Nonlinear Functional Analysis and Applications
Vol. 25, No. 3 (2020), pp. 401-409
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2020.25.03.01
http://nfaa.kyungnam.ac.kr/journal-nfaa

SOME ASPECTS OF FEASIBILITY AND SOLVABILITY FOR CONE LINEAR COMPLEMENTARITY PROBLEMS

Khadije Bypour[1] and Mehdi Roohi[2]

[1]Department of Mathematics, Faculty of Sciences
Golestan University, Gorgan, Iran
e-mail: kh.bypour@stu.gu.ac.ir

[2]Department of Mathematics, Faculty of Sciences
Golestan University, Gorgan, Iran
e-mail: m.roohi@gu.ac.ir

Abstract. In this paper, we introduce a new class of linear operators for which cone linear complementarity problem (LCP) is feasible. For feasible LCP, it is important to know, when it is solvable. We give a result which feasibility implies solvability. We describe structure of the solution set of the LCP with positive operator. Moreover, the class of linear operators which are constant on the solution set of LCP is characterized.

1. Introduction

Let H be a real Hilbert space, $K \subseteq H$ a cone and $L : H \to H$ an operator. Given an arbitrary element $q \in H$, the linear complementarity problem defined by L, K and q is

$$\mathrm{LCP}(L, K, q) : \begin{cases} \text{find } x \in K \text{ such that} \\ L(x) + q \in K^* \text{ and } \langle x, L(x) + q \rangle = 0, \end{cases} \tag{1.1}$$

where $K^* = \{y \in H : \langle y, x \rangle \geqslant 0 \text{ for all } x \in K\}$ is the *dual cone* of K [7].

Given $\mathrm{LCP}(L, K, q)$, we write

$$\mathrm{FEA}(L, K, q) = \{x : x \in K, L(x) + q \in K^*\}$$

[0]Received September 21, 2018. Revised October 14, 2019. Accpted November 24, 2019.
[0]2010 Mathematics Subject Classification: 90C33, 47H07.
[0]Keywords: Linear complementarity problem, feasibility, solvability.
[0]Corresponding author: M. Roohi(m.roohi@gu.ac.ir).

and

$$\mathrm{SOL}(L, K, q) = \{x : x \in K, L(x) + q \in K^*, \langle x, L(x) + q\rangle = 0\}.$$

These are the *feasible set* and *solution set* of $\mathrm{LCP}(L, K, q)$, respectively. Obviously, we have $\mathrm{SOL}(L, K, q) \subseteq \mathrm{FEA}(L, K, q)$ for each problem, but generally the converse is not true. For a given complementarity problem, when $\mathrm{SOL}(L, K, q) \neq \emptyset$ (resp. $\mathrm{FEA}(L, K, q) \neq \emptyset$), the problem is said to be *solvable* (resp. *feasible*).

Observe that $\mathrm{LCP}(L, K, q)$ defined in (1.1) has the following quadratic form:

$$\begin{array}{ll} \text{minimize} & \langle x, L(x) + q\rangle \\ \text{subject to} & L(x) + q \in K^* \\ & x \in K. \end{array} \tag{1.2}$$

This problem has extensive applications in engineering, economic, game theory, and other fields [4, 7, 13, 15]. The LCP with an arbitrary linear operator is difficult to solve. Observe that if $q \in K^*$, then $\mathrm{LCP}(L, K, q)$ is always solvable with the zero vector being a trivial solution.

A key issue in cone linear complementarity problems is finding necessary and sufficient conditions on the linear operator L that ensures nonemptyness of the solution set.

This paper is organized as follows. In the next section, we express the basic material about the Lorentz cone in a Hilbert space H and Jordan algebra associated to the Lorentz cone in H and some of their basic properties are described. In Section 3, we introduce the class of F-operators and show that feasibility of the LCP on the selfdual cone is equivalent to the fact that the linear operator is an F-operator. We consider some properties of solution set of LCP with positive operator. Also, we give a characterization for the class of linear operators which are constant on the solution set.

2. Notation and preliminaries

In this section, we briefly introduce some basic concepts in real Hilbert space H, and review some basic materials. These concepts and materials play important roles in subsequent analysis. More details and related results can be found in [10, 12].

Let H be a real Hilbert space with the inner product $\langle ., .\rangle$, and the norm $\|.\|$. Let K be a cone in H with the vertex at 0. A cone K is said to be *pointed* if $K \cap (-K) = \{0\}$, and is *solid* if $\mathrm{int}(K) \neq \emptyset$ where int is topological interior. A cone K is said to be *selfdual* if $K^* = K$.

Proposition 2.1. ([6, Proposition 1.1.4]) *For a nonempty closed convex cone K in H:*

$$\mathrm{int}(K^*) = \{y \in H : \langle y, x\rangle > 0 \text{ for all } x \in K \setminus \{0\}\}.$$

Furthermore, the following properties are equivalent:

(i) *K is pointed;*
(ii) $\mathrm{int}(K^*) \neq \emptyset$.

Consider a pointed, closed, convex cone K in H. This K induces a partial order on H:

$$x \geqslant y \Leftrightarrow x - y \in K.$$

We use the notation $x > y$ when $x - y \in \mathrm{int}(K)$ (if exists). Beside this, we write $z \leqslant 0$ when $z \in -K$.

Proposition 2.2. *Let K be a nonempty closed convex selfdual cone in H. Then $x \in K$ if and only if $\langle x, y\rangle \geq 0$ holds for all $y \in K$. Moreover, $x \in \mathrm{int}(K)$ if and only if $\langle x, y\rangle > 0$ holds for all $y \in K \setminus \{0\}$.*

Proof. It follows from the definition of K^* and Proposition 2.1. □

For a closed convex cone K in H, let P_K denote the metric projection onto K [1]. For $z \in H$, $z^+ := P_K(z)$ if and only if $z^+ \in K$ and $\|z - z^+\| \leqslant \|z - y\|$ for all $y \in K$. This is also equivalent to $\langle z - z^+, y - z^+\rangle \leqslant 0$ for any $y \in K$. Since K is a closed convex cone, z^+ is unique, and is called *positive part* of z. Similarly, z^- means $P_K(-z)$, and is called *negative part* of z.

Let e be an unit vector in H (i.e., $\|e\| = \sqrt{\langle e, e\rangle} = 1$). In [10], the following closed convex cone

$$K(e, r) = \{z \in H : \langle z, e\rangle \geqslant r\|z\|\},$$

is considered, where $r \in \mathbb{R}$ with $0 < r < 1$. It is easy to prove that $K(e, r)$ is pointed, i.e., $K(e, r) \cap (-K(e, r)) = \{0\}$. Define the orthogonal complementarity space of $\{e\}$ by

$$\langle e\rangle^\perp := \{x \in H : \langle x, e\rangle = 0\}.$$

For any element $z \in H$, we have the orthogonal decomposition $z = x + \lambda e$ with unique $x \in \langle e\rangle^\perp$ and $\lambda \in \mathbb{R}$ (in fact, $\lambda = \langle z, e\rangle$). With this, it can be verified that

$$K(e, r) = \{x + \lambda e : \lambda \geqslant \frac{r}{\sqrt{1 - r^2}}\|x\|\}.$$

Proposition 2.3. ([11, Proposition 2.1]) *For any $e \in H$ with $\|e\| = 1$ and $0 < r < 1$, the dual cone of $K(e, r)$ can be written as*

$$K^*(e, r) = \{z \in H : \langle z, w\rangle \geqslant 0, \text{ for all } w \in K(e, r)\} = K(e, \sqrt{1 - r^2}).$$

Consequently, $K(e, \frac{1}{\sqrt{2}})$ is a selfdual closed convex cone.

From the above proposition, we may write

$$K\left(e, \frac{1}{\sqrt{2}}\right) = \{x + \lambda e \in H : x \in \langle e\rangle^{\perp} \text{ and } \lambda \in \mathbb{R} \text{ with } \lambda \geqslant \|x\|\}. \tag{2.1}$$

When $H = \mathbb{R}^n$ and $e = (1, 0) \in \mathbb{R} \times \mathbb{R}^{n-1}$, the set $K\left(e, \frac{1}{\sqrt{2}}\right)$ coincides with the *Lorentz cone* (also called *second order cone*) K^n in $\mathbb{R}^n$. Hence, $K\left(e, \frac{1}{\sqrt{2}}\right)$ is called the Lorentz cone in H determined by e.

For the sake of simplicity, we denote $\mathbb{K} := K\left(e, \frac{1}{\sqrt{2}}\right)$.

Lemma 2.4. ([10, Lemma 2.2]) *For any $z = x + \lambda e \in H$ with $x \in \langle e\rangle^{\perp}$ and $\lambda \in \mathbb{R}$, the following results hold:*

(a) *If $z \geqslant 0$, $z^+ = z$ and $z^- = 0$.*
(b) *If $z \leqslant 0$, $z^+ = 0$ and $z^- = -z$.*
(c) *If $z \notin \mathbb{K}$ and $-z \notin \mathbb{K}$, then*

$$z^+ = \frac{\|x\| + \lambda}{2\|x\|} x + \frac{\|x\| + \lambda}{2} e \quad \text{and} \quad z^- = \frac{\lambda - \|x\|}{2\|x\|} x + \frac{\|x\| - \lambda}{2} e.$$

(d) *For any $z \in H$, we have $z^+, z^- \in \mathbb{K}$, $z = z^+ - z^-$, $\langle z^+, z^-\rangle = 0$ and $\|z\|^2 = \|z^+\|^2 + \|z^-\|^2$.*

Now, we introduce the concept of Jordan product in a Hilbert space H and some related conclusions. For any $z, w \in H$ with $z = x + \lambda e$ and $w = y + \mu e$, where $x, y \in \langle e\rangle^{\perp}$ and $\lambda, \mu \in \mathbb{R}$, the *Jordan product* $z \bullet w$ of z and w is defined by

$$z \bullet w = \mu x + \lambda y + \langle z, w\rangle e = \mu x + \lambda y + (\langle x, y\rangle + \lambda\mu)e. \tag{2.2}$$

Note that the Jordan product is not associative even in the finite dimensional Euclidean spaces. It is easy to show that $\mathbb{K}$ is the cone of squares w.r.t. $\bullet$ multiplication. Thus, H becomes a Jordan algebra after introducing the Jordan product (for more details see [6]).

Definition 2.5. For any $z, w \in H$, we say that z and w *operator commute* if $z \bullet (w \bullet u) = w \bullet (z \bullet u)$ holds for any $u \in H$.

Definition 2.6. A linear operator $L : H \to H$ is said to have the *cross commutative property* if for any $q \in H$ and any two solutions z_1 and z_2 of LCP(L, K, q), it follows that z_1 operator commutes with w_2 and z_2 operator commutes with w_1, where $w_i = L(z_i) + q$ $(i = 1, 2)$.

The following lemmas give the conditions and properties of operator commuting z and w.

Lemma 2.7. ([11, Lemma 2.1]) *Let $z, w \in H$ with $z = x + \lambda e$, $w = y + \mu e$, $x, y \in \langle e\rangle^{\perp}$ and $\lambda, \mu \in \mathbb{R}$. Then z and w operator commute if and only if there is $\beta \in \mathbb{R}$ (depends on x and y) such that $y = \beta x$ or $x = \beta y$. In particular, if $x \neq 0$ (respectively, $y \neq 0$), then z and w operator commute if and only if there is an $\alpha \in \mathbb{R}$ such that $y = \alpha x$ (respectively, $x = \alpha y$).*

Lemma 2.8. ([11, Lemma 2.2]) *Let $z = x + \lambda e$, $w = y + \mu e$ with $x, y \in \langle e\rangle^{\perp}$ and $\lambda, \mu \in \mathbb{R}$. Then the following two conditions are equivalent:*

(a) *$z \in \mathbb{K}, w \in \mathbb{K}$ and $\langle z, w\rangle = 0$;*
(b) *$z \in \mathbb{K}, w \in \mathbb{K}$ and $z \bullet w = 0$.*

In each case, we may get that z and w operator commute.

Lemma 2.9. ([11, Lemma 2.3]) *Let $z, w \in \mathbb{K}$. If z and w operator commute, then $z \bullet w \in \mathbb{K}$.*

Given a linear operator $L : H \to H$, the linear operator $L^* : H \to H$ denotes the *adjoint* of L, which is defined by $\langle L(x), y\rangle = \langle x, L^*(y)\rangle$ for all $x, y \in H$. A linear operator L is *selfadjoint* if $L = L^*$.

Definition 2.10. A linear operator $L : H \to H$ is called

(a) *positive*, if $\langle z, L(z)\rangle \geqslant 0$ for all $z \in H$;
(b) *strictly positive*, if $\langle z, L(z)\rangle > 0$ for all $0 \neq z \in H$;

3. Main results

Throughout this section, we assume that K is a nonempty, pointed, closed, convex cone in H.

Properties of linear operator L play strong roles in the analysis of feasibility of the LCP. Here, we introduce a class of linear operators related to the feasibility of the LCP.

Definition 3.1. An operator $L : H \to H$ is called *F-operator*, if there exists $z \in H$ such that

$$z \in \operatorname{int}(K) \text{ and } L(z) \in \operatorname{int}(K). \tag{3.1}$$

The class of all F-operators is denoted by $\mathcal{F}$.

Example 3.2. Clearly, every strictly positive operator is F-operator. But the converse is not true. For example, consider $L : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$L(x, y) = (x + 2y, x + y).$$

Then $z = (2, 1) \in \operatorname{int}(K)$ and $L(z) = (4, 3) \in \operatorname{int}(K)$ (where, K is the Lorentz cone of $\mathbb{R}^2$). However, by letting $u = (-1, 1)$, $\langle u, L(u)\rangle = -1 < 0$, we obtain that L is not strictly positive.

It should be noticed that, if L is a continuous linear operator, then (3.1) is feasible if and only if

$$z \in K \text{ and } L(z) \in \operatorname{int}(K). \tag{3.2}$$

Clearly, (3.2) is implied by (3.1). On the other hand, suppose a vector $z \in K$ is given such that $L(z) \in \operatorname{int}(K)$. Since L is continuous at z, it follows that $L(z + \lambda u) \in \operatorname{int}(K)$ for small enough $\lambda > 0$ (where $u = (1, 1)$). One can take λ small enough to get $z + \lambda u \in \operatorname{int}(K)$; we have (3.1).

Theorem 3.3. *Let K be selfdual. Then the continuous linear operator $L : H \to H$ is an F-operator if and only if the* LCP(L, K, q) *is feasible for all $q \in H$.*

Proof. Consider an arbitrary LCP(L, K, q) with $L \in \mathcal{F}$. If z^* satisfy in (3.1), then $L(z^*) \in \operatorname{int}(K)$. Therefore, by Proposition 2.2, we have $\langle x, L(z^*)\rangle > 0$ for all $x \in K$. Then for all $q \in H$, by Archimedean property of real numbers, we find a large enough positive scalar t, such that

$$t\langle x, L(z^*)\rangle = \langle x, L(tz^*)\rangle \geq \langle x, -q\rangle,$$

and of course $tz^* \in \operatorname{int}(K)$. Thus $\langle x, L(tz^*) + q\rangle \geq 0$ for all $x \in K$. Therefore, $L(tz^*) + q \in K$ by Proposition 2.2, and so $tz^* \in \mathrm{FEA}(L, K, q)$.

Conversely, let $d \in \operatorname{int}(K)$. By assumption $\mathrm{FEA}(L, K, -d) \neq \emptyset$, that is, there exists $x \in K$ such that $y = L(x) - d \in K$. From this we get $L(x) = y + d \in K + \operatorname{int}(K) = \operatorname{int}(K)$. Hence $L \in \mathcal{F}$. □

Certainly, if a complementarity problem is solvable, then it is feasible. Given a complementarity problem, it may be feasible but not solvable. Thus, it is important to know under what conditions a feasible vector is a solution.

Theorem 3.4. *Let $L : H \to H$ be a continuous linear operator, $q \in H$ and $z \in \mathrm{FEA}(L, K, q)$. If there exists an $u \in K$ such that*

$$\langle u, L(z) + q\rangle = 0, \tag{3.3}$$

$$\langle z - u, L(z) + q\rangle \leq 0. \tag{3.4}$$

Then $z \in \mathrm{SOL}(L, K, q)$.

Proof. It follows from $z \in \mathrm{FEA}(L, K, q)$ that $z \in K$ and $L(z) + q \in K$. Thus $\langle z, L(z) + q\rangle \geq 0$. On the other hand by conditions (3.3) and (3.4) we have $\langle z, L(z) + q\rangle \leq \langle u, L(z) + q\rangle = 0$. Therefore $\langle z, L(z) + q\rangle = 0$, i.e., $z \in \mathrm{SOL}(L, K, q)$. □

The following theorems describe properties of the solution set of the LCP with a positive operator.

Theorem 3.5. *Let* $L : H \to H$ *be a positive operator and* $q \in H$ *be arbitrary. If* $\mathrm{LCP}(L, K, q)$ *has a solution, then* $\mathrm{SOL}(L, K, q)$ *equals to*

$$C = \{z \in K : L(z) + q \in K^*, \langle z - z_*, q\rangle = 0, (L + L^*)(z - z_*) = 0\},$$

where z_* *is an arbitrary solution.*

Proof. Let z_* be a given solution and z an arbitrary solution. We have $\langle z - z_*, L(z - z_*)\rangle = 0$, because L is positive and z_* and z are solutions of $\mathrm{LCP}(L, K, q)$. From this we obtain $(L + L^*)(z - z_*) = 0$. Thus, we have

$$\langle z, (L + L^*)(z)\rangle = \langle z, (L + L^*)(z_*)\rangle,$$

and

$$\langle z_*, (L + L^*)(z_*)\rangle = \langle z_*, (L + L^*)(z)\rangle.$$

The last two equalities imply that $\langle z, L(z)\rangle = \langle z_*, L(z_*)\rangle$. At the same time, we have

$$0 = \langle z_*, L(z_*) + q\rangle = \langle z, L(z) + q\rangle.$$

Consequently, $\langle z, q\rangle = \langle z_*, q\rangle$ and so $z \in C$.

Conversely, suppose that $z \in C$. To prove that z solves $\mathrm{LCP}(L, K, q)$, it suffices to show that $\langle z, L(z) + q\rangle = 0$. From $(L + L^*)(z - z_*) = 0$, by similar argument we get $\langle z, L(z)\rangle = \langle z_*, L(z_*)\rangle$. Therefore, as $\langle (z - z_*), q\rangle = 0$, we obtain $\langle z, L(z) + q\rangle = \langle z_*, L(z_*) + q\rangle = 0$, because z_* is a given solution of $\mathrm{LCP}(L, K, q)$. □

Theorem 3.6. *Let* $L : H \to H$ *be a positive operator and* $q \in H$ *be arbitrary. If* $z_1, z_2 \in \mathrm{SOL}(L, \mathbb{K}, q)$*, then*

$$\langle z_1, L(z_2) + q\rangle = \langle z_2, L(z_1) + q\rangle = 0. \tag{3.5}$$

Proof. Let $q \in H$, and z_1, z_2 be two solutions of $\mathrm{LCP}(L, \mathbb{K}, q)$. Define $z := z_1 - z_2$ and $w_i := L(z_i) + q\,(i = 1, 2)$. We have

$$0 \leqslant \langle z, L(z)\rangle = \langle z_1 - z_2, w_1 - w_2\rangle = -(\langle z_1, w_2\rangle + \langle z_2, w_1\rangle) \leqslant 0,$$

because L is positive and $z_i, w_i \in \mathbb{K}\,(i = 1, 2)$. Hence

$$\langle z_1, w_2\rangle + \langle z_2, w_1\rangle = 0.$$

We have $\langle z_1, w_2\rangle = \langle z_2, w_1\rangle = 0$, because $\langle z_1, w_2\rangle \geqslant 0$ and $\langle z_2, w_1\rangle \geqslant 0$. □

From Lemma 2.8 and Theorem 3.6, we conclude that if a linear operator L is positive, then L has cross commutative property.

It is interesting to note that in Theorem 3.5, the set $\mathrm{SOL}(L, K, q)$ is completely determined provided that one solution of $\mathrm{LCP}(L, K, q)$ is known. This points out one particular feature of a positive LCP.

Theorem 3.5 shows that for positive LCP, the scalar $\langle z, q\rangle$ and the vector $(L + L^*)(z)$ are constant for all $z \in \mathrm{SOL}(L, K, q)$.

In the following theorem, we see that $\mathrm{LCP}(L, K, q)$ of selfadjoint positive type has the property that $L(z)$ is constant for all solutions z.

Theorem 3.7. *Let $L : H \to H$ be a selfadjoint positive operator. Then $L(z)$ is constant for all $z \in \mathrm{SOL}(L, K, q)$.*

Proof. The hypotheses of this theorem include those of Theorem 3.5. The desired conclusion now follows from the selfadjointness assumption of L and the condition $(L + L^*)(z - z_*) = 0$ in the definition of the solution set C. □

The following theorem gives a characterization for the class of linear operators L with the property that $L(z)$ is constant for all $z \in \mathrm{SOL}(L, \mathbb{K}, q)$, where $\mathbb{K}$ is the Lorentz cone in H determined by e.

Theorem 3.8. *Let $L : H \to H$ be a continuous positive operator and $L(\mathbb{K}) \subseteq \mathbb{K}$. The following statements are equivalent:*

(a) *$L(z)$ is constant for all $z \in \mathrm{SOL}(L, \mathbb{K}, q)$.*

(b) *[z and $L(z)$ operator commute, $z \bullet L(z) \leqslant 0$] $\Rightarrow L(z) = 0$.*

Proof. (a) $\Rightarrow$ (b): Suppose that z and $L(z)$ operator commute and $z \bullet L(z) \leqslant 0$. We aim to prove $L(z) = 0$. To this purpose, we consider the following three cases:

Case I: $z \in \mathbb{K}$: Set $\hat{q} = L(z)^+ - L(z^+) = L(z)^- - L(z^-)$. We show that $z^+, z^- \in \mathrm{SOL}(L, \mathbb{K}, \hat{q})$. Clearly, $z^+, z^- \in \mathbb{K}$, $L(z^+) + \hat{q} = L(z)^+ \in \mathbb{K}$ and $L(z^-) + \hat{q} = L(z)^- \in \mathbb{K}$. Now, we must show that complementarity condition holds. By Lemma 2.4, $z^+ = z$ and $z^- = 0$. By assumption, $L(z) \in \mathbb{K}$. Thus, $L(z)^+ = L(z)$ and $L(z)^- = 0$. Hence, $\langle z^-, L(z)^- \rangle = 0$. By Lemma 2.9, we have $z \bullet L(z) \in \mathbb{K}$, i.e., $z \bullet L(z) \geqslant 0$. Therefore, $z \bullet L(z) = 0$. Thus, by Lemma 2.8, $\langle z, L(z) \rangle = 0$. Hence, z^+ and z^- are two solutions of $\mathrm{LCP}(L, \mathbb{K}, \hat{q})$. So, by (a), we have $L(z^+) = L(z^-)$. Therefore, $L(z) = L(z^+) - L(z^-) = 0$.

Case II: $z \in -\mathbb{K}$: In a similar way, as in the case I, we can show that z^+ and z^- are two solutions of $\mathrm{LCP}(L, \mathbb{K}, \hat{q})$, with $\hat{q} = L(z)^+ - L(z^+) = L(z)^- - L(z^-)$, and obtain the desired result.

Case III: $z \notin \mathbb{K}, z \notin -\mathbb{K}$: Let $z = x + \lambda e$ and $L(z) = y + \mu e$, where $x, y \in \langle e \rangle^\perp$ and $\lambda, \mu \in \mathbb{R}$. We obtain that $x \neq 0$ and $-\|x\| < \lambda < \|x\|$. Since z and $L(z)$ operator commute, by Lemma 2.7, we get that there is an $\alpha \in \mathbb{R}$ such that $y = \alpha x$. Note that

$$z \bullet L(z) = \mu x + \lambda y + (\langle x, y \rangle + \lambda \mu) e \leqslant 0.$$

It follows from the definition of $\mathbb{K}$ that $-(\langle x, y \rangle + \lambda \mu) \geqslant \|\mu x + \lambda y\| \geqslant 0$. This, together with the positivity of L, imply that $\langle x, y \rangle + \lambda \mu = 0$ and $\mu x + \lambda y = 0$. By $y = \alpha x$, it is easy to verify that $\mu = -\alpha \lambda$. This and $\langle x, y \rangle + \lambda \mu = 0$ imply that $\alpha = 0$. Furthermore, we have $L(z) = y + \mu e$. Thus $L(z) = 0$.

(b) $\Rightarrow$ (a): Suppose that z_1 and z_2 are two distinct solutions of $\mathrm{LCP}(L, \mathbb{K}, q)$ and $w_i = L(z_i) + q\,(i = 1, 2)$. Define $z := z_1 - z_2$. By Lemma 2.8 and Theorem 3.6, we conclude that z_1 (resp. z_2) operator commute with w_2 (resp. w_1). It is easy to verify that z and $L(z)$ are operator commute, and

$$z \bullet L(z) = (z_1 - z_2) \bullet (w_1 - w_2) = -(z_1 \bullet w_2 + z_2 \bullet w_1) \leqslant 0.$$

Hence by (b), we have $L(z) = 0$, and so $L(z_1) = L(z_2)$. □

References

[1] H.H. Bauschke and P.L. Combettes, Convex Analysis and Monotone Operator Theory in Hilbert Spaces, Springer, (2011).

[2] J.M. Borwein, *Generalized linear complementarity problems treated without fixed-point theory*, J. Optim. Theory. Appl., **43** (1984), 343–356.

[3] Y.J. Cho, J. Li and N.-J. Huang, *Solvability of implicit complementarity problems*, Math. Comput. Model., **45** (2007), 1001–1009.

[4] R.W. Cottle, J.S. Pang and R.E. Stone, The Linear Complementarity Problem, Classics in Applied Mathematics, Society for Industrial & Applied Mathematics, (2009).

[5] F. Facchinei, J.S. Pang, Finite dimensional variational inequalities and complementarity problems I, Springer, New York (2003).

[6] J. Faraut and A. Koranyi, Analysis on Symmetric Cones, Oxford University Press Inc., New York (1994).

[7] G. Isac, Topological Methods in Complementarity Theory, Nonconvex Optimization and Its Applications, Springer, (2000).

[8] C. Kanzow, *Global convergence properties of some iterative methods for linear complementarity problems*, SIAM J. Optim., **6** (1996), 326–341.

[9] Y.Y. Lin and J.S. Pang, *Iterative methods for larg convex quadratic programs: a survey*, SIAM J. Control Optim., **25** (1987), 383–411.

[10] X.H. Miao and J.S. Chen, *On the lorentz cone complementarity problems in infinite-dimensional real Hilbert spaces*, Numer. Funct. Anal. Optim., **32** (2011), 507–523.

[11] X.H. Miao and Z.H. Huang, *GUS-property for Lorentz cone linear complementarity problems on Hilbert spaces*, Sci. China Math., **54** (2011), 1259–1268.

[12] X.H. Miao, Z.H. Huang and J.Y. Han, *Some w-unique and w-P properties for linear transformation on Hilbert spaces*, Acta Math. Appl. Sin. Engl. Ser., **26** (2010), 23–32.

[13] K.G. Murty, Linear Complementarity, Linear and Non-linear Programming, Sigma Series in Applied Mathematics, Heldermann Verlag, Berlin, (1988).

[14] J.S. Pang and D. Chan, *Iterative methods for variational and complementarity problems*, Math. Programm., **27** (1982), 284–313.

[15] L.X. Qin, L.C. Kong and J. Han, *Sufficiency of linear transformations on Euclidean Jordan algebras*, Optim. Lett., **3** (2009), 265–276.