x)))\rho_d((u,v),(T(u,v),T(v,u))}{1+\rho_d((x,y),(u,v))},\\
&\qquad \rho_d((x,y),(u,v)),\\
&\qquad [\rho_d((u,v),(T(x,y),T(y,x)))+\rho_d((x,y),(T(u,v),T(v,u))],\\
&\qquad [\rho_d((x,y),(T(x,y),T(y,x)))+\rho_d((u,v),(T(u,v),T(v,u))]\Big\}.
\end{aligned}
$$

Hence all the conditions of Corollary 2.5 are satisfied. Therefore, T has a coupled fixed point in $Y\times Y$. In other words, the system (3.1) of Fredhelom type integral equations has a solution in $Y\times Y$.

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