Nonlinear Functional Analysis and Applications
Vol. 25, No. 3 (2020), pp. 411-436
ISSN: 1229-1595(print), 2466-0973(online)
https://doi.org/10.22771/nfaa.2020.25.03.02
http://nfaa.kyungnam.ac.kr/journal-nfaa
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FIXED POINT THEOREMS IN COMPLEX VALUED BANACH SPACES WITH APPLICATIONS TO A NONLINEAR INTEGRAL EQUATION

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Abstract. We approximate the common fixed point of two nonself operators satisfying rational contractive conditions via Jungck-type iterative schemes in complex valued Banach spaces. Moreover, we prove that the Jungck-CR iterative sequence is (S, T)-stable and give some numerical examples to validate our analytical results. Furthermore, we apply our results in solving certain mixed type Volterra-Fredholm functional nonlinear integral equation.

1. Introduction

Fixed point theory became one of the most interesting area of research in the last sixty years, for instance it has shown the importance of theoretical subjects, which are directly applicable in different applied fields of science. Other areas of applications include optimization problems, control theory, economics

[0]Received July 21, 2019. Revised November 20, 2019. Accepted November 21, 2019.
[0]2010 Mathematics Subject Classification: 47H09, 47H10, 49M05, 54H25.
[0]Keywords: Complex valued Banach spaces, common fixed point, weakly compatible mappings, coincidence point of mappings S and T, strong convergence, Jungck-type iterative process, (S, T)-stable, mixed type Volterra-Fredholm functional nonlinear integral equation.
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and a host of others. In particular, it plays an important role in the investigation of existence of solutions to differential and integral equations, which direct the behaviour of several real life problems for which the existence of solution is critical (see, e.g. [15], [44]). In 1922, Banach [7] provided a general iterative method to construct a fixed point result and proved its uniqueness under linear contraction in complete metric spaces.

The notion of complex valued metric spaces was introduced by Azam *et al.* [6] in 2011. They established some fixed point theorems for a pair of mappings satisfying rational inequality. Their results is intended to define rational expressions which are meaningless in cone metric spaces, hence results in this direction cannot be generalized to cone metric spaces, but to complex valued metric spaces. It is known that complex valued metric space is useful in many branches of Mathematics, including number theory, algebraic geometry, applied Mathematics as well as in physics including hydrodynamics, mechanical engineering, thermodynamics and electrical engineering (see, e.g. [43]). Several authors have obtained interesting and applicable results in complex valued metric spaces (see, e.g. [2], [3], [6], [16], [32, 33], [37], [38], [43], [44]).

It is known that there is a close relationship between the problem of solving a nonlinear equation and that of approximating fixed points of a corresponding contractive type operator (see, e.g. [8], [9], [29], [31], [40]). Hence, there is a practical and theoretical interests in approximating fixed points of several contractive type operators. Since, the introduction of the notion of complex valued metric spaces by Azam *et al.* [6] in 2011, most results obtained in literature by many authors are existential in nature (see, e.g. [2], [3], [6], [12], [16], [37], [38], [43], [44]). Consequently, there is a gap in literature with respect to the approximation of the fixed point of several nonlinear mappings in this type of space. Recently, Okeke [29] initiated the idea of approximating the fixed point of nonlinear mappings in complex valued Banach spaces. It is our purpose in the present study to continue in this research direction and thereby filling this gap that exist in literature. We study the approximation of fixed points of some mappings satisfying certain contractive conditions of rational type in complex valued Banach spaces. We approximate the common fixed point of two nonself operators satisfying rational contractive conditions via Jungck-type iterative schemes in complex valued Banach spaces. Moreover, we prove that the Jungck-CR iterative sequence is (S, T)-stable and give some numerical examples to validate our analytical results. Furthermore, we apply our results in solving certain mixed type Volterra-Fredholm functional nonlinear integral equation.

The theory of integral and differential equations is an important aspect of nonlinear analysis and the most applied tool for proving the existence of the solutions of such equations is the fixed point technique (see, e.g. [7], [11], [13], [30]). One of the most frequent and difficult problems faced by scientists in mathematical sciences is nonlinear problems. This is because nature is intrinsically nonlinear (see, e.g. [13]). Solving nonlinear equations is cumbersome but important to mathematicians and applied mathematicians such as engineers and physicist. Some authors have used the fixed point iterative methods in solving such equations (see, e.g. [11], [13], [30]).

In this paper, we apply our results in solving certain mixed type Volterra-Fredholm functional nonlinear integral equation in complex valued Banach spaces. Our results unify, generalize and extend several known results to complex valued Banach spaces, including the results of [4], [5], [11], [13], [14], [17], [25], [26], [27], [28], [30], [36] among others.

2. Preliminaries

The following symbols, notations and definitions which can be found in [6] will be useful in this study.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2),\ Im(z_1) \leq Im(z_2).$$

It follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2),\ Im(z_1) < Im(z_2)$,
(ii) $Re(z_1) < Re(z_2),\ Im(z_1) = Im(z_2)$,
(iii) $Re(z_1) < Re(z_2),\ Im(z_1) < Im(z_2)$,
(iv) $Re(z_1) = Re(z_2),\ Im(z_1) = Im(z_2)$.

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii), and (iii) is satisfied and we will write $z_1 \prec z_2$ if only (iii) is satisfied.

Note that

(a) $a, b \in \mathbb{R}$ and $a \leq b \implies az \precsim bz$ for all $z \in \mathbb{C}$;
(b) $0 \precsim z_1 \precnsim z_2 \implies |z_1| < |z_2|$;
(c) $z_1 \precsim z_2$ and $z_2 \prec z_3 \implies z_1 \prec z_3$.

Definition 2.1. ([6]) Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies:

(1) $0 \precsim d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, y) \precsim d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a complex valued metric on X, and (X, d) is called a complex valued metric space.

Recently, Okeke [29] defined a complex valued Banach space and proved some interesting fixed point theorems in the framework of complex valued Banach spaces.

Definition 2.2. ([29]) Let E be a linear space over a field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ (the set of real numbers) or $\mathbb{C}$ (the set of complex numbers). A complex valued norm on E is a complex valued function $\|.\| : E \to \mathbb{C}$ satisfying the following conditions:

(1) $\|x\| = 0$ if and only if $x = 0,\ x \in E$;
(2) $\|kx\| = |k|.\|x\|$ for all $k \in \mathbb{K},\ x \in E$;
(3) $\|x + y\| \precsim \|x\| + \|y\|$ for all $x, y \in E$.

A linear space with a complex valued norm defined on it is called a *complex valued normed linear space,* denoted by $(E, \|.\|)$. A point $x \in E$ is called an *interior point* of a set $A \subseteq E$ if there exist $0 \prec r \in \mathbb{C}$ such that

$$B(x, r) = \{y \in E : \|x - y\| \prec r\} \subseteq A.$$

A point $x \in E$ is called a limit point of the set A whenever for each $0 \prec r \in \mathbb{C}$, we have

$$B(x, r) \cap (AnE) \neq \emptyset.$$

The set A is said to be open if each element of A is an interior point of A. A subset $B \subseteq E$ is said to be closed if it contains each of its limit point. The family

$$F = \{B(x, r) : x \in E,\ 0 \prec r\}$$

is a sub-basis for a Hausdorff topology τ on E.

Suppose $\{x_n\}$ is a sequence in E and $x \in E$. $\{x_n\}$ is called convergent to x, if for every $c \in \mathbb{C}$, with $0 \prec c$ there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $\|x_n - x\| \prec c$. In this case, we denoted by $x_n \to x$ or $\lim_{n\to\infty} x_n = x$, and x is called the limit of $\{x_n\}$.

$\{x_n\}$ is called a Cauchy sequence in $(E, \|.\|)$, if for all $c \in \mathbb{C}$, with $0 \prec c$ there exists $n_0 \in \mathbb{N}$ such that for all $n, m > n_0$, $\|x_n - x_m\| \prec c$, If every Cauchy sequence is convergent in $(E, \|.\|)$, then $(E, \|.\|)$ is called a complex valued Banach space.

Example 2.3. ([29]) Let $E = \mathbb{C}$ be the set of complex numbers. Define $\|.\| : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by

$$\|z_1 - z_2\| = |x_1 - x_2| + i|y_1 - y_2|,\ \forall z_1, z_2 \in \mathbb{C},$$

where $z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2$. Clearly, $(\mathbb{C}, \|.\|)$ is a complex valued normed linear space.

Example 2.4. ([29]) Let $E = \mathbb{C}$ be the set of complex numbers. Define a mapping $\|.\| : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by

$$\|z_1 - z_2\| = e^{ik}|z_1 - z_2|, \ \forall z_1, z_2 \in \mathbb{C},$$

where $k \in [0, \frac{\pi}{2}]$, $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$. Then $(\mathbb{C}, \|.\|)$ is a complex valued normed linear space.

Example 2.5. ([29]) Let $(C[a,b], \|.\|_\infty)$ be the space of all continuous complex valued functions on a closed interval $[a,b]$, endowed with the Chebyshev norm

$$\|x - y\|_\infty = \max_{t \in [a,b]} |x(t) - y(t)|e^{ik}, \ x, y \in C[a,b], \ k \in [0, \frac{\pi}{2}].$$

Then $(C[a,b], \|.\|_\infty)$ is a complex valued Banach space, since the elements of $C[a,b]$ are continuous functions, and convergence with respect to the Chebyshev norm $\|.\|_\infty$ corresponds to uniform convergence. We can easily show that every Cauchy sequence of continuous functions converges to a continuous function, i.e. an element of the space $C[a,b]$.

The following lemma will be useful in this study.

Lemma 2.6. ([29]) *Let $(E, \|.\|)$ be a complex valued Banach space and let $\{x_n\}$ be a sequence in E. Then $\{x_n\}$ converges to x if and only if $|\|x_n - x\|| \to 0$ as $n \to \infty$.*

Lemma 2.7. ([29]) *Let $(E, \|.\|)$ be a complex valued Banach space and let $\{x_n\}$ be a sequence in E. Then $\{x_n\}$ is a Cauchy sequence if and only if $|\|x_n - x_{n+m}\|| \to 0$ as $n \to \infty$.*

Definition 2.8. ([9]) A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called a comparison function if it satisfies the following conditions:

(i) ψ is monotone increasing;
(ii) $\lim_{n\to\infty} \psi^n(t) = 0$, $\forall t \geq 0$.

Remark 2.9. Every comparison function satisfies $\psi(0) = 0$ (see, e.g. [36]).

Olatinwo [36] proved some strong convergence and stability results using the following contractive conditions satisfying rational inequality.

Condition 2.10. ([36]) *For two nonself mappings $S, T : Y \to E$ with $T(Y) \subseteq S(Y)$, where $S(Y)$ is a complete subspace of E, there exist:*

(1) *a real number $L \geq 0$, a sublinear comparison function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ and a monotone increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(0) = 0$ and for all $x, y \in Y$, we have*

$$\|Tx - Ty\| \leq \frac{\varphi(\|Sx - Tx\|) + \psi(\|Sx - Sy\|)}{1 + L\|Sx - Tx\|}; \tag{2.1}$$

(2) *real numbers* $k \geq 0$, $L \geq 0$, $a \in [0,1)$ *and a monotone increasing function* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\varphi(0) = 0$ *and for all* $x, y \in Y$, *we have*

$$\|Tx - Ty\| \leq \left(\frac{\varphi(\|Sx - Tx\|) + a\|Sx - Sy\|}{1 + L\|Sx - Tx\|} \right) e^{k\|Sx - Tx\|}. \tag{2.2}$$

Definition 2.11. Let S and T be self mappings of a nonempty set X.

(i) A point $x \in X$ is said to be a fixed point of T if $Tx = x$.
(ii) A point $x \in X$ is said to be a coincidence point of S and T if $Sx = Tx$ and point w is called a point of coincidence of S and T if $w = Sx = Tx$.
(iii) A point $x \in X$ is said to be a common fixed point of S and T if $x = Sx = Tx$.

In 1976, Jungck [19] introduced concept of commuting mappings as follows:

Definition 2.12. ([19]) Let X be a non-empty set. The mappings S and T are commuting if $TSx = STx$ for all $x \in X$.

The concept of weakly commuting mappings which are more general than commuting mappings was introduced by Sessa [41] as follows:

Definition 2.13. ([41]) Let S and T be mappings from a metric space (X, d) into itself. The mappings S and T are said to be weakly commuting if

$$d(STx, TSx) \leq d(Sx, Tx)$$

for all $x \in X$.

Jungck [20] introduced the more generalized commuting mappings in metric spaces, called compatible mappings, which also are more general than the concept of weakly commuting mappings as follows:

Definition 2.14. ([20]) Let S and T be mappings from a metric space (X, d) into itself. The mapping S and T are said to be compatible if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

Remark 2.15. Generally, commuting mappings are weakly commuting and weakly commuting mappings are compatible, but the converses are not necessarily true and some examples can be found in [19], [20], [21], [22].

In 1996, Jungck [23] introduced the concept of weakly compatible mappings as follows:

Definition 2.16. ([23]) Let S and T be self mappings of a nonempty set X. The mappings S and T are said to be weakly compatible if $STx = TSx$ whenever $Sx = Tx$.

There exists weakly compatible mappings which are not compatible mappings in metric spaces (see, e.g. [24]).

Suppose that X is a Banach space, Y an arbitrary set, and $S, T : Y \to X$ such that $T(Y) \subseteq S(Y)$. For $x_0 \in Y$, consider the following iterative process:

$$Sx_{n+1} = Tx_n, \ n = 0, 1, 2, \cdots . \tag{2.3}$$

The iterative process (2.3) is called Jungck iterative process and was introduced in 1976 by Jungck [19]. If $S = I_d$ (identity mapping) and $Y = X$, then the Jungck iteration process (2.3) becomes the Picard iterative process.

Given $\alpha_n \in [0, 1]$, Singh *et al.* [42] defined the Jungck-Mann iterative process as follows:

$$Sx_{n+1} = (1 - \alpha_n)Sx_n + \alpha_n Tx_n. \tag{2.4}$$

For $\alpha_n, \beta_n, \gamma_n \in [0, 1]$, Olatinwo [34] defined the Jungck-Ishikawa and Olatinwo [35] defined the Junck-Noor iterative processes, respectively as follows:

$$\begin{cases} Sx_{n+1} = (1 - \alpha_n)Sx_n + \alpha_n Ty_n \\ Sy_n = (1 - \beta_n)Sx_n + \beta_n Tx_n, \end{cases} \tag{2.5}$$

$$\begin{cases} Sx_{n+1} = (1 - \alpha_n)Sx_n + \alpha_n Ty_n \\ Sy_n = (1 - \beta_n)Sx_n + \beta_n Tz_n \\ Sz_n = (1 - \gamma_n)Sx_n + \gamma_n Tx_n. \end{cases} \tag{2.6}$$

In 2013, Hussain *et al.* [14] introduced the Jungck-CR iterative process, which is the Jungck version of the CR iterative process introduced by Chugh *et al.* [10]. The following is the Jungck-CR iterative process as defined in [14],

$$\begin{cases} Sx_{n+1} = (1 - \alpha_n)Sy_n + \alpha_n Ty_n \\ Sy_n = (1 - \beta_n)Tx_n + \beta_n Tz_n \\ Sz_n = (1 - \gamma_n)Sx_n + \gamma_n Tx_n. \end{cases} \tag{2.7}$$

Remark 2.17. If we put $\alpha_n = 0$ for all $n \in \mathbb{N}$ and $\alpha_n = 0$, $\beta_n = 1$ for all $n \in \mathbb{N}$ in Jungck-CR iterative process (2.7), then we obtain the Jungck versions of Agarwal *et al.* [1] and Sahu and Petrusel [39], respectively as follows:

$$\begin{cases} Sx_{n+1} = (1 - \beta_n)Tx_n + \beta_n Ty_n \\ Sy_n = (1 - \gamma_n)Sx_n + \gamma_n Tx_n, \end{cases} \tag{2.8}$$

$$\begin{cases} Sx_{n+1} = Ty_n \\ Sy_n = (1 - \gamma_n)Sx_n + \gamma_n Tx_n. \end{cases} \tag{2.9}$$

They proved analytically and with some numerical examples that the Jungck-CR iterative process converges faster in the sense of Berinde [9] than all of

Jungck-S iterative process, Jungck-Agarwal iterative process, Jungck-SP iterative process, Jungck-Noor iterative process and Jungck-Ishikawa iterative process (see [14]).

In this paper, we approximate the common fixed point of two nonlinear operators satisfying contractive conditions (2.1) and (2.2) in complex valued Banach spaces via Jungck-CR iterative process and other Jungck-type iterations.

Definition 2.18. ([9]) Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be two sequences of positive numbers that converge to a, respectively b. Assume there exists

$$l = \lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|}. \tag{2.10}$$

(1) If $l = 0$, then it is said that the sequence $\{a_n\}_{n=0}^{\infty}$ converges to a faster than the sequence $\{b_n\}_{n=0}^{\infty}$ to b.
(2) If $0 < l < \infty$, then we say that the sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ have the same rate of convergence.

Definition 2.19. ([42]) Let $S, T : Y \to X$ be nonself operators for an arbitrary set Y such that $T(Y) \subseteq S(Y)$ and p be a point of coincidence of S and T. Let $\{Sx_n\}_{n=0}^{\infty} \subset X$, be the sequence generated by an iterative procedure

$$Sx_{n+1} = f(T, x_n),\ n = 0, 1, 2, \cdots, \tag{2.11}$$

where $x_0 \in X$ is the initial approximation and f is some function. Suppose that $\{Sx_n\}_{n=0}^{\infty}$ converges to p. Let $\{Sy_n\}_{n=0}^{\infty} \subset X$ be an arbitrary sequence and set

$$\varepsilon_n = d(Sy_n, f(T, y_n)),\ n = 0, 1, 2, \cdots. \tag{2.12}$$

Then, the iterative procedure (2.11) is said to be (S, T)-stable or stable if $\lim_{n\to\infty} \varepsilon_n = 0$ then $\lim_{n\to\infty} Sy_n = p$.

Lemma 2.20. ([9]) *If δ is a real number such that $0 \leq \delta < 1$ and $\{\epsilon_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \epsilon_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying*

$$u_{n+1} \leq \delta u_n + \epsilon_n,\ n = 0, 1, 2, \cdots,$$

we have $\lim_{n\to\infty} u_n = 0$.

Lemma 2.21. ([18]) *If $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a subadditive comparison function and $\{\epsilon_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \epsilon_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying*

$$u_{n+1} \leq \sum_{k=0}^{m} \delta_k \psi^k(u_n) + \epsilon_n,\ n = 0, 1, 2, \cdots,$$

where $\delta_0, \delta_1, \cdots, \delta_m \in [0,1]$ *with* $0 \leq \sum_{k=0}^{m} \delta_k \leq 1$*, we have* $\lim_{n\to\infty} u_n = 0$.

3. Strong convergence theorems in complex valued Banach spaces

In this section, we approximate the common fixed point of two generalized operators satisfying contractive conditions(cf. Condition 2.10) of rational type via some Jungck-type iterative processes. Our results unify, extend and generalize several known results from real Banach spaces to complex valued Banach spaces.

Theorem 3.1. *Let* D *be a nonempty closed convex subset of a complex valued Banach space* $(E, \|.\|)$ *and let* $S, T : D \to E$ *be nonself operators on* D *satisfying the following contractive condition:*

(C) *For all real numbers* $k \geq 0$, $L \geq 0$, $a \in [0,1)$ *and a monotone increasing function* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\varphi(0) = 0$ *and for all* $x, y \in D$*, we have*

$$\|Tx - Ty\| \precsim \left(\frac{\varphi(\|Sx - Tx\|) + a\|Sx - Sy\|}{1 + L\|Sx - Tx\|}\right) e^{k\|Sx - Tx\|}. \tag{3.1}$$

Assume that $T(D) \subseteq S(D)$*,* $S(D) \subseteq E$ *is a complex valued Banach space and* $Sx^* = Tx^* = p$ *(say). For* $x_0 \in D$*, let* $\{Sx_n\}_{n=0}^{\infty}$ *be the Jungck-CR iterative process* (2.7)*, where* $\{\alpha_n\}$*,* $\{\beta_n\}$*,* $\{\gamma_n\}$ *are sequences of positive numbers in* $[0,1]$ *with* $\{\alpha_n\}$ *satisfying* $\sum_{n=0}^{\infty} \alpha_n = \infty$*. Then the Jungck-CR iterative process* $\{Sx_n\}_{n=0}^{\infty}$ *converges strongly to* p*. Moreover,* p *will be the unique common fixed point of* S, T *provided* $D = E$*, and* S *and* T *are weakly compatible.*

Proof. We want to prove that the Jungck-CR iterative process (2.7) converges strongly to p. Using (2.7) and (3.1), we have

$$\begin{aligned}
\|Sx_{n+1} - p\| &= \|(1-\alpha_n)Sy_n + \alpha_n Ty_n - p\| \\
&\precsim (1-\alpha_n)\|Sy_n - p\| + \alpha_n\|Ty_n - p\| \\
&= (1-\alpha_n)\|Sy_n - p\| + \alpha_n\|Tx^* - Ty_n\| \\
&\precsim (1-\alpha_n)\|Sy_n - p\| + \alpha_n \left(\tfrac{\varphi(\|Sx^* - Tx^*\|) + a\|Sx^* - Sy_n\|}{1 + L\|Sx^* - Tx^*\|}\right) \\
&\quad \times e^{k\|Sx^* - Tx^*\|} \\
&= (1-\alpha_n)\|Sy_n - p\| + \alpha_n \left(\tfrac{\varphi(\|0\|) + a\|Sy_n - p\|}{1 + L\|0\|}\right) e^{k\|0\|} \\
&= (1-\alpha_n)\|Sy_n - p\| + a\alpha_n\|Sy_n - p\| \\
&= (1 - \alpha_n(1-a))\|Sy_n - p\|.
\end{aligned} \tag{3.2}$$

Next, we obtain the following estimate.

$$\begin{aligned}
\|Sx_n - p\| &= \|(1-\beta_n)Tx_n + \beta_n Tz_n - p\| \\
&\precsim (1-\beta_n)\|Tx_n - p\| + \beta_n\|Tz_n - p\| \\
&= (1-\beta_n)\|Tx^* - Tx_n\| + \beta_n\|Tx^* - Tz_n\|
\end{aligned}$$

$$\begin{aligned}
&\precsim (1-\beta_n)\left(\tfrac{\varphi(\|Sx^*-Tx^*\|)+a\|Sx^*-Sx_n\|}{1+L\|Sx^*-Tx^*\|}\right)e^{k\|Sx^*-Tx^*\|} \\
&\quad +\beta_n\left(\tfrac{\varphi(\|Sx^*-Tx^*\|)+a\|Sx^*-Sz_n\|}{1+L\|Sx^*-Tx^*\|}\right)e^{k\|Sx^*-Tx^*\|} \\
&= (1-\beta_n)\left(\tfrac{\varphi(\|0\|)+a\|Sx_n-p\|}{1+L\|0\|}\right)e^{k\|0\|}+\beta_n\left(\tfrac{\varphi(\|0\|)+a\|Sz_n-p\|}{1+L\|0\|}\right)e^{k\|0\|} \\
&= a(1-\beta_n)\|Sx_n-p\|+a\beta_n\|Sz_n-p\|. \qquad (3.3)
\end{aligned}$$

Similarly, we have the following estimate,

$$\begin{aligned}
\|Sz_n-p\| &= \|(1-\gamma_n)Sx_n+\gamma_nTx_n-p\| \\
&\precsim (1-\gamma_n)\|Sx_n-p\|+\gamma_n\|Tx_n-p\| \\
&= (1-\gamma_n)\|Sx_n-p\|+\gamma_n\|Tx^*-Tx_n\| \\
&\precsim (1-\gamma_n)\|Sx_n-p\| \\
&\quad +\gamma_n\left(\tfrac{\varphi(\|Sx^*-Tx^*\|)+a\|Sx^*-Sx_n\|}{1+L\|Sx^*-Tx^*\|}\right)e^{k\|Sx^*-Tx^*\|} \\
&= (1-\gamma_n)\|Sx_n-p\|+\gamma_n\left(\tfrac{\varphi(\|0\|)+a\|Sx_n-p\|}{1+L\|0\|}\right)e^{k\|0\|} \\
&= (1-\gamma_n)\|Sx_n-p\|+a\gamma_n\|Sx_n-p\| \\
&= (1-\gamma_n(1-a))\|Sx_n-p\|. \qquad (3.4)
\end{aligned}$$

Using (3.4) in (3.3), we have

$$\begin{aligned}
\|Sy_n-p\| &\precsim a(1-\beta_n)\|Sx_n-p\|+a\beta_n(1-\gamma_n(1-a))\|Sx_n-p\| \\
&\precsim (1-\beta_n)\|Sx_n-p\|+a\beta_n\|Sx_n-p\| \\
&= (1-\beta_n(1-a))\|Sx_n-p\|. \qquad (3.5)
\end{aligned}$$

Using (3.5) in (3.2), we have

$$\begin{aligned}
\|Sx_{n+1}-p\| &\precsim (1-\alpha_n(1-a))(1-\beta_n(1-a))\|Sx_n-p\| \\
&\precsim (1-\alpha_n(1-a))\|Sx_n-p\|. \qquad (3.6)
\end{aligned}$$

Using the fact that $(1-\alpha_n(1-a))<1$, we obtain the following inequalities from relation (3.6).

$$\begin{cases}
\|Sx_{n+1}-p\| \precsim (1-\alpha_n(1-a))\|Sx_n-p\|, \\
\|Sx_n-p\| \precsim (1-\alpha_{n-1}(1-a))\|Sx_{n-1}-p\|, \\
\|Sx_{n-1}-p\| \precsim (1-\alpha_{n-2}(1-a))\|Sx_{n-2}-p\|, \\
\quad\vdots \\
\|Sx_1-p\| \precsim (1-\alpha_0(1-a))\|Sx_0-p\|.
\end{cases} \qquad (3.7)$$

From relation (3.7), we have

$$\|Sx_{n+1}-p\| \precsim \|Sx_0-p\|\prod_{k=0}^{n}(1-\alpha_k(1-a)), \qquad (3.8)$$

where $(1-\alpha_k(1-a))\in(0,1)$, since $\alpha_k\in[0,1]$ for all $k\in\mathbb{N}$. It is well known in classical analysis that $1-x\le e^{-x}$ for all $x\in[0,1]$. Using these facts together with relation (3.8), we have

$$\|Sx_{n+1} - p\| \precsim \frac{\|Sx_0 - p\|}{e^{(1-a)\sum_{k=0}^{n}\alpha_k}}. \tag{3.9}$$

Therefore,

$$\lim_{n\to\infty} |\|Sx_{n+1} - p\|| \leq \left(\frac{|\|Sx_0 - p\||}{|e^{(1-a)\sum_{k=0}^{n}\alpha_k}|}\right) \longrightarrow 0 \text{ as } n \to \infty. \tag{3.10}$$

Using Lemma 2.6, this means that $\lim_{n\to\infty} \|Sx_{n+1} - p\| = 0$. Therefore, $\{Sx_n\}_{n=0}^{\infty}$ converges strogly to p.

Next, we prove that p is the unique common fixed point of S and T.

Suppose that p^* is another coincidence point of S and T. Then, there exists $y^* \in D$ such that $Sy^* = Ty^* = p^*$. Using (3.1), we have

$$\begin{aligned} 0 \preceq \|p - p^*\| &= \|Ty - Ty^*\| \\ &\precsim \left(\frac{\varphi(\|Sy-Ty\|)+a\|Sy-Sy^*\|}{1+L\|Sy-Ty\|}\right) e^{k\|Sy-Ty\|} \\ &= \left(\frac{\varphi(\|0\|)+a\|Sy-Sy^*\|}{1+L\|0\|}\right) e^{k\|0\|} \\ &= a\|p - p^*\|. \end{aligned} \tag{3.11}$$

Therefore,

$$0 \leq |\|p - p^*\|| \leq |a\|p - p^*\||. \tag{3.12}$$

Since $a \in [0, 1)$, by Lemma 2.6, we have $p = p^*$.

Since S and T are weakly compatible and $p = Ty = Sy$, so that $Tp = TTy = TSy = STy$ and hence $Tp = Sp = p$, and therefore p is the unique common fixed point of S and T. The proof of Theorem 3.1 is completed. □

Remark 3.2. Theorem 3.1 extends and generalizes several known results including the results of Hussain *et al.* [14] from real Banach spaces to complex valued Banach spaces.

Next, we obtain the following corollaries as consequencies of Theorem 3.1.

Corollary 3.3. *Let D be a nonempty closed convex subset of a complex valued Banach space $(E, \|.\|)$ and let $S, T : D \to E$ be nonself operators on D satisfying the following contractive condition:*

(C) *For all real numbers $k \geq 0$, $L \geq 0$, $a \in [0, 1)$ and a monotone increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(0) = 0$ and for all $x, y \in D$, we have*

$$\|Tx - Ty\| \precsim \left(\frac{\varphi(\|Sx - Tx\|) + a\|Sx - Sy\|}{1 + L\|Sx - Tx\|}\right) e^{k\|Sx-Tx\|}. \tag{3.13}$$

Assume that $T(D) \subseteq S(D)$, $S(D) \subseteq E$ is a complex valued Banach space and $Sx^* = Tx^* = p$ (say). For $x_0 \in D$, let $\{Sx_n\}_{n=0}^{\infty}$ be the iterative process (2.8), *where $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences of positive numbers in* $[0, 1]$ *with $\{\beta_n\}$*

satisfying $\sum_{n=0}^{\infty}\beta_n=\infty$. *Then* $\{Sx_n\}_{n=0}^{\infty}$ *converges strongly to* p. *Moreover,* p *will be the unique common fixed point of* S,T *provided* $D=E$, *and* S *and* T *are weakly compatible.*

Proof. The proof of Corollary 3.3 follows the same lines as in the proof of Theorem 3.1. □

Corollary 3.4. *Let* D *be a nonempty closed convex subset of a complex valued Banach space* $(E,\|.\|)$ *and let* $S,T:D\to E$ *be nonself operators on* D *satisfying the following contractive condition:*

(C) *For all real numbers* $k\geq 0$, $L\geq 0$, $a\in[0,1)$ *and a monotone increasing function* $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ *such that* $\varphi(0)=0$ *and for all* $x,y\in D$, *we have*

$$\|Tx-Ty\|\precsim\left(\frac{\varphi(\|Sx-Tx\|)+a\|Sx-Sy\|}{1+L\|Sx-Tx\|}\right)e^{k\|Sx-Tx\|}. \tag{3.14}$$

Assume that $T(D)\subseteq S(D)$, $S(D)\subseteq E$ *is a complex valued Banach space and* $Sx^*=Tx^*=p$ *(say). For* $x_0\in D$, *let* $\{Sx_n\}_{n=0}^{\infty}$ *be the iterative process* (2.9), *where* $\{\gamma_n\}$ *is a sequence of positive number in* $[0,1]$ *with* $\{\gamma_n\}$ *satisfying* $\sum_{n=0}^{\infty}\gamma_n=\infty$. *Then* $\{Sx_n\}_{n=0}^{\infty}$ *converges strongly to* p. *Moreover,* p *will be the unique common fixed point of* S,T *provided* $D=E$, *and* S *and* T *are weakly compatible.*

Proof. The proof of Corollary 3.4 follows the same lines as in the proof of Theorem 3.1. □

Theorem 3.5. *Let* D *be a nonempty closed convex subset of a complex valued Banach space* $(E,\|.\|)$ *and let* $S,T:D\to E$ *be nonself operators on* D *satisfying the following contractive condition:*

(C) *For a real number* $L\geq 0$, *a sublinear comparison function* $\psi:\mathbb{R}^+\to\mathbb{R}^+$ *and a monotone increasing function* $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ *such that* $\varphi(0)=0$ *and for all* $x,y\in D$, *we have*

$$\|Tx-Ty\|\precsim\frac{\varphi(\|Sx-Tx\|)+\psi(\|Sx-Sy\|)}{1+L\|Sx-Tx\|}. \tag{3.15}$$

Assume that $T(D)\subseteq S(D)$, $S(D)\subseteq E$ *is a complex valued Banach space and* $Sz=Tz=p$ *(say). For* $x_0\in D$, *let* $\{Sx_n\}_{n=0}^{\infty}$ *be the Jungck-Ishikawa iterative process* (2.5), *where* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences of positive numbers in* $[0,1]$ *with* $\{\alpha_n\}$ *satisfying* $\sum_{n=0}^{\infty}\alpha_n=\infty$. *Then* $\{Sx_n\}_{n=0}^{\infty}$ *converges strongly to* p. *Moreover,* p *will be the unique common fixed point of* S,T *provided* $D=E$, *and* S *and* T *are weakly compatible.*

Proof. We now prove that the Jungck-Ishikawa iterative process $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p. Using (2.5) and (3.15), we have

$$
\begin{array}{rcl}
\|Sx_{n+1}-p\| & = & \|(1-\alpha_n)Sx_n+\alpha_n Ty_n-p\| \\
& \precsim & (1-\alpha_n)\|Sx_n-p\|+\alpha_n\|Ty_n-p\| \\
& = & (1-\alpha_n)\|Sx_n-p\|+\alpha_n\|Tz-Ty_n\| \\
& \precsim & (1-\alpha_n)\|Sx_n-p\|+\alpha_n\left(\frac{\varphi(\|Sz-Tz\|)+\psi(\|Sz-Sy_n\|)}{1+L\|Sz-Tz\|}\right) \\
& = & (1-\alpha_n)\|Sx_n-p\|+\alpha_n\left(\frac{\varphi(\|0\|)+\psi(\|p-Sy_n\|)}{1+L\|0\|}\right) \\
& = & (1-\alpha_n)\|Sx_n-p\|+\alpha_n\psi(\|Sy_n-p\|) \\
& = & (1-\alpha_n)\|Sx_n-p\|+\alpha_n\psi(\|(1-\beta_n)Sx_n+\beta_nTx_n-p\|) \\
& \precsim & (1-\alpha_n)\|Sx_n-p\|+\alpha_n(1-\beta_n)\psi(\|Sx_n-p\|) \\
& & +\alpha_n\beta_n\psi(\|Tx_n-p\|) \\
& = & (1-\alpha_n)\|Sx_n-p\|+\alpha_n(1-\beta_n)\psi(\|Sx_n-p\|) \\
& & +\alpha_n\beta_n\psi(\|Tz-Tx_n\|) \\
& \precsim & (1-\alpha_n)\|Sx_n-p\|+\alpha_n(1-\beta_n)\psi(\|Sx_n-p\|) \\
& & +\alpha_n\beta_n\psi\left(\frac{\varphi(\|Sz-Tz\|)+\psi(\|Sz-Sx_n\|)}{1+L\|Sz-Tz\|}\right) \\
& = & (1-\alpha_n)\|Sx_n-p\|+\alpha_n(1-\beta_n)\psi(\|Sx_n-p\|) \\
& & +\alpha_n\beta_n\psi\left(\frac{\varphi(\|0\|)+\psi(\|p-Sx_n\|)}{1+L\|0\|}\right) \\
& = & (1-\alpha_n)\|Sx_n-p\|+\alpha_n(1-\beta_n)\psi(\|Sx_n-p\|) \\
& & +\alpha_n\beta_n\psi^2(\|Sx_n-p\|).
\end{array}
\tag{3.16}
$$

From (3.16), set

$$r_n=\alpha_n(1-\beta_n)\psi(\|Sx_n-p\|)+\alpha_n\beta_n\psi^2(\|Sx_n-p\|). \tag{3.17}$$

Hence, (3.16) becomes

$$\|Sx_{n+1}-p\|\precsim(1-\alpha_n)\|Sx_n-p\|+r_n. \tag{3.18}$$

Using the fact that $(1-\alpha_n)<1$, we obtain the following inequalities from (3.18).

$$
\left\{
\begin{array}{l}
\|Sx_{n+1}-p\|\precsim(1-\alpha_n)\|Sx_n-p\|+r_n, \\
\|Sx_n-p\|\precsim(1-\alpha_{n-1})\|Sx_{n-1}-p\|+r_{n-1}, \\
\|Sx_{n-1}-p\|\precsim(1-\alpha_{n-2})\|Sx_{n-2}-p\|+r_{n-2}, \\
\quad\vdots \\
\|Sx_1-p\|\precsim(1-\alpha_0)\|Sx_0-p\|+r_0.
\end{array}
\right.
\tag{3.19}
$$

Using relation (3.19), we derive

$$\|Sx_{n+1}-p\|\precsim\|Sx_0-p\|\prod_{k=0}^{n}(1-\alpha_k)+r_n, \tag{3.20}$$

where $(1-\alpha_k) \in (0,1)$ and $\alpha_k \in [0,1]$ for each $k \in \mathbb{N}$. It is well known in classical analysis that $1-x \leq e^{-x}$ for all $x \in [0,1]$. Using these facts together with relation (3.20), we have

$$\|Sx_{n+1} - p\| \precsim \frac{\|Sx_0 - p\|}{e^{\sum_{k=0}^{n} \alpha_k}} + r_n. \tag{3.21}$$

Hence,

$$|\|Sx_{n+1} - p\|| \leq \frac{|\|Sx_0 - p\||}{|e^{\sum_{k=0}^{n} \alpha_k}|} + |r_n|. \tag{3.22}$$

Therefore, using Lemma 2.6 and Lemma 2.21, we have that

$$\lim_{n\to\infty} |\|Sx_{n+1} - p\|| \leq \frac{|\|Sx_0 - p\||}{|e^{\sum_{k=0}^{n} \alpha_k}|} + |r_n| \longrightarrow 0 \text{ as } n \to \infty. \tag{3.23}$$

By Lemma 2.6, we have

$$\lim_{n\to\infty} \|Sx_n - p\| = 0. \tag{3.24}$$

This means that $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p.

Next, we prove that p is the unique common fixed point of S and T.

Suppose that p^* is another coincidence point of S and T. Then, there exists $y^* \in D$ such that $Sy^* = Ty^* = p^*$. Using (3.15), we have

$$\begin{aligned} 0 \precsim \|p - p^*\| &= \|Ty - Ty^*\| \\ &\precsim \frac{\varphi(\|Sy-Ty\|)+\psi(\|Sy-Sy^*\|)}{1+L\|Sy-Ty\|} \\ &= \frac{\varphi(\|0\|)+\psi(\|Sy-Sy^*\|)}{1+L\|0\|} \\ &= \psi(\|p-p^*\|). \end{aligned} \tag{3.25}$$

Therefore,

$$0 \leq |\|p - p^*\|| \leq |\psi(\|p - p^*\|)|. \tag{3.26}$$

By Lemma 2.6 and Lemma 2.21, we have

$$0 \leq |\|p - p^*\|| \leq |\psi(\|p - p^*\|)| \longrightarrow 0 \text{ as } n \to \infty, \tag{3.27}$$

so that $p = p^*$ by Lemma 2.6. Since S and T are weakly compatible and $p = Ty = Sy$, so that $Tp = TTy = TSy = STy$ and hence $Tp = Sp = p$, and therefore p is the unique common fixed point of S and T. The proof of Theorem 3.5 is completed. □

Remark 3.6. Theorem 3.5 is the extension and generalization of several known results from real Banach spaces to complex valued Banach spaces, including the results of Olatinwo [36].

Next, we obtain the following corollary as a consequence of Theorem 3.5.

Corollary 3.7. *Let D be a nonempty closed convex subset of a complex valued Banach space $(E, \|.\|)$ and let $S, T : D \to E$ be nonself operators on D satisfying the following contractive condition:*

(C) *For a real number $L \geq 0$, a sublinear comparison function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ and a monotone increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(0) = 0$ and for all $x, y \in D$, we have*

$$\|Tx - Ty\| \precsim \frac{\varphi(\|Sx - Tx\|) + \psi(\|Sx - Sy\|)}{1 + L\|Sx - Tx\|}. \tag{3.28}$$

Assume that $T(D) \subseteq S(D)$, $S(D) \subseteq E$ is a complex valued Banach space and $Sz = Tz = p$ (say). For $x_0 \in D$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck-Mann iterative process (2.4)*, where $\{\alpha_n\}$ is a sequence of positive number in $[0, 1]$ with $\{\alpha_n\}$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{Sx_n\}_{n=0}^{\infty}$ converges strongly to p. Moreover, p will be the unique common fixed point of S, T provided $D = E$, and S and T are weakly compatible.*

Proof. The proof of Corollary 3.7 follows the same lines as in the proof of Theorem 3.5. □

Next, we give the following numerical examples to validate our analytical results.

Example 3.8. Let $D = [0, 1]$ and define $\|.\| : D \to \mathbb{C}$ by

$$\|x - y\| = |x - y|e^{\frac{i\pi}{3}}.$$

Then $(D, \|.\|)$ is a complex valued Banach space. Next, we define the nonself operators $S, T : D \to \mathbb{C}$ by $Sx = Tx = \frac{x}{2}$. Suppose $\{Sx_n\}_{n=0}^{\infty}$ is the Jungck-CR iterative process (2.7) and choose $\{\alpha_n\} = \{\beta_n\} = \{\gamma_n\} = \frac{1}{\sqrt{5}}$, for each $n = 0, 1, 2, 3, \cdots$. Suppose $a = \frac{1}{2}$, $L = 4$, $\varphi = \frac{t^2}{2}$ for all $t \in \mathbb{R}^+$ and $k = \frac{1}{3}$. Clearly, we see that 0 is the unique common fixed point of S and T. Define $\{x_n\}_{n=0}^{\infty} = \frac{1}{n+1}$, then $Sx_0 = \frac{1}{2} \in D$. Hence, by relation (3.1) for all $x, y \in D$, we have

$$\begin{aligned}\|Tx - Ty\| &\precsim \left(\tfrac{\varphi(\|Sx-Tx\|)+a\|Sx-Sy\|}{1+L\|Sx-Tx\|}\right) e^{k\|Sx-Tx\|} \\ &= \left(\frac{\varphi(|Sx-Tx|e^{\frac{i\pi}{3}})+\frac{1}{2}|Sx-Sy|e^{\frac{i\pi}{3}}}{1+4|Sx-Tx|e^{\frac{i\pi}{3}}}\right) e^{\frac{1}{2}|Sx-Tx|e^{\frac{i\pi}{3}}} \\ &= \left(\frac{\varphi(|0|)+\frac{1}{2}|\frac{x}{2}-\frac{y}{2}|e^{\frac{i\pi}{3}}}{1+4|0|e^{\frac{i\pi}{3}}}\right) e^{0} \\ &= \frac{1}{4}|x-y|e^{\frac{i\pi}{3}}. \end{aligned} \tag{3.29}$$

Using relation (3.6), we have

$$\begin{aligned}
\|Sx_{n+1}-p\| &\precsim (1-\alpha_n(1-a))\|Sx_n-p\| \\
&= (1-\tfrac{1}{\sqrt{5}}(1-\tfrac{1}{2})|\tfrac{1}{2}(\tfrac{1}{n+1})-0|e^{\frac{i\pi}{3}} \\
&= (\tfrac{10-\sqrt{5}}{10})|\tfrac{1}{2}(\tfrac{1}{n+1})|e^{\frac{i\pi}{3}} \\
&= (\tfrac{10-\sqrt{5}}{20})|\tfrac{1}{n+1}|e^{\frac{i\pi}{3}} \\
&\to 0 \text{ as } n\to\infty. \qquad (3.30)
\end{aligned}$$

Clearly, we see that all the conditions of Theorem 3.1 are satisfied and 0 is the unique common fixed point of S and T.

Example 3.9. Let $D=[0,1]$ and define $\|.\|: D\to\mathbb{C}$ by

$$\|x-y\| = i|x-y|.$$

Then $(D,\|.\|)$ is a complex valued Banach space. Next, we define the nonself operators $S,T: D\to\mathbb{C}$ by $Sx=Tx=\frac{x}{4}$. Suppose $\{Sx_n\}_{n=0}^{\infty}$ is the Jungck-CR iterative process (2.7) and choose $\{\alpha_n\}=\{\beta_n\}=\{\gamma_n\}=\frac{1}{\sqrt{7}}$, for each $n=0,1,2,3,\cdots$. Suppose $a=\frac{1}{2}$, $L=4$, $\varphi=\frac{t^2}{2}$ for all $t\in\mathbb{R}^+$ and $k=\frac{1}{3}$. Clearly, we see that 0 is the unique common fixed point of S and T. Define $\{x_n\}_{n=0}^{\infty}=\frac{1}{n+1}$, then $Sx_0=\frac{1}{2}\in D$.

By similar computations as in Example 3.8, we see that all the conditions of Theorem 3.1 are satisfied and 0 is the unique common fixed point of S and T.

4. Some stability results in complex valued Banach spaces

We begin this section with the following stability results, which extends and generalizes several known results from real Banach spaces to complex valued Banach spaces, including the results of Hussain *et al.* [14].

Theorem 4.1. *Let D be a nonempty closed convex subset of a complex valued Banach space $(E,\|.\|)$ and let $S,T: D\to E$ be nonself operators on D satisfying the following contractive condition:*

(C) *For all real numbers $k\geq 0$, $L\geq 0$, $a\in[0,1)$ and a monotone increasing function $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ such that $\varphi(0)=0$ and for all $x,y\in D$, we have*

$$\|Tx-Ty\| \precsim \left(\frac{\varphi(\|Sx-Tx\|)+a\|Sx-Sy\|}{1+L\|Sx-Tx\|}\right)e^{k\|Sx-Tx\|}. \qquad (4.1)$$

Assume that $T(D)\subseteq S(D)$, $S(D)\subseteq E$ is a complex valued Banach space and $Sx^*=Tx^*=p$ (say). For $x_0\in D$ and $\alpha\in(0,1)$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck-CR iterative process (2.7) *converging to p, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences of positive numbers in $[0,1]$ with $\{\alpha_n\}$ satisfying $\alpha\leq\alpha_n$ for all $n\in\mathbb{N}$. Then the Jungck-CR iterative process $\{Sx_n\}_{n=0}^{\infty}$ is (S,T)-stable.*

Proof. Suppose that $\{Sg_n\}_{n=0}^{\infty} \subset D$ is an arbitrary sequence, set

$$\varepsilon_n = \|Sg_{n+1} - (1-\alpha_n)Sb_n - \alpha_n Tb_n\|, \ n = 0, 1, 2, \cdots, \tag{4.2}$$

where

$$\begin{cases} Sb_n = (1-\beta_n)Tg_n + \beta_n Tc_n \\ Sc_n = (1-\gamma_n)Sg_n + \gamma_n Tg_n, \end{cases} \tag{4.3}$$

and let $\lim_{n\to\infty} \varepsilon_n = 0$. We want to prove that $\lim_{n\to\infty} Sg_{n+1} = p$. Using relations (2.7) and (4.1), we have

$$\begin{aligned} \|Sg_{n+1} - p\| &\precsim \|Sg_{n+1} - (1-\alpha_n)Sb_n - \alpha_n Tb_n\| \\ &\quad + \|(1-\alpha_n)Sb_n + \alpha_n Tb_n - p\| \\ &\precsim \varepsilon_n + (1-\alpha_n)\|Sb_n - p\| + \alpha_n \|Tx^* - Tb_n\| \\ &\precsim \varepsilon_n + (1-\alpha_n)\|Sb_n - p\| + \alpha_n \left(\tfrac{\varphi(\|Sx^*-Tx^*\|)+a\|Sx^*-Sb_n\|}{1+L\|Sx^*-Tx^*\|}\right) \\ &\quad \times e^{k\|Sx^*-Tx^*\|} \\ &= \varepsilon_n + (1-\alpha_n)\|Sb_n - p\| + \alpha_n \left(\tfrac{\varphi(\|0\|)+a\|p-Sb_n\|}{1+L\|0\|}\right) e^{k\|0\|} \\ &= \varepsilon_n + (1-\alpha_n)\|Sb_n - p\| + a\alpha_n \|Sb_n - p\| \\ &= (1-\alpha_n(1-a))\|Sb_n - p\| + \varepsilon_n. \end{aligned} \tag{4.4}$$

Next, we compute the following estimates:

$$\begin{aligned} \|Sb_n - p\| &= \|(1-\beta_n)Tg_n + \beta_n Tc_n - p\| \\ &\precsim (1-\beta_n)\|Tg_n - p\| + \beta_n \|Tc_n - p\| \\ &= (1-\beta_n)\|Tx^* - Tg_n\| + \beta_n \|Tx^* - Tc_n\| \\ &\precsim (1-\beta_n)\left(\tfrac{\varphi(\|Sx^*-Tx^*\|)+a\|Sx^*-Sg_n\|}{1+L\|Sx^*-Tx^*\|}\right) e^{k\|Sx^*-Tx^*\|} \\ &\quad + \beta_n \left(\tfrac{\varphi(\|Sx^*-Tx^*\|)+a\|Sx^*-Sc_n\|}{1+L\|Sx^*-Tx^*\|}\right) e^{k\|Sx^*-Tx^*\|} \\ &= (1-\beta_n)\left(\tfrac{\varphi(\|0\|)+a\|p-Sg_n\|}{1+L\|0\|}\right) e^{k\|0\|} + \beta_n \left(\tfrac{\varphi(\|0\|)+a\|p-Sc_n\|}{1+L\|0\|}\right) e^{k\|0\|} \\ &= a(1-\beta_n)\|Sg_n - p\| + a\beta_n \|Sc_n - p\|. \end{aligned} \tag{4.5}$$

$$\begin{aligned} \|Sc_n - p\| &= \|(1-\gamma_n)Sg_n + \gamma_n Tg_n - p\| \\ &\precsim (1-\gamma_n)\|Sg_n - p\| + \gamma_n \|Tg_n - p\| \\ &= (1-\gamma_n)\|Sg_n - p\| + \gamma_n \|Tx^* - Tg_n\| \\ &\precsim (1-\gamma_n)\|Sg_n - p\| + \gamma_n \left(\tfrac{\varphi(\|Sx^*-Tx^*\|)+a\|Sx^*-Sg_n\|}{1+L\|Sx^*-Tx^*\|}\right) \\ &\quad \times e^{k\|Sx^*-Tx^*\|} \\ &= (1-\gamma_n)\|Sg_n - p\| + \gamma_n \left(\tfrac{\varphi(\|0\|)+a\|p-Sg_n\|}{1+L\|0\|}\right) e^{k\|0\|} \\ &= (1-\gamma_n)\|Sg_n - p\| + a\gamma_n \|Sg_n - p\| \\ &= (1-\gamma_n(1-a))\|Sg_n - p\|. \end{aligned} \tag{4.6}$$

Using (4.6) in (4.5), we have

$$\begin{aligned} \|Sb_n - p\| &\precsim a(1-\beta_n)\|Sg_n - p\| + a\beta_n(1-\gamma_n(1-a))\|Sg_n - p\| \\ &\precsim (1-\beta_n(1-a))\|Sg_n - p\|. \end{aligned} \tag{4.7}$$

Using (4.7) in (4.4), we have

$$\begin{array}{rcl}\|Sg_{n+1}-p\| & \precsim & (1-\alpha_n(1-a))(1-\beta_n(1-a))\|Sg_n-p\|+\varepsilon_n \\ & \precsim & (1-\alpha_n(1-a))\|Sg_n-p\|+\varepsilon_n.\end{array} \tag{4.8}$$

Therefore, from relation (4.8), we have

$$\begin{array}{rcl}|\|Sg_{n+1}-p\|| & \leq & |(1-\alpha_n(1-a))\|Sg_n-p\|+\varepsilon_n| \\ & \leq & |(1-\alpha_n(1-a))\|Sg_n-p\||+|\varepsilon_n| \\ & \to & 0 \text{ as } n\to\infty.\end{array} \tag{4.9}$$

Since $0<\alpha\leq\alpha_n$ and $a\in[0,1)$, we have $(1-\alpha_n(1-a))<1$. Hence, by Lemma 2.6 and Lemma 2.20, we have from relation (4.9) that $\lim_{n\to\infty} Sg_{n+1}=p$.

Conversely, suppose $\lim_{n\to\infty} Sg_{n+1}=p$. Then, using (2.7), (4.1) and triangle inequality, we have

$$\begin{array}{rcl}\varepsilon_n & = & \|Sg_{n+1}-(1-\alpha_n)Sb_n-\alpha_n Tb_n\| \\ & \precsim & \|Sg_{n+1}-p\|+\|p-(1-\alpha_n)Sb_n-\alpha_n Tb_n\| \\ & \precsim & \|Sg_{n+1}-p\|+(1-\alpha_n)\|p-Sb_n\|+\alpha_n\|p-Tb_n\| \\ & = & \|Sg_{n+1}-p\|+(1-\alpha_n)\|Sb_n-p\|+\alpha_n\|Tx^*-Tb_n\| \\ & \precsim & \|Sg_{n+1}-p\|+(1-\alpha_n)\|Sb_n-p\| \\ & & +\alpha_n\left(\frac{\varphi(\|Sx^*-Tx^*\|)+a\|Sx^*-Sb_n\|}{1+L\|Sx^*-Tx^*}\right)e^{k\|Sx^*-Tx^*\|} \\ & = & \|Sg_{n+1}-p\|+(1-\alpha_n)\|Sb_n-p\|+\alpha_n\left(\frac{\varphi(\|0\|)+a\|p-Sb_n\|}{1+L\|0\|}\right)e^{k\|0\|} \\ & = & \|Sg_{n+1}-p\|+(1-\alpha_n)\|Sb_n-p\|+a\alpha_n\|Sb_n-p\| \\ & = & \|Sg_{n+1}-p\|+(1-\alpha_n(1-a))\|Sb_n-p\|.\end{array} \tag{4.10}$$

Using relation (4.7) in (4.10), we have

$$\begin{array}{rcl}\varepsilon_n & \precsim & \|Sg_{n+1}-p\|+(1-\alpha_n(1-a))(1-\beta_n(1-a))\|Sg_n-p\| \\ & \precsim & \|Sg_{n+1}-p\|+(1-\alpha_n(1-a))\|Sg_n-p\|.\end{array} \tag{4.11}$$

Therefore, from (4.11) we have

$$\begin{array}{rcl}|\varepsilon_n| & \leq & |\|Sg_{n+1}-p\|+(1-\alpha_n(1-a))\|Sg_n-p\|| \\ & \leq & |\|Sg_{n+1}-p\||+|(1-\alpha_n(1-a))\|Sg_n-p\|| \\ & \to & 0 \text{ as } n\to\infty.\end{array} \tag{4.12}$$

Hence, by Lemma 2.6 and Lemma 2.20, we have

$$\lim_{n\to\infty}\varepsilon_n=0. \tag{4.13}$$

This means that the Jungck-CR iterative process $\{Sx_n\}_{n=0}^{\infty}$ is (S,T)-stable. This completes the proof. □

Remark 4.2. Theorem 4.1 is the unification, extension and generalization of the results in ([14], Theorem 13) from real Banach spaces to complex valued Banach spaces.

Next, we obtain the following corollaries as consequencies of Theorem 4.1.

Corollary 4.3. *Let D be a nonempty closed convex subset of a complex valued Banach space $(E, \|.\|)$ and let $S, T : D \to E$ be nonself operators on D satisfying the following contractive condition:*

(C) *For all real numbers $k \geq 0$, $L \geq 0$, $a \in [0, 1)$ and a monotone increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(0) = 0$ and for all $x, y \in D$, we have*

$$\|Tx - Ty\| \precsim \left(\frac{\varphi(\|Sx - Tx\|) + a\|Sx - Sy\|}{1 + L\|Sx - Tx\|} \right) e^{k\|Sx - Tx\|}. \tag{4.14}$$

Assume that $T(D) \subseteq S(D)$, $S(D) \subseteq E$ is a complex valued Banach space and $Sx^* = Tx^* = p$ (say). For $x_0 \in D$ and $\alpha \in (0, 1)$, let $\{Sx_n\}_{n=0}^{\infty}$ be the iterative process defined by (2.8) *converging to p, where $\{\beta_n\}$, $\{\gamma_n\}$ are sequences of positive numbers in $[0, 1]$ with $\{\beta_n\}$ satisfying $\beta \leq \beta_n$ for all $n \in \mathbb{N}$. Then $\{Sx_n\}_{n=0}^{\infty}$ is (S, T)-stable.*

Proof. The proof of Corollary 4.3 follows similar lines as in the proof of Theorem 4.1. □

Corollary 4.4. *Let D be a nonempty closed convex subset of a complex valued Banach space $(E, \|.\|)$ and let $S, T : D \to E$ be nonself operators on D satisfying the following contractive condition:*

(C) *For all real numbers $k \geq 0$, $L \geq 0$, $a \in [0, 1)$ and a monotone increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(0) = 0$ and for all $x, y \in D$, we have*

$$\|Tx - Ty\| \precsim \left(\frac{\varphi(\|Sx - Tx\|) + a\|Sx - Sy\|}{1 + L\|Sx - Tx\|} \right) e^{k\|Sx - Tx\|}. \tag{4.15}$$

Assume that $T(D) \subseteq S(D)$, $S(D) \subseteq E$ is a complex valued Banach space and $Sx^* = Tx^* = p$ (say). For $x_0 \in D$ and $\alpha \in (0, 1)$, let $\{Sx_n\}_{n=0}^{\infty}$ be the iterative process defined by (2.9) *converging to p, where $\{\gamma_n\}$ is a sequence of positive number in $[0, 1]$ with $\{\gamma_n\}$ satisfying $\gamma \leq \gamma_n$ for all $n \in \mathbb{N}$. Then $\{Sx_n\}_{n=0}^{\infty}$ is (S, T)-stable.*

Proof. The proof of Corollary 4.4 follows similar lines as in the proof of Theorem 4.1. □

We now give the following numerical examples to validate our analytical results.

Example 4.5. Let $D = [0, 1]$ and define $\|.\| : D \to \mathbb{C}$ by

$$\|x - y\| = i|x - y|.$$

Then $(D, \|.\|)$ is a complex valued Banach space. Next, we define the nonself operators $S, T : D \to \mathbb{C}$ by $Sx = Tx = \frac{x}{4}$. Suppose $\{Sx_n\}_{n=0}^{\infty}$ is the Jungck-CR iterative process (2.7) and choose $\{\alpha_n\} = \{\beta_n\} = \{\gamma_n\} = \frac{1}{\sqrt{7}}$, for each

$n = 0, 1, 2, 3, \cdots$. Suppose $a = \frac{1}{2}$, $L = 5$, $\varphi = \frac{t^2}{2}$ for all $t \in \mathbb{R}^+$ and $k = \frac{1}{3}$. Clearly, we see that 0 is the unique common fixed point of S and T. Define $\{g_n\}_{n=0}^{\infty} = \frac{1}{n+1}$, then $Sg_0 = \frac{1}{2} \in D$. Hence, by relation (4.1) $\forall x, y \in D$, we have

$$\begin{array}{rcl} \|Tx - Ty\| & \precsim & \left(\frac{\varphi(\|Sx-Tx\|)+a\|Sx-Sy\|}{1+L\|Sx-Tx\|}\right) e^{k\|Sx-Tx\|} \\ & = & \left(\frac{\varphi(|Sx-Tx|i)+\frac{1}{2}|Sx-Sy|i}{1+5|Sx-Tx|i}\right) e^{\frac{1}{2}|Sx-Tx|i} \\ & = & \left(\frac{\varphi(|0|i)+\frac{1}{2}|\frac{x}{4}-\frac{y}{4}|i}{1+5|0|i}\right) e^{0} \\ & = & \frac{1}{8}|x-y|i. \end{array} \tag{4.16}$$

By relation (4.9), we have

$$\begin{array}{rcl} \|Sg_{n+1} - p\| & \precsim & |(1-\alpha_n(1-a))\|Sg_n - p\|| + |\varepsilon_n| \\ & = & |(1-\frac{1}{\sqrt{7}}(1-\frac{1}{2}))|\frac{1}{4}(\frac{1}{n+1}) - 0|i| + |\varepsilon_n| \\ & = & |(1-\frac{1}{2\sqrt{7}})|\frac{1}{4}(\frac{1}{n+1})|i| + |\varepsilon_n| \\ & = & |\frac{14-\sqrt{7}}{56}||\frac{1}{n+1}| + |\varepsilon_n| \\ & \to & 0 \text{ as } n \to \infty. \end{array} \tag{4.17}$$

Similarly, using (4.12), we obtain

$$\begin{array}{rcl} |\varepsilon_n| & \leq & ||\frac{1}{4}(\frac{1}{n+2}) - 0|i| + |(1-\frac{1}{\sqrt{7}}(1-\frac{1}{2}))|\frac{1}{4}(\frac{1}{n+1}) - 0|i| \\ & = & \frac{1}{4}|\frac{1}{n+2}| + |\frac{14-\sqrt{7}}{56}||\frac{1}{n+1}| \\ & \to & 0 \text{ as } n \to \infty. \end{array} \tag{4.18}$$

Clearly, all the conditions of Theorem 4.1 are satisfied. Therefore, the Jungck-CR iterative process $\{Sx_n\}_{n=0}^{\infty}$ is (S,T)-stable.

Example 4.6. Let $D = [0, 1]$ and define $\|.\| : D \to \mathbb{C}$ by

$$\|x - y\| = |x - y|e^{\frac{i\pi}{3}}.$$

Then $(D, \|.\|)$ is a complex valued Banach space. Next, we define the nonself operators $S, T : D \to \mathbb{C}$ by $Sx = Tx = \frac{x}{2}$. Suppose $\{Sx_n\}_{n=0}^{\infty}$ is the Jungck-CR iterative process (2.7) and choose $\{\alpha_n\} = \{\beta_n\} = \{\gamma_n\} = \frac{1}{\sqrt{11}}$, for each $n = 0, 1, 2, 3, \cdots$. Suppose $a = \frac{1}{2}$, $L = 10$, $\varphi = \frac{t^2}{2}$ for all $t \in \mathbb{R}^+$ and $k = \frac{1}{3}$. Clearly, we see that 0 is the unique common fixed point of S and T. Define $\{g_n\}_{n=0}^{\infty} = \frac{1}{n+1}$, then $Sg_0 = \frac{1}{2} \in D$.

By similar computations as in Example 4.5, we see that all the conditions of Theorem 4.1 are satisfied. Therefore, the Jungck-CR iterative process $\{Sx_n\}_{n=0}^{\infty}$ is (S,T)-stable.

5. Applications to a nonlinear integral equation

In this section, we show that the Jungck-Mann iterative process (2.4) converges strongly to the solution of a mixed type Volterra-Fredholm functional nonlinear integral equation in complex valued Banach spaces.

In 2011, Crăciun and Şerban [11] considered the following mixed type Volterra-Fredholm functional nonlinear integral equation:

$$x(t) = F\left(t, x(t), \int_{a_1}^{t_1} \cdots \int_{a_m}^{t_m} K(t,s,x(s))ds, \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} H(t,s,x(s))ds\right), \tag{5.1}$$

where $[a_1; b_1] \times \cdots \times [a_m; b_m]$ is an interval in $\mathbb{R}^m$, $K, H : [a_1; b_1] \times \cdots \times [a_m; b_m] \times [a_1; b_1] \times \cdots \times [a_m; b_m] \times \mathbb{R} \to \mathbb{R}$ continuous functions and $F : [a_1; b_1] \times \cdots \times [a_m; b_m] \times \mathbb{R}^3 \to \mathbb{R}$. They established the following results.

Theorem 5.1. ([11]) *We assume that:*

(i) $K, H \in C([a_1, b_1] \times \cdots \times [a_m, b_m] \times [a_1, b_1] \times \cdots \times [a_m, b_m] \times \mathbb{R})$;
(ii) $F \in C([a_1, b_1] \times \cdots \times [a_m, b_m] \times \mathbb{R}^3)$;
(iii) *there exist* α, β, γ *nonnegative constants such that*

$$|F(t, u_1, v_1, w_1) - F(t, u_2, v_2, w_2)| \le \alpha|u_1 - u_2| + \beta|v_1 - v_2| + \gamma|w_1 - w_2|,$$

for all $t \in [a_1, b_1] \times \cdots \times [a_m, b_m]$, $u_1, u_2, v_1, v_2, w_1, w_2 \in \mathbb{R}$;
(iv) *there exist* L_K *and* L_H *nonnegative constants such that*

$$|K(t,s,u) - K(t,s,v)| \le L_K|u - v|,$$

$$|H(t,s,u) - H(t,s,v)| \le L_H|u - v|,$$

for all $t, s \in [a_1, b_1] \times \cdots \times [a_m, b_m]$, $u, v \in \mathbb{R}$;
(v) $\alpha + (\beta L_K + \gamma L_H)(b_1 - a_1) \cdots (b_m - a_m) < 1$.

Then, the equation (5.1) *has a unique solution* $x^* \in C([a_1, b_1] \times \cdots \times [a_m, b_m])$.

Remark 5.2. ([11]) Let $(\mathbb{B}, |.|)$ be a Banach space. Then Theorem 5.1 remains also true if we consider the mixed type Volterra-Fredholm functional nonlinear integral equation (5.1) in the Banach space $\mathbb{B}$ instead of Banach space $\mathbb{R}$.

Consequently, we obtain the following results in complex valued Banach spaces.

Theorem 5.3. *We assume that:*

(A_1) $K, H \in C([a_1, b_1] \times \cdots \times [a_m, b_m] \times [a_1, b_1] \times \cdots \times [a_m, b_m] \times \mathbb{C})$;
(A_2) $F \in C([a_1, b_1] \times \cdots \times [a_m, b_m] \times \mathbb{C}^3)$;
(A_3) *there exist* α, β, γ *nonnegative constants such that:*

$$|F(t, u_1, v_1, w_1) - F(t, u_2, v_2, w_2)| \le \alpha|u_1 - u_2| + \beta|v_1 - v_2| + \gamma|w_1 - w_2|,$$

for all $t \in [a_1, b_1] \times \cdots \times [a_m, b_m]$, $u_1, u_2, v_1, v_2, w_1, w_2 \in \mathbb{C}$;

(A_4) *there exist L_K and L_H nonnegative constants such that:*

$$|K(t,s,u) - K(t,s,v)| \leq L_K|u-v|,$$
$$|H(t,s,u) - H(t,s,v)| \leq L_H|u-v|,$$

for all $t,s \in [a_1,b_1] \times \cdots \times [a_m,b_m]$, $u,v \in \mathbb{C}$;

(A_5) $\alpha + (\beta L_K + \gamma L_H)(b_1-a_1)\cdots(b_m-a_m) < 1$.

Suppose that the sequence $\{Sx_n\}_{n=0}^{\infty}$ is the Jungck-Mann iterative process defined by

$$Sx_{n+1} = (1-\lambda_n)Sx_n + \lambda_n Tx_n, \tag{5.2}$$

where $\{\lambda_n\}_{n=0}^{\infty} \subset [0,1]$ is a real sequence satisfying $\sum_{n=0}^{\infty}\lambda_n = \infty$. Then, the equation (5.1) *has a unique solution, say $x^* \in C([a_1,b_1]\times\cdots\times[a_m,b_m])$ and the Jungck-Mann iterative process* (5.2) *converges to x^*.*

Proof. We consider the complex valued Banach space $\mathcal{B}_{\mathbb{C}} = C([a_1,b_1]\times\cdots\times[a_m,b_m], \|.\|_{\mathbb{C}})$, where $\|.\|_{\mathbb{C}}$ is the Chebyshev's norm defined by

$$\|x-y\|_{\mathbb{C}} = |x-y|i, \ \forall x,y \in \mathcal{B}_{\mathbb{C}}.$$

Let $\{Sx_n\}_{n=0}^{\infty}$ be an iterative sequence generated by the Jungck-Mann iterative process (5.2) for the operators $A_1, A_2 : \mathcal{B}_{\mathbb{C}} \to \mathcal{B}_{\mathbb{C}}$ defined by

$$\begin{aligned} A_1(x)(t) &= A_2(x)(t) \\ &= F\left(t, x(t), \int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m} K(t,s,x(s))ds, \int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m} H(t,s,x(s))ds\right). \end{aligned} \tag{5.3}$$

We want to show that $Sx_n \longrightarrow x^*$ as $n \to \infty$. Using relation (5.1), (5.2) and assumptions $(A_1)-(A_4)$, we have

$$\begin{aligned} \|Sx_{n+1} - x^*\| &= \|(1-\lambda_n)A_1x_n + \lambda_n A_2x_n - x^*\| \\ &\precsim (1-\lambda_n)\|A_1x_n - x^*\| + \lambda_n\|A_2x_n - x^*\|. \end{aligned} \tag{5.4}$$

Next, we compute the following estimates:

$$\begin{aligned} &\|A_1x_n - x^*\| \\ &= |A_1(x_n)(t) - A_1(x^*)(t)|i \\ &= |F\left(t, x_n(t), \textstyle\int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m} K(t,s,x_n(s))ds, \int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m} H(t,s,x_n(s))ds\right) \\ &\quad -F\left(t, x^*(t), \textstyle\int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m} K(t,s,x^*(s))ds, \int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m} H(t,s,x^*(s))ds\right)|i \end{aligned}$$

$$\begin{aligned}
&\precsim \alpha|x_n(t)-x^*(t)|i \\
&\quad +\beta|\int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m}K(t,s,x_n(s))ds-\int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m}K(t,s,x^*(s))ds|i \\
&\quad +\gamma|\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}H(t,s,x_n(s))ds-\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}H(t,s,x^*(s))ds|i \\
&\precsim \alpha|x_n(t)-x^*(t)|i+\beta i\int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m}|K(t,s,x_n(s))-K(t,s,x^*(s))|ds \\
&\quad +\gamma i\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}|H(t,s,x_n(s))-H(t,s,x^*(s))|ds \\
&\precsim \alpha|x_n(t)-x^*(t)|i+\beta i\int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m}L_K|x_n(s)-x^*(s)|ds \\
&\quad +\gamma i\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}L_H|x_n(s)-x^*(s)|ds \\
&\precsim [\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(b_i-a_i)]\,\|Sx_n-x^*\|.
\end{aligned} \tag{5.5}$$

Similarly, we have

$$\|A_2x_n-x^*\| \precsim \left[\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(b_i-a_i)\right]\|Sx_n-x^*\|. \tag{5.6}$$

Using (5.5) and (5.6) in (5.4), we have

$$\|Sx_{n+1}-x^*\| \precsim \left[\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(b_i-a_i)\right]\|Sx_n-x^*\|. \tag{5.7}$$

This implies that

$$|||Sx_{n+1}-x^*||| \leq \left[\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(b_i-a_i)\right]|||Sx_n-x^*|||. \tag{5.8}$$

By assumption (A_5), we have

$$\left[\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(b_i-a_i)\right]<1.$$

Therefore by Lemma 2.6 and Lemma 2.20, we obtain

$$\begin{aligned}
|||Sx_{n+1}-x^*||| &\leq [\alpha+(\beta L_K+\gamma L_H)\prod_{i=1}^{m}(b_i-a_i)]\,|||Sx_n-x^*||| \\
&\to 0 \text{ as } n\to\infty.
\end{aligned} \tag{5.9}$$

Hence, we have

$$\lim_{n\to\infty}\|Sx_n-x^*\|=0. \tag{5.10}$$

This completes the proof. □

Acknowledgments This paper was completed while the first author was visiting the Abdus Salam School of Mathematical Sciences (ASSMS), Government College University Lahore, Pakistan as a postdoctoral fellow and the second author was supported by the Basic Science Research Program through the National Research Foundation(NRF) Grant funded by Ministry of Education of the republic of Korea (2018R1D1A1B07045427).

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