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# FURTHER INVESTIAGATION ON COUPLED BEST PROXIMITY POINT RESULTS OF SOME PROXIMAL CONTRACTIVE MULTIVALUED MAPPINGS

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**Abstract.** In this paper, we establish new coupled best proximity point theorems for $(F, \varphi, \alpha, \psi)$-proximal contractive multimaps via $C$-class funtions. Our results extend and generalized the results previously obtained in [21] as well as some known results in the literature. We provide examples to analyze and support our main results.

## 1. Introduction

Suppose $U, V$ are nonempty subsets of a metric space $(X, d)$. Let $T : U \to V$ be a given map. A point $x^* \in U$ is called to be a fixed point of $T$ if $Tx^* = x^*$. Clearly, $T(U) \cap U \neq \emptyset$ is a necessary (but not sufficient) condition for the existence of a fixed point of $T$. If $T(U) \cap U = \emptyset$, then $d(x, Tx) > 0$ for all $x \in U$, that is the set $\{x : Tx = x\} = \emptyset$. In a such situation, one attempts to find an element $x$ which is closest to $Tx$. Best proximity point theory have been developed in this direction. For more details on this approach, we refer the readers to [14, 15, 17, 22, 23, 24] and references therein.

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One of the most remarkable and powerful tool in nonlinear analysis, due to Banach [11], is known as the Banach contraction principle. This principle has been generalized by a large number of mathematicians, in many different ways (see e.g. [2, 12, 13, 19, 27]). Recently, Samet *et al.* [25] introduced the class of $\alpha$-$\psi$-contractive type mappings and established some fixed point results for such mappings within the framework of complete metric spaces.

More recently, Jleli and Samet [18] introduced the notion of $\alpha$-$\psi$-proximal contractive type mappings and established certain best proximity point theorems. A number of researchers have obtained best proximity point theorems in many different settings; see e.g. [3, 5, 8, 9, 10, 16, 17, 20].

Abkar and Gbeleh [5] and Al-Thagafi and Shahzad [8] investigated best proximity points for multivalued mappings. The notion of coupled best proximity points was introduced by Sintunavarat and Kumam [26] and proved coupled best proximity point theorems for cyclic contractions in metric spaces.

Recently, Nantadilok [21] established the coupled best proximity point theorems for $\alpha$-$\psi$-proximal contractive multimaps. Later, Ansari and Shukla [4] introduced the notions of ordered $F$-$(F,\varphi)$-contraction and subcontraction in the setting of partial metric spaces. Some fixed point theorems for ordered $F$-$(F,\varphi)$-contraction were obtained and proved.

In this paper, combining the ideas of Ansari *et al.* [4] and Nantadilok [21], we establish coupled best proximity point theorems for $(F,\varphi,\alpha,\psi)$-proximal contractive multivalued mappings.

For the sake of completeness, let $(X,d)$ be a metric space. For $U,V\subset X$, we use the following notations subsequently:

- $\operatorname{dist}(U,V)=\inf\{d(a,b): a\in U, b\in V\}$,
- $D(x,V)=\inf\{d(x,b): b\in V\}$,
- $U_0=\{a\in U: d(a,b)=\operatorname{dist}(U,V) \text{ for some } b\in V\}$,
- $V_0=\{b\in V: d(a,b)=\operatorname{dist}(U,V) \text{ for some } a\in U\}$,
- $2^X\backslash\emptyset$ is the set of all nonempty subsets of $X$,
- $\mathrm{CL}(X)$ is the set of all nonempty closed subsets of $X$,
- $\mathrm{K}(X)$ is the set of all nonempty compact subsets of $X$.

For every $U,V\in\mathrm{CL}(X)$, the map $H$ which is called the generalized Hausdorff metric induced by $d$, is defined by

$$H(U,V)=\begin{cases}\max\left\{\sup_{x\in U} d(x,V), \sup_{y\in V} d(y,U)\right\} & \text{if the maximum exists;}\\ \infty & \text{otherwise.}\end{cases} \tag{1.1}$$

A point $x^* \in X$ is said to be the best proximity point of a mapping $T : U \to V$ if $d(x^*, Tx^*) = \mathrm{dist}(U, V)$. When $U = V$, the best proximity point is essentially the fixed point of the mapping $T$.

## 2. Preliminaries

We collect some definitions and results which will be necessary and useful in the sequel.

**Definition 2.1.** ([28]) Let $(U, V)$ be a pair of nonempty subsets of a metric space $(X, d)$ with $U_0 \neq \emptyset$. Then the pair $(U, V)$ is said to have the weak $P$-property if for any $x_1, x_2 \in U$ and $y_1, y_2 \in V$,

$$\left.\begin{array}{l} d(x_1, y_1) = \mathrm{dist}(U, V) \\ d(x_2, y_2) = \mathrm{dist}(U, V) \end{array}\right\} \quad \Rightarrow \quad d(x_1, x_2) \leq d(y_1, y_2). \tag{2.1}$$

Let $\Psi$ denote the set of all functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following properties:

(1) $\psi$ is monotone nondecreasing;

(2) $\sum\limits_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$.

**Definition 2.2.** ([6]) An element $x^* \in U$ is said to be the best proximity point of a multivalued nonself mapping $T$, if $D(x^*, Tx^*) = \mathrm{dist}(U, V)$.

**Definition 2.3.** ([10]) Let $U$ and $V$ be two nonempty subsets of a metric space $(X, d)$. A mapping $T : U \to 2^V \backslash \emptyset$ is called $\alpha$-proximal admissible if there exists a mapping $\alpha : U \times U \to [0, \infty)$ such that

$$\left.\begin{array}{l} \alpha(x_1, x_2) \geq 1 \\ d(u_1, y_1) = \mathrm{dist}(U, V) \\ d(u_2, y_2) = \mathrm{dist}(U, V) \end{array}\right\} \quad \Rightarrow \quad \alpha(u_1, u_2) \geq 1, \tag{2.2}$$

where $x_1, x_2, u_1, u_2 \in U, y_1 \in Tx_1$ and $y_2 \in Tx_2$.

**Definition 2.4.** ([10]) Let $U$ and $V$ be two nonempty subsets of a metric space $(X, d)$. A mapping $T : U \to \mathrm{CL}(V)$ is said to be an $\alpha$-$\psi$-proximal contraction, if there exist two functions $\psi \in \Psi$ and $\alpha : U \times U \to [0, \infty)$ such that

$$\alpha(x, y) H(Tx, Ty) \ \leq \ \psi\big(d(x, y)\big), \quad \forall x, y \in U \tag{2.3}$$

**Lemma 2.5.** ([7]) *Let $(X, d)$ be a metric space and $V \in \mathrm{CL}(X)$. Then for each $x \in X$ with $d(x, V) > 0$ and $q > 1$, there exists an element $b \in V$ such that*

$$d(x, b) \ < \ q d(x, V). \tag{2.4}$$

**(C)** : If $\{x_n\}$ is a sequence in $U$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n$ and $x_n \to x \in U$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all $k$.

**Definition 2.6.** ([1, 4]) We say that the function $\varphi\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a function of subclass of type I, if $x \geq 1$, then $\varphi(1, y) \leq \varphi(x, y)$ for all $y \in \mathbb{R}^+$.

**Example 2.7.** ([1, 4]) Define $\varphi\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:

(a) $\varphi(a, b) = a^n b$, $n \in \mathbb{N}$;

(b) $\varphi(a, b) = \left[\frac{1}{n+1}\left(\sum_{i=0}^{n} a^i\right) + l\right]^b, l > 1$, $n \in \mathbb{N}$

for all $a, b \in \mathbb{R}^+$. Then each $\varphi$ is a function of subclass of type I.

**Definition 2.8.** ([1, 4]) Let $\varphi, F\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$. Then we say that the pair $(F, \varphi)$ is an upper class of type I, if $\varphi$ is a function of subclass of type I and

(i) $0 \leq s \leq 1 \Longrightarrow F(s, t) \leq F(1, t)$;

(ii) $\varphi(1, y) \leq F(s, t) \Longrightarrow y \leq st$ for all $s, t, y \in \mathbb{R}^+$.

**Example 2.9.** ([1, 4]) Define $\varphi, F\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ by:

(a) $\varphi(a, b) = (b + l)^a, l > 1$ and $F(s, t) = st + l$;

(b) $\varphi(a, b) = (a + l)^b, l > 1$ and $F(s, t) = (1 + l)^{st}$;

(c) $\varphi(a, b) = a^m b$, $m \in \mathbb{N}$ and $F(s, t) = st$;

(d) $\varphi(a, b) = b$ and $F(s, t) = t$;

(e) $\varphi(a, b) = \frac{1}{n+1}\left(\sum_{i=0}^{n} a^i\right) b$, $n \in \mathbb{N}$ and $F(s, t) = st$;

(f) $\varphi(a, b) = \left[\frac{1}{n+1}\left(\sum_{i=0}^{n} a^i\right) + l\right]^b, l > 1$, $n \in \mathbb{N}$ and $F(s, t) = (1 + l)^{st}$

for all $a, b, s, t \in \mathbb{R}^+$. Then the each pair $(F, \varphi)$ is an upper class of type I.

We note that for the notion of $(F, \varphi)$ where it is an upper class of type II, we refer the readers to [1, 4].

**Definition 2.10.** ([21]) Let $U$ and $V$ be two nonempty subsets of a metric space $(X, d)$. A mapping $T : U \times U \to 2^V \backslash \emptyset$ is called $\alpha$-proximal admissible if there exists a mapping $\alpha : U \times U \to [0, \infty)$ such that

$$\left.\begin{array}{l} \alpha(x_1, x_2) \geq 1 \\ d(w_1, u_1) = \mathrm{dist}(U, V) \\ d(w_2, u_2) = \mathrm{dist}(U, V) \end{array}\right\} \quad \Rightarrow \quad \alpha(w_1, w_2) \geq 1, \tag{2.5}$$

where $x_1, x_2, w_1, w_2, y_1, y_2 \in U, u_1 \in T(x_1, y_1)$ and $u_2 \in T(x_2, y_2)$, and

$$\left.\begin{array}{l} \alpha(y_1, y_2) \geq 1 \\ d(w_1', v_1) = \mathrm{dist}(U, V) \\ d(w_2', v_2) = \mathrm{dist}(U, V) \end{array}\right\} \quad \Rightarrow \quad \alpha(w_1', w_2') \geq 1, \tag{2.6}$$

where $y_1, y_2, w'_1, w'_2, x_1, x_2 \in U, v_1 \in T(y_1, x_1)$ and $v_2 \in T(y_2, x_2)$.

**Definition 2.11.** ([21]) Let $U$ and $V$ be two nonempty subsets of a metric space $(X, d)$. A mapping $T : U \times U \to \mathrm{CL}(V)$ is said to be an $\alpha$-$\psi$-proximal contraction, if there exist two functions $\psi \in \Psi$ and $\alpha : U \times U \to [0, \infty)$ such that

$$\alpha(x, y)H(T(x, x'), T(y, y')) \leq \psi\big(d(x, y)\big), \quad \forall\, x, x', y, y' \in U. \tag{2.7}$$

**Definition 2.12.** ([21]) An element $(r^*, s^*) \in U \times U$ is said to be the coupled best proximity point of a multivalued nonself mapping $T$, if

$$D(r^*, T(r^*, s^*)) = \mathrm{dist}(U, V)$$

and

$$D(s^*, T(s^*, r^*)) = \mathrm{dist}(U, V).$$

The results concerning Definitions 2.10, 2.11, 2.12, one can refer [21]. Inspired and motivated by the recent results of Ansari and Shukla [1, 4], Ali *et al.* [10], we establish the coupled best proximity points for $(F, \varphi, \alpha, \psi)$-proximal contractive multimaps. Our results extend the recent results of Nantadilok [21] and many others in the literature. We also give some examples to support our main results.

## 3. Main results

We begin this section by introducing the following definition.

**Definition 3.1.** Let $U$ and $V$ be two nonempty subsets of a metric space $(X, d)$. A mapping $T : U \times U \to \mathrm{CL}(V)$ is said to be an $(F, \varphi, \alpha, \psi)$-proximal contraction, if there exist two functions $\psi \in \Psi$ and $\alpha : U \times U \to [0, \infty)$ such that

$$\varphi\Big(\alpha(x, y), H(T(x, x'), T(y, y'))\Big) \leq F\Big(1, \psi\big(d(x, y)\big)\Big), \ \forall\, x, x', y, y' \in U, \tag{3.1}$$

where the pair $(F, \varphi)$ is an upper class of type $I$.

Now we are in a position to introduce the main results.

**Theorem 3.2.** *Let $U$ and $V$ be two nonempty closed subsets of a complete metric space $(X, d)$ such that $U_0$ is nonempty. Let $\alpha : U \times U \to [0, \infty)$ and let $\psi \in \Psi$ be a strictly increasing map. Suppose that $T : U \times U \to \mathrm{CL}(V)$ is a mapping satisfying the following conditions:*

(1) *$T(x, y) \subseteq V_0$ for each $x, y \in U_0$ and $(U, V)$ satisfies the weak $P$-property;*
(2) *$T$ is an $\alpha$-proximal admissible map;*

(3) *there exist elements* $(x_0, y_0), (x_1, y_1)$ *in* $U_0 \times U_0$ *and* $u_1 \in T(x_0, y_0), v_1 \in T(y_0, x_0)$ *such that*

$$\begin{aligned} d(x_1, u_1) &= \operatorname{dist}(U, V), \quad \alpha(x_0, x_1) \geq 1 \quad \textit{and} \\ d(y_1, v_1) &= \operatorname{dist}(U, V), \quad \alpha(y_0, y_1) \geq 1; \end{aligned} \tag{3.2}$$

(4) $T$ *is a continuous* $(F, \varphi, \alpha, \psi)$*-proximal contraction.*

*Then there exists an element* $(r^*, s^*) \in U_0 \times U_0$ *such that*

$$\begin{aligned} D\big(r^*, T(r^*, s^*)\big) &= \operatorname{dist}(U, V) \quad \textit{and} \\ D\big(s^*, T(s^*, r^*)\big) &= \operatorname{dist}(U, V). \end{aligned}$$

*Proof.* From condition (3), there exist elements $(x_0, y_0), (x_1, y_1)$ in $U_0 \times U_0$ and $u_1 \in T(x_0, y_0), v_1 \in T(y_0, x_0)$ such that

$$\begin{aligned} d(x_1, u_1) &= \operatorname{dist}(U, V), \quad \alpha(x_0, x_1) \geq 1 \quad \text{and} \\ d(y_1, v_1) &= \operatorname{dist}(U, V), \quad \alpha(y_0, y_1) \geq 1. \end{aligned} \tag{3.3}$$

Assume that $u_1 \notin T(x_1, y_1), v_1 \notin T(y_1, x_1)$; for otherwise $(x_1, y_1)$ is the coupled best proximity point. From condition (4) and Definition 2.8, we have

$$\begin{aligned} \varphi\Big(1, H\big(T(x_0, y_0), T(x_1, y_1)\big)\Big) &\leq \varphi\Big(\alpha(x_0, x_1), H\big(T(x_0, y_0), T(x_1, y_1)\big)\Big) \\ &\leq F\big(1, \psi\big(d(x_0, x_1)\big)\big) \\ \Longrightarrow \quad 0 < d(u_1, T(x_1, y_1) &\leq H\big(T(x_0, y_0), T(x_1, y_1)\big) \\ &\leq \psi\big(d(x_0, x_1)\big)\big) \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \varphi\Big(1, H\big(T(y_0, x_0), T(y_1, x_1)\big)\Big) &\leq \varphi\Big(\alpha(y_0, y_1), H\big(T(y_0, x_0), T(y_1, x_1)\big)\Big) \\ &\leq F\big(1, \psi\big(d(y_0, y_1)\big)\big) \\ \Longrightarrow \quad 0 < d\big(v_1, T(y_1, x_1)\big) &\leq H\big(T(y_0, x_0), T(y_1, x_1)\big) \\ &\leq \psi\big(d(y_0, y_1)\big). \end{aligned} \tag{3.5}$$

For $q, q' > 1$, it follows from Lemma 2.5 that there exist $u_2 \in T(x_1, y_1)$ and $v_2 \in T(y_1, x_1)$ such that

$$\begin{aligned} 0 < d(u_1, u_2) &< q d\big(u_1, T(x_1, y_1)\big) \quad \text{and} \\ 0 < d(v_1, v_2) &< q' d\big(v_1, T(y_1, x_1)\big). \end{aligned} \tag{3.6}$$

From (3.4), (3.5) and (3.6), we have

$$0 < d(u_1, u_2) < q d\big(u_1, T(x_1, y_1)\big) \leq q \psi\big(d(x_0, x_1)\big) \tag{3.7}$$

and

$$0 < d(v_1, v_2) < q' d\big(v_1, T(y_1, x_1)\big) \leq q' \psi\big(d(y_0, y_1)\big). \tag{3.8}$$

As $u_2 \in T(x_1, y_1) \subseteq V_0$, there exists $x_2 \neq x_1 \in U_0$ such that

$$d(x_2, u_2) = \operatorname{dist}(U, V) \tag{3.9}$$

and as $v_2 \in T(y_1, x_1) \subseteq V_0$, there exists $y_2 \neq y_1 \in U_0$ such that

$$d(y_2, v_2) = \operatorname{dist}(U, V), \tag{3.10}$$

for otherwise $(x_1, y_1)$ is the coupled best proximity point. As $(U, V)$ satisfies the weak $P$-property, from (3.3), (3.9) and (3.10), we have

$$\begin{aligned} 0 < d(x_1, x_2) \le d(u_1, u_2) \quad \text{and} \\ 0 < d(y_1, y_2) \le d(v_1, v_2). \end{aligned} \tag{3.11}$$

From (3.7), (3.8) and (3.11), we have

$$\begin{aligned} 0 < d(x_1, x_2) \le d(u_1, u_2) < qd\big(u_1, T(x_1, y_1)\big) \le q\psi\big(d(x_0, x_1)\big) \quad \text{and} \\ 0 < d(y_1, y_2) \le d(v_1, v_2) < q'd\big(v_1, T(y_1, x_1)\big) \le q'\psi\big(d(y_0, y_1)\big). \end{aligned} \tag{3.12}$$

Since $\psi$ is strictly increasing, we have

$$\begin{aligned} \psi\big(d(x_1, x_2)\big) &< \psi\Big(q\psi\big(d(x_0, x_1)\big)\Big) \quad \text{and} \\ \psi\big(d(y_1, y_2)\big) &< \psi\Big(q'\psi\big(d(y_0, y_1)\big)\Big). \end{aligned}$$

Put

$$\begin{aligned} q_1 &= \psi\Big(q\psi\big(d(x_0, x_1)\big)\Big)/\psi\big(d(x_1, x_2)\big), \\ q_1' &= \psi\Big(q'\psi\big(d(y_0, y_1)\big)\Big)/\psi\big(d(y_1, y_2)\big). \end{aligned}$$

We also have

$$\alpha(x_0, x_1) \ge 1, \;\; d(x_1, u_1) = \operatorname{dist}(U, V) \;\; \text{and} \;\; d(x_2, u_2) = \operatorname{dist}(U, V)$$

and

$$\alpha(y_0, y_1) \ge 1, \;\; d(y_1, v_1) = \operatorname{dist}(U, V) \;\; \text{and} \;\; d(y_2, v_2) = \operatorname{dist}(U, V).$$

Since $T$ is an $\alpha$-proximal admissible, $\alpha(x_1, x_2) \ge 1$ and $\alpha(y_1, y_2) \ge 1$. Thus we have

$$\begin{aligned} d(x_2, u_2) = \operatorname{dist}(U, V), \quad \alpha(x_1, x_2) \ge 1 \quad \text{and} \\ d(y_2, v_2) = \operatorname{dist}(U, V), \quad \alpha(y_1, y_2) \ge 1. \end{aligned} \tag{3.13}$$

Assume that $u_2 \notin T(x_2, y_2)$ and $v_2 \notin T(y_2, x_2)$, for otherwise $(x_2, y_2)$ is the coupled best proximity point. From condition (4) and Definition 2.8, we have

$$\begin{aligned} \varphi\Big(1, H\big(T(x_1, y_1), T(x_2, y_2)\big)\Big) &\le \varphi\Big(\alpha(x_1, x_2), H\big(T(x_1, y_1), T(x_2, y_2)\big)\Big) \\ &\le F\big(1, \psi\big(d(x_1, x_2)\big)\big) \end{aligned}$$

$$\Longrightarrow \quad 0 < d\big(u_2, T(x_2, y_2)\big) \leq H\big(T(x_1, y_1), T(x_2, y_2)\big) \leq \psi\big(d(x_1, x_2)\big), \tag{3.14}$$

and

$$\varphi\Big(1, H\big(T(y_1, x_1), T(y_2, x_2)\big)\Big) \leq \varphi\Big(\alpha(y_1, y_2), H\big(T(y_1, x_1), T(y_2, x_2)\big)\Big) \leq F\big(1, \psi\big(d(y_1, y_2)\big)\big)$$

$$\Longrightarrow \quad 0 < d\big(v_2, T(y_2, x_2)\big) \leq H\big(T(y_1, x_1), T(y_2, x_2)\big) \leq \psi\big(d(y_1, y_2)\big). \tag{3.15}$$

For $q_1, q_1' > 1$, it follows from Lemma 2.5 that there exist $u_3 \in T(x_2, y_2)$ and $v_3 \in T(y_2, x_2)$ such that

$$\begin{aligned} 0 < d(u_2, u_3) < q_1 d\big(u_2, T(x_2, y_2)\big), \\ 0 < d(v_2, v_3) < q_1' d\big(v_2, T(y_2, x_2)\big). \end{aligned} \tag{3.16}$$

From (3.14), (3.15) and (3.16) we have

$$\begin{aligned} 0 < d(u_2, u_3) &< q_1 d\big(u_2, T(x_2, y_2)\big) \\ &\leq q_1 \psi\big(d(x_1, x_2)\big) \\ &= \psi\Big(q\psi\big(d(x_0, x_1)\big)\Big) \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} 0 < d(v_2, v_3) &< q_1' d\big(v_2, T(y_2, x_2)\big) \\ &\leq q_1' \psi\big(d(y_1, y_2)\big) \\ &= \psi\Big(q' \psi\big(d(y_0, y_1)\big)\Big). \end{aligned} \tag{3.18}$$

As $u_3 \in T(x_2, y_2) \in V_0$, there exists $x_3 \neq x_2 \in U_0$ such that

$$d(x_3, u_3) \;=\; \operatorname{dist}(U, V), \tag{3.19}$$

and as $v_3 \in T(y_2, x_2) \in V_0$, there exists $y_3 \neq y_2 \in U_0$ such that

$$d(y_3, v_3) \;=\; \operatorname{dist}(U, V), \tag{3.20}$$

for otherwise $(x_2, y_2)$ is the coupled best proximity point. As $(U, V)$ satisfies the weak $P$-property, from (3.13), (3.19) and (3.20) we have

$$\begin{aligned} 0 < d(x_2, x_3) \leq d(u_2, u_3), \\ 0 < d(y_2, y_3) \leq d(v_2, v_3). \end{aligned} \tag{3.21}$$

From (3.17), (3.18) and (3.21) we have

$$\begin{aligned} 0 < d(x_2,x_3) &< q_1 d\big(u_2, T(x_2,y_2)\big) \\ &\le q_1 \psi\big(d(x_1,x_2)\big) \\ &= \psi\Big(q\psi\big(d(x_0,x_1)\big)\Big) \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} 0 < d(y_2,y_3) &< q'_1 d\big(v_2, T(y_2,x_2)\big) \\ &\le q'_1 \psi\big(d(y_1,y_2)\big) \\ &= \psi\Big(q'\psi\big(d(y_0,y_1)\big)\Big). \end{aligned} \tag{3.23}$$

Since $\psi$ is strictly increasing, we have

$$\psi\big(d(x_2,x_3)\big) \quad < \quad \psi^2\Big(q\psi\big(d(x_0,x_1)\big)\Big) \quad \text{and} \tag{3.24}$$

$$\psi\big(d(y_2,y_3)\big) \quad < \quad \psi^2\Big(q'\psi\big(d(y_0,y_1)\big)\Big). \tag{3.25}$$

Put

$$\begin{aligned} q_2 &= \psi^2\Big(q\psi\big(d(x_0,x_1)\big)\Big)/\psi\big(d(x_2,x_3)\big), \\ q'_2 &= \psi^2\Big(q'\psi\big(d(y_0,y_1)\big)\Big)/\psi\big(d(y_2,y_3)\big). \end{aligned}$$

We also have

$$\alpha(x_1,x_2) \ge 1, \;\; d(x_2,u_2) = \operatorname{dist}(U,V) \;\; \text{and} \;\; d(x_3,u_3) = \operatorname{dist}(U,V)$$

and

$$\alpha(y_1,y_2) \ge 1, \;\; d(y_2,v_2) = \operatorname{dist}(U,V) \;\; \text{and} \;\; d(y_3,v_3) = \operatorname{dist}(U,V).$$

Since $T$ is an $\alpha$-proximal admissible, $\alpha(x_2,x_3) \ge 1$ and $\alpha(y_2,y_3) \ge 1$, respectively. Thus we have

$$\begin{aligned} d(x_3,u_3) &= \operatorname{dist}(U,V), \quad \alpha(x_2,x_3) \ge 1 \quad \text{and} \\ d(y_3,v_3) &= \operatorname{dist}(U,V), \quad \alpha(y_2,y_3) \ge 1. \end{aligned} \tag{3.26}$$

Continuing in the same process, we get sequences $\{x_n\}, \{y_n\}$ in $U_0$ and $\{u_n\}$, $\{v_n\}$ in $V_0$, where $u_n \in T(x_{n-1},y_{n-1})$ and $v_n \in T(y_{n-1},x_{n-1})$ for each $n \in \mathbb{N}$, such that

$$\begin{aligned} d(x_{n+1},u_{n+1}) &= \operatorname{dist}(U,V), \quad \alpha(x_n,x_{n+1}) \ge 1 \quad \text{and} \\ d(y_{n+1},v_{n+1}) &= \operatorname{dist}(U,V), \quad \alpha(y_n,y_{n+1}) \ge 1, \end{aligned} \tag{3.27}$$

and

$$d(u_{n+1}, u_{n+2}) < \psi^n\Big(q\psi\big(d(x_0, x_1)\big)\Big) \quad \text{and}$$
$$d(v_{n+1}, v_{n+2}) < \psi^n\Big(q'\psi\big(d(y_0, y_1)\big)\Big). \tag{3.28}$$

As $u_{n+2} \in T(x_{n+1}, y_{n+1}) \in V_0$, there exists $x_{n+2} \neq x_{n+1} \in U_0$ such that

$$d(x_{n+2}, u_{n+2}) \;=\; \operatorname{dist}(U, V) \tag{3.29}$$

and as $v_{n+2} \in T(y_{n+1}, x_{n+1}) \in V_0$, there exists $y_{n+2} \neq y_{n+1} \in U_0$ such that

$$d(y_{n+2}, v_{n+2}) \;=\; \operatorname{dist}(U, V). \tag{3.30}$$

Since $(U, V)$ satisfies the weak $P$-property, from (3.27), (3.29) and (3.30), we have

$$d(x_{n+1}, x_{n+2}) \;\leq\; d(u_{n+1}, u_{n+2})$$

and

$$d(y_{n+1}, y_{n+2}) \;\leq\; d(v_{n+1}, v_{n+2}).$$

Thus, from (3.28), we have

$$d(x_{n+1}, x_{n+2}) < \psi^n\Big(q\psi\big(d(x_0, x_1)\big)\Big) \quad \text{and}$$
$$d(y_{n+1}, y_{n+2}) < \psi^n\Big(q'\psi\big(d(y_0, y_1)\big)\Big). \tag{3.31}$$

Now, we shall prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in $U$. Let $\epsilon > 0$ be fixed. Since $\sum_{n=1}^{\infty} \psi^n\Big(q\psi\big(d(x_0, x_1)\big)\Big) < \infty$ and $\sum_{n=1}^{\infty} \psi^n\Big(q'\psi\big(d(y_0, y_1)\big)\Big) < \infty$, there exist some positive integers $\varphi = \varphi(\epsilon)$ and $\varphi' = \varphi'(\epsilon)$ such that

$$\sum_{k\geq\varphi}^{\infty} \psi^k\Big(q\psi\big(d(x_0, x_1)\big)\Big) < \epsilon$$

and

$$\sum_{k\geq\varphi'}^{\infty} \psi^k\Big(q'\psi\big(d(y_0, y_1)\big)\Big) < \epsilon,$$

respectively. For $m > n > \varphi$, using the triangular inequality, we obtain

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \\ &\leq \sum_{k=n}^{m-1} \psi^k\Big(q\psi\big(d(x_0, x_1)\big)\Big) \\ &\leq \sum_{k\geq\varphi}^{\infty} \psi^k\Big(q\psi\big(d(x_0, x_1)\big)\Big) < \epsilon \end{aligned} \tag{3.32}$$

and

$$\begin{aligned} d(y_n, y_m) &\leq \sum_{k=n}^{m-1} d(y_k, y_{k+1}) \\ &\leq \sum_{k=n}^{m-1} \psi^k\Big(q'\psi\big(d(y_0, y_1)\big)\Big) \\ &\leq \sum_{k\geq\varphi'}^{\infty} \psi^k\Big(q'\psi\big(d(y_0, y_1)\big)\Big) < \epsilon, \end{aligned} \tag{3.33}$$

respectively. Hence $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in $U$.

Similarly, we can show that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences in $V$. Since $U$ and $V$ are closed subsets of a complete metric space, there exists $(r^*, s^*)$ in $U \times U$ such that $x_n \to r^*$, $y_n \to s^*$ as $n \to \infty$ and there exist $u^*, v^*$ in $V$ such that $u_n \to u^*$, $v_n \to v^*$ as $n \to \infty$. By (3.29) and (3.30), we conclude that

$$\begin{aligned} d(r^*, u^*) &= \operatorname{dist}(U, V), \text{ as } n \to \infty \quad \text{and} \\ d(s^*, v^*) &= \operatorname{dist}(U, V), \text{ as } n \to \infty. \end{aligned}$$

Since $T$ is continuous and $u_n \in T(x_{n-1}, y_{n-1})$, we have $u^* \in T(r^*, s^*)$ and $v_n \in T(y_{n-1}, x_{n-1})$, we have $v^* \in T(s^*, r^*)$. Hence,

$$\begin{aligned} \operatorname{dist}(U, V) &\leq D\big(r^*, T(r^*, s^*)\big) \\ &\leq d(r^*, u^*) \\ &= \operatorname{dist}(U, V) \end{aligned}$$

and

$$\begin{aligned} \operatorname{dist}(U, V) &\leq D\big(s^*, T(s^*, r^*)\big) \\ &\leq d(s^*, v^*) \\ &= \operatorname{dist}(U, V). \end{aligned}$$

Therefore, $(r^*, s^*)$ is the coupled best proximity point of the mapping $T$. □

**Remark 3.3.** If we take $\varphi(x, y) = xy$ and $F(s, t) = st$ in Theorem 3.2, then our result reduces to Theorem 2.4 in [21].

**Theorem 3.4.** *Let $U$ and $V$ be two nonempty closed subsets of a complete metric space $(X, d)$ such that $U_0$ is nonempty. Let $\alpha : U \times U \to [0, \infty)$ and let $T : U \times U \to \mathrm{K}(V)$ be a mapping satisfying the following conditions:*

(1) *$T(x, y) \subseteq V_0$ for each $(x, y) \in U_0 \times U_0$ and $(U, V)$ satisfies the weak $P$-property;*
(2) *$T$ is an $\alpha$-proximal admissible map;*

(3) *there exist elements* $(x_0,y_0),(x_1,y_1)$ *in* $U_0\times U_0$ *and* $u_1\in T(x_0,y_0), v_1\in T(y_0,x_0)$ *such that*

$$\begin{aligned} d(x_1,u_1)&=\operatorname{dist}(U,V), \quad \alpha(x_0,x_1)\geq 1 \quad \textit{and}\\ d(y_1,v_1)&=\operatorname{dist}(U,V), \quad \alpha(y_0,y_1)\geq 1; \end{aligned} \tag{3.34}$$

(4) $T$ *is a continuous* $(F,\varphi,\alpha,\psi)$*-proximal contraction.*

*Then there exists an element* $(r^*,s^*)\in U_0\times U_0$ *such that*

$$\begin{aligned} D\big(r^*,T(r^*,s^*)\big)&=\operatorname{dist}(U,V) \quad \textit{and}\\ D\big(s^*,T(s^*,r^*)\big)&=\operatorname{dist}(U,V). \end{aligned}$$

**Theorem 3.5.** *Let* $U$ *and* $V$ *be two nonempty closed subsets of a complete metric space* $(X,d)$ *such that* $U_0$ *is nonempty. Let* $\alpha:U\times U\to[0,\infty)$ *and let* $\psi\in\Psi$ *be a strictly increasing map. Suppose that* $T:U\times U\to \mathrm{CL}(V)$ *is a mapping satisfying the following conditions:*

(1) $T(x,y)\subseteq V_0$ *for each* $(x,y)\in U_0\times U_0$ *and* $(U,V)$ *satisfies the weak* $P$*-property;*

(2) $T$ *is an* $\alpha$*-proximal admissible map;*

(3) *there exist elements* $(x_0,y_0),(x_1,y_1)$ *in* $U_0\times U_0$ *and* $u_1\in T(x_0,y_0), v_1\in T(y_0,x_0)$ *such that*

$$\begin{aligned} d(x_1,u_1)&=\operatorname{dist}(U,V), \quad \alpha(x_0,x_1)\geq 1 \quad \textit{and}\\ d(y_1,v_1)&=\operatorname{dist}(U,V), \quad \alpha(y_0,y_1)\geq 1; \end{aligned} \tag{3.35}$$

(4) *property* $(C)$ *holds and* $T$ *is an* $(F,\varphi,\alpha,\psi)$*-proximal contraction.*

*Then there exists an element* $(x^*,y^*)\in U_0\times U_0$ *such that*

$$\begin{aligned} D\big(x^*,T(x^*,y^*)\big)&=\operatorname{dist}(U,V) \quad \textit{and}\\ D\big(y^*,T(y^*,x^*)\big)&=\operatorname{dist}(U,V). \end{aligned}$$

*Proof.* Similar to the proof of Theorem 3.2, there exist Cauchy sequences $\{x_n\}$ and $\{y_n\}$ in $U$ and Cauchy sequences $\{u_n\}$ and $\{v_n\}$ in $V$ such that

$$\begin{aligned} d(x_{n+1},u_{n+1})&=\operatorname{dist}(U,V), \quad \alpha(x_n,x_{n+1})\geq 1 \quad \text{and}\\ d(y_{n+1},v_{n+1})&=\operatorname{dist}(U,V), \quad \alpha(y_n,y_{n+1})\geq 1; \end{aligned} \tag{3.36}$$

and $x_n\to r^*\in U$, $y_n\to s^*\in U$ as $n\to\infty$ and $u_n\to u^*\in V$, $v_n\to v^*\in V$ as $n\to\infty$.

From condition (C), there exist subsequences $\{x_{n_k}\}$ of $\{x_n\}$, $\{y_{n_k}\}$ of $\{y_n\}$ such that $\alpha(x_{n_k},r^*)\geq 1$, $\alpha(y_{n_k},s^*)\geq 1$ for all $k$. Since $T$ is an $(F,\varphi,\alpha,\psi)$-proximal contraction, we have

$$\begin{aligned} \varphi\Big(1,H\big(T(x_{n_k},y_{n_k}),T(r^*,s^*)\big)\Big)&\leq \varphi\Big(\alpha(x_{n_k},r^*),H\big(T(x_{n_k},y_{n_k}),T(x^*,s^*)\big)\Big)\\ &\leq F\big(1,\psi\big(d(x_{n_k},r^*)\big)\big) \end{aligned}$$

$$\implies \quad H\big(T(x_{n_k}, y_{n_k}), T(r^*, s^*)\big)) \leq \psi\big(d(x_{n_k}, r^*)\big)$$

and

$$\begin{aligned}\varphi\Big(1, H\big(T(y_{n_k}, x_{n_k}), T(s^*, r^*)\big)\Big) &\leq \varphi\Big(\alpha(y_{n_k}, s^*), H\big(T(y_{n_k}, x_{n_k}), T(y^*, r^*)\big)\Big) \\ &\leq F\big(1, \psi\big(d(y_{n_k}, s^*)\big)\big) \\ \implies \quad H\big(T(y_{n_k}, x_{n_k}), T(s^*, r^*)\big)) &\leq \psi\big(d(y_{n_k}, s^*)\big)\end{aligned}$$

Letting $k \to \infty$ in the above inequalities, we get $T(x_{n_k}, y_{n_k}) \to T(x^*, y^*)$ and $T(y_{n_k}, x_{n_k}) \to T(y^*, x^*)$ respectively. By the continuity of the metric $d$, we have

$$\begin{aligned} d(x^*, u^*) &= \lim_{k\to\infty} d(x_{n_k+1}, u_{n_k+1}) = \operatorname{dist}(U, V), \\ d(y^*, v^*) &= \lim_{k\to\infty} d(y_{n_k+1}, v_{n_k+1}) = \operatorname{dist}(U, V). \end{aligned} \tag{3.37}$$

Since $u_{n_k+1} \in T(x_{n_k}, y_{n_k})$, $u_{n_k} \to u^*$ and $T(x_{n_k}, y_{n_k}) \to T(x^*, y^*)$, $u^* \in T(x^*, y^*)$ and since $v_{n_k+1} \in T(y_{n_k}, x_{n_k})$, $v_{n_k} \to v^*$ and $T(y_{n_k}, x_{n_k}) \to T(y^*, x^*)$, $v^* \in T(y^*, x^*)$, we have

$$\begin{aligned}\operatorname{dist}(U, V) &\leq D\big(x^*, T(x^*, y^*)\big) \\ &\leq d(x^*, u^*) \\ &= \operatorname{dist}(U, V)\end{aligned}$$

and

$$\begin{aligned}\operatorname{dist}(U, V) &\leq D\big(y^*, T(y^*, x^*)\big) \\ &\leq d(y^*, v^*) \\ &= \operatorname{dist}(U, V).\end{aligned}$$

Therefore, $(x^*, y^*)$ is the coupled best proximity point of the mapping $T$. □

**Remark 3.6.** If we take $\varphi(x, y) = xy$ and $F(s, t) = st$ in Theorem 3.5, then our result reduces to Theorem 2.6 in [21].

**Theorem 3.7.** *Let $U$ and $V$ be two nonempty closed subsets of a complete metric space $(X, d)$ such that $U_0$ is nonempty. Let $\alpha : U \times U \to [0, \infty)$ and let $T : U \times U \to \mathrm{K}(V)$ be a mapping satisfying the following conditions:*

(1) *$T(x, y) \subseteq V_0$ for each $(x, y) \in U_0 \times U_0$ and $(U, V)$ satisfies the weak $P$-property;*
(2) *$T$ is an $\alpha$-proximal admissible map;*
(3) *there exist elements $(x_0, y_0), (x_1, y_1)$ in $U_0 \times U_0$ and $u_1 \in T(x_0, y_0), v_1 \in T(y_0, x_0)$ such that*

$$\begin{aligned} d(x_1, u_1) = \operatorname{dist}(U, V), \quad \alpha(x_0, x_1) \geq 1 \quad \textit{and} \\ d(y_1, v_1) = \operatorname{dist}(U, V), \quad \alpha(y_0, y_1) \geq 1; \end{aligned} \tag{3.38}$$

(4) *property* $(C)$ *holds and* $T$ *is an* $(F,\varphi,\alpha,\psi)$*-proximal contraction.*
*Then there exists an element* $(x^*,y^*)\in U_0\times U_0$ *such that*

$$D\big(x^*,T(x^*,y^*)\big)=\operatorname{dist}(U,V)\quad \textit{and}$$
$$D\big(y^*,T(y^*,x^*)\big)=\operatorname{dist}(U,V).$$

We give the following examples to support our main results.

**Example 3.8.** Let $X=[0,\infty)\times[0,\infty)$ be endowed with the usual metric $d$. Let $2<a\le 3$ be any fixed real number, $U=\{(a,x):0\le x<\infty\}$ and $V=\{(0,x):0\le x<\infty\}$. Define $T:U\times U\to \operatorname{CL}(V)$ by

$$T\left((a,x),(a,y)\right)=\left\{(0,b^2):0\le b\le \max\{x,y\}\right\}, \tag{3.39}$$

and $\alpha:U\times U\to[0,\infty)$ by

$$\alpha\left((a,x),(a,y)\right)=\begin{cases}1 & \text{if } x=y=0,\\ \dfrac{1}{a(x+y)} & \text{otherwise.}\end{cases} \tag{3.40}$$

Let $\varphi(x,y)=xy$, $F(s,t)=st$ and let $\psi(t)=\dfrac{t}{2}$ for all $t\ge 0$. Note that $U_0=U, V_0=V$ and $T(x,y)\in V_0$ for each $x,y\in U_0$. If $w_1=(a,y_1), w_1'=(a,y_1'), w_2=(a,y_2), w_2'=(a,y_2')\in U$ with either $y_1\neq 0$ or $y_2\neq 0$ or both are nonzero, we have

$$\begin{aligned}\varphi\Big(\alpha(w_1,w_2),H\left(T(w_1,w_1'),T(w_2,w_2')\right)\Big)&=\frac{1}{a(y_1+y_2)}\left|y_1^2-y_2^2\right|\\&<\frac{1}{2}|y_1-y_2|\\&=\psi\left(d(w_1,w_2)\right)\\&=F\Big(1,\psi\left(d(w_1,w_2)\right)\Big)\end{aligned}$$

for otherwise

$$\varphi\Big(\alpha(w_1,w_2),H\left(T(w_1,w_1'),T(w_2,w_2')\right)\Big)=0=F\Big(1,\psi\left(d(w_1,w_2)\right)\Big).$$

For $x_0=(a,\frac{1}{2a}), x_1=(a,\frac{1}{4a^2}), y_0=(a,\frac{1}{3a})\in U_0$ and $u_1=(0,\frac{1}{4a^2})\in T(x_0,y_0)$ such that $d(x_1,u_1)=a=\operatorname{dist}(U,V)$ and $\alpha(x_0,x_1)=\dfrac{4a}{1+2a}>1$. And for $x_1=(a,\frac{1}{3a}), y_1=(a,\frac{1}{9a^2})\in U_0$ and $v_1=(0,\frac{1}{9a^2})\in T(x_1,y_1)$ such that $d(y_1,v_1)=a=\operatorname{dist}(U,V)$ and $\alpha(y_0,y_1)=\dfrac{9a}{1+3a}>1$. Furthermore, one can see that the remaining conditions of Theorem 3.2 also hold. Therefore, $T$ has the coupled best proximity point.

**Example 3.9.** Let $X = [0,\infty) \times [0,\infty)$ be a product space endowed with the usual metric $d$. Suppose that $U = \left\{(\frac{1}{2}, x) : 0 \le x < \infty\right\}$ and $V = \{(0,x) : 0 \le x < \infty\}$.

Define $T : U \times U \to \mathrm{CL}(V)$ by

$$T\left((\frac{1}{2},a),(\frac{1}{2},b)\right) = \begin{cases} \left\{(0,\frac{x}{2}) : 0 \le x \le \max\{a,b\}\right\} & \text{if } a,b \le 1, \\ \left\{(0,x^2) : 0 \le x \le \max\{a^2,b^2\}\right\} & \text{if } a,b > 1, \end{cases} \tag{3.41}$$

and define $\alpha : U \times U \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x,y \in \left\{(\frac{1}{2},a) : 0 \le a \le 1\right\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\psi(t) = \dfrac{t}{2}$ for all $t \ge 0$. Note that $U_0 = U, V_0 = V$, and $T(x,y) \subseteq V_0$ for each $(x,y) \in U_0 \times U_0$. Also, the pair $(U,V)$ satisfies the weak $P$-property.

Let $(x_0,y_0),(x_1,y_1) \in \left\{(\frac{1}{2},x) : 0 \le x \le 1\right\}^2$. Then

$$T(x_0,y_0), T(x_1,y_1) \subseteq \left\{(0,\frac{x}{2}) : 0 \le x \le 1\right\}.$$

Consider $u_1 \in T(x_0,y_0), u_2 \in T(x_1,y_1)$ and $w_1,w_2 \in U$ such that $d(w_1,u_1) = \mathrm{dist}(U,V)$ and $d(w_2,u_2) = \mathrm{dist}(U,V)$. Then we have

$$w_1, w_2 \in \left\{(\frac{1}{2},x) : 0 \le x \le \frac{1}{2}\right\},$$

so $\alpha(w_1,w_2) = 1$. And, for $v_1 \in T(y_0,x_0), v_2 \in T(y_1,x_1)$ and $w'_1, w'_2 \in U$ such that $d(w'_1,v_1) = \mathrm{dist}(U,V)$ and $d(w'_2,v_2) = \mathrm{dist}(U,V)$. Then we have

$$w'_1, w'_2 \in \left\{(\frac{1}{2},x) : 0 \le x \le \frac{1}{2}\right\},$$

so $\alpha(w'_1,w'_2) = 1$. Therefore, $T$ is an $\alpha$-proximal admissible map. For $(x_0,y_0) = \left((\frac{1}{2},1),(\frac{1}{2},1)\right) \in U_0 \times U_0$ and $u_1 = (0,\frac{1}{2}) \in T(x_0,y_0), v_1 = (0,\frac{1}{4}) \in T(y_0,x_0)$ in $V_0$, we have

$$(x_1,y_1) = \left((\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{4})\right) \in U_0 \times U_0$$

such that

$$d(x_1,u_1) = \mathrm{dist}(U,V), \quad \alpha(x_0,x_1) = \alpha\left((\frac{1}{2},1),(\frac{1}{2},\frac{1}{2})\right) = 1$$

and

$$d(y_1,v_1) = \mathrm{dist}(U,V), \quad \alpha(y_0,y_1) = \alpha\left((\frac{1}{2},1),(\frac{1}{2},\frac{1}{4})\right) = 1.$$

Let $\varphi(x,y) = xy$ and $F(s,t) = st$. If $x, x', y, y' \in \left\{(\frac{1}{2}, a) : 0 \le a \le 1\right\}^2$. Then we have

$$\begin{aligned}\varphi\Big(\alpha(x,y), H\left(T(x,x'), T(y,y')\right)\Big) &= \frac{|x-y|}{2} \\ &= \frac{1}{2}d(x,y) \\ &= F\Big(1, \psi\left(d(x,y)\right)\Big),\end{aligned}$$

for otherwise

$$\varphi\Big(\alpha(x,y), H\left(T(x,x'), T(y,y')\right)\Big) \le F\Big(1, \psi\left(d(x,y)\right)\Big).$$

Hence, $T$ is an $(F\varphi, \alpha, \psi)$-proximal contraction. Moreover, if $\{x_n\}$ is a sequence in $U$ such that $\alpha(x_n, x_{n+1}) = 1$ for all $n$ and $x_n \to x \in U$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) = 1$ for all $k$. Therefore, all the conditions of Theorem 3.5 hold and $T$ has the coupled best proximity point.

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