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PROXIMAL-LIKE SUBGRADIENT METHODS FOR SOLVING MULTI-VALUED VARIATIONAL INEQUALITIES

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Abstract. In this paper, we propose and analyze the convergence of a new algorithm for solving monotone and Lipschitz continuous multi-valued variational inequalities by using proximal operator. By choosing suitable parameters of proximal steps and of subgradient stepsizes, we show that the convergence of the algorithm does not require the prior knowledge of Lipschitz continuous constant of cost operator.

1. Introduction

Let C be a nonempty, closed and convex subset of a finite dimensional vector space $\mathcal{R}^s$. Let $F : \mathcal{R}^s \to 2^{\mathcal{R}^s}$ be a multi-valued mapping and $g : C \to \mathcal{R} \cup \{+\infty\}$ be a convex function.

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We consider a problem of the multi-valued generalized variational inequalities in a general form: Find $x^* \in C, w^* \in F(x^*)$ such that

$$\langle w^*, x - x^* \rangle + g(x) - g(x^*) \geq 0, \quad \forall x \in C. \tag{1.1}$$

As usual, F is said to be a *cost operator*. In the case F is single-valued, Problem (1.1) is the well-known general variational inequality problem, shortly $VI(C, F, g)$, which is an interesting problem of nonlinear analysis [3, 6, 7, 15, 28, 34].

Let $g : C \to \mathcal{R}$ be proper, convex and lower semicontinuous. The proximal mapping of g with parameter $\lambda > 0$ on C is formulated as follows:

$$prox_{\lambda g}(y) = \text{argmin} \left\{ \lambda g(x) + \frac{1}{2}\|y - x\|^2 : x \in C \right\}.$$

An application of the proximal mapping is to solve the problem

$$\min\{f_1(x) + f_2(x) : x \in C\},$$

where $f_1 : \mathcal{R}^s \to \mathcal{R} \cup \{+\infty\}$ is differentiable and convex, and $f_2 : \mathcal{R}^s \to \mathcal{R} \cup \{+\infty\}$ is proper, convex and lower semicontinuous. Under the restriction that ∇f_1 is $\hat{L}-$Lipschitz continuous, one of the most simple methods for solving this problem is the proximal gradient method as the follows:

$$x^{k+1} = prox_{\lambda_k f_2} \left[x^k - \lambda_k \nabla f_1(x^k) \right],$$

where $\lambda_k \in \left(1, \frac{2}{\hat{L}}\right)$.

The variational inequality theory is an important tool in studying a wide class of obstacle of solution methods for equilibrium problems, mathematical programs with equilibrium constraints arising in several branches of pure and applied sciences [22, 29, 30]. Several numerical methods have been developed for solving multi-valued mixed variational inequalities and related optimization problems, see [11, 21, 33] and the references therein.

In the case $g = 0$, a popular solution method for solving the variational inequalities $VI(C, F, g)$ is the proximal method. Set $h(x, y) = \langle F(x), y - x \rangle$. The iteration sequence of the method is defined by a projection

$$x^{k+1} = Pr_C(x^k - \lambda F(x^k)),$$

where Pr_C is the metric projection onto C. Otherwise, it is well known that the method is only convergent in the case that the cost operator F is strongly monotone and Lipschitz continuous. In general, it is not convergent for monotone variational inequalities. In order to avoid the hypothesis of the strongly monotonicity of F, the extragradient method is first introduced by Korpelevich in [23] and Antipin et al. later in [1, 2], which is extended to pseudomonotone and Lipschitz continuous variational inequalities for a finite dimensional vector space. The iteration sequence $\{x^k\}$ is defined by

$$\begin{cases} y^k = Pr_C(x^k - \lambda F(x^k)), \\ x^{k+1} = Pr_C(x^k - \lambda F(y^k)). \end{cases}$$

However, the extragradient method requires computing two proximal points, which is computational expensive costs except when C has a special structure.

In 2011, in order to avoid the efforts for the second proximal point, Censor et al. [12] proposed the subgradient extragradient method in a real Hilbert space $\mathcal{H}$, in here the second proximal point is given by a specific form which is a projection onto a half-space T_k. The iterative sequence is defined as the following:

$$\begin{cases} y^k = Pr_C(x^k - \lambda F(x^k)), \\ T_k = \{w \in \mathcal{H} : \langle x^k - \lambda F(x^k) - y^k, w - y^k \rangle \leq 0\}, \\ x^{k+1} = Pr_{T_k}[x^k - \lambda F(x^k)], \end{cases} \tag{1.2}$$

where $\lambda > 0$. Under the main assumption that the cost operator F is pseudomonotone and Lipschitz continuous on C, the sequence $\{x^k\}$ strongly converges to a solution of $VI(C, F)$ in $\mathcal{H}$.

In [26], Malitsky introduced a proximal extrapolated gradient method for solving monotone and Lipschitz continuous problem $VI(C, F, g)$ in $\mathcal{R}^s$. In this method, the main step is given as follows:

$$\begin{cases} y^k = x^k + \tau_k(x^k - x^{k-1}), \\ x^{k+1} = prox_{\lambda_k g}(x^k - \lambda_k F(y^k)), \end{cases} \tag{1.3}$$

where parameters τ_k, λ_k and y^k are defined from local properties of $F(y^k)$. Inspired of the subgradient method of Malitsky [26], Cho et al. in [16] introduced a more general version of process (1.3) for solving the problem of finding a common point of the equilibrium problem and the fixed point problem in $\mathcal{R}^s$.

Recently, Anh and Hieu in [3] proposed a multi-step proximal method for solving the equilibrium problem: Find $x \in C$ such that

$$f(x, y) \geq 0, \quad \forall y \in C, \tag{1.4}$$

where $f : C \times C \to \mathcal{R}$.

One is a variant of the gradient-type projection method and the proximal method in a real Hilbert space $\mathcal{H}$. They established sufficient conditions for the convergence of the proposed methods and derived a new estimate of the rates of the convergence. The iteration sequence is given by the form:

$$\begin{cases} y^k = prox_{\lambda_k f(x^k,\cdot)}(x^k), \\ z^k = prox_{\rho_k f(y^k,\cdot)}(y^k), \\ t^k = prox_{\rho_k f(z^k,\cdot)}(x^k), \\ x^{k+1} = \alpha_k x^k + (1-\alpha_k)t^k, \end{cases} \tag{1.5}$$

where $\{\lambda_k\} \subset (0,1), \{\rho_k\} \subset (0,1)$ and $\{\alpha_k\} \subset (0,1)$. Under pseudomonotone and Lipschitz-type continuous assumptions of the bifunction f, the sequence $\{x^k\}$ definded by (1.5) strongly converges to a solution $x^* \in C$ of (1.4). There are several modifications of the proximal method for solving quasivariational inequalities, equilibrium problem, fixed point problems and other related problems; see for instance [20, 24, 25, 31, 35].

Motivated by ideas of the proximal extrapolated gradient methods (1.3) of Malitsky, the contraction and projection method of Dong et al. in [16], and the proximal methods in [4, 5], our interest in this paper is to propose a proximal-like subgradient algorithm and obtain its convergence for solving Problem (1.1), when F is monotone and Lipschitz continuous.

Comparing with current methods such as the relaxed proximal point methods of Huebner and Tichatschke in [19], the proximal method of Cholamjiak and Cholamjiak [14] and other [8], the fundamental difference here is that, our algorithm is very simple, only requires one proximal-like step without strongly monotone assumptions of the cost operator, where Lipschitz continuous constant can be known or unknown.

The paper is organized as follows. In the next section, we present some lemmas which will be used in the main results. In Section 3, we give a proximal-like subgradient algorithm for solving Problem (1.1) and the proof of its convergence. Section 4 is devoted to an ergodic rate of convergence of the iteration sequence by using the gap function for multi-valued mixed variational inequalities.

2. Preliminaries

Let C be a nonempty, closed and convex subset of $\mathcal{R}^s$. For each $x \in \mathcal{R}^s$, there exists a unique solution of the strongly convex quadratic problem, denoted by $Pr_C(x)$,

$$\min\left\{\|x-y\|^2 : y \in C\right\}.$$

Pr_C is usually called the *metric projection* onto C. It is easy to see that $y = Pr_C(x)$ if and only if $\langle v-y, x-y\rangle \le 0$ for all $v \in C$. An important property of Pr_C is 1-inverse strongly monotone on $\mathcal{R}^s$, i.e.,

$$\|Pr_C(x) - Pr_C(y)\|^2 \le \langle Pr_C(x) - Pr_C(y), x-y\rangle, \quad \forall x, y \in \mathcal{R}^s.$$

A mapping $F : C \to 2^{\mathcal{R}^s}$ is said to be

(i) *strongly monotone* with constant $\beta > 0$, if

$$\langle w_x - w_y, x - y\rangle \geq \beta \|x - y\|^2 \quad \forall x, y \in C, w_x \in F(x), w_y \in F(y);$$

(ii) *monotone*, if

$$\langle w_x - w_y, x - y\rangle \geq 0 \quad \forall x, y \in C, w_x \in F(x), w_y \in F(y).$$

Note that the Hausdorff distance of two sets $\mathcal{A}$ and $\mathcal{B}$ is defined as

$$\rho(\mathcal{A}, \mathcal{B}) := \max\{d(\mathcal{A}, \mathcal{B}), d(\mathcal{B}, \mathcal{A})\},$$

where $d(\mathcal{A}, \mathcal{B}) := \sup_{a\in\mathcal{A}} \inf_{b\in\mathcal{B}} \|a - b\|$ and $d(\mathcal{B}, \mathcal{A}) := \sup_{b\in\mathcal{B}} \inf_{a\in\mathcal{A}} \|a - b\|$.

Let $T : C \to 2^{\mathcal{R}^s}$. As usual, T is said to be *Lipschitz continuous* on C with constant $L > 0$ if

$$\rho(T(x), T(y)) \leq L\|x - y\|, \quad \forall x, y \in C.$$

In the case $L \in (0, 1)$, the mapping T is said to be *contractive* with constant L on C.

To investigate the convergence of our algorithms, we recall the following technical lemmas which will be used in the sequel.

Lemma 2.1. ([13], Lemma 2.1) *Let $\{a_k\}, \{b_k\}$ and $\{c_k\}$ be three sequences of nonnegative real numbers satisfying the inequality*

$$a_{k+1} \leq (1 + b_k)a_k + c_k, \quad \forall k \geq k_0,$$

for some integer $k_0 \geq 1$, where $\sum_{k=k_0}^{\infty} b_k < \infty$ and $\sum_{k=k_0}^{\infty} c_k < \infty$. Then, $\lim_{k\to\infty} a_k$ exists. In addition, if $\{a_k\}$ has a subsequence which converges to zero, then $\lim_{k\to\infty} a_k = 0$.

Lemma 2.2. ([10], Lemma 2.39) *Let $\{x^k\}$ be a sequence in a real Hilbert space $\mathcal{H}$ and let C be a nonempty subset of $\mathcal{H}$. Suppose that, for every $x \in C$, $\{\|x^k - x\|\}$ converges and that every weak sequential cluster point of $\{x^k\}$ belongs to C. Then, $\{x^k\}$ converges weakly to a point in C.*

3. Proximal-like subgradient algorithm

In this section, we introduce a new iteration algorithm and present its convergence which is called *proximal-like subgradient algorithm.*

We assume that the cost operator F is monotone and Lipschitz continuous with constant $L > 0$ in $\mathcal{R}^s$. Note that the subdifferential $\partial g(x)$ of g at $x \in C$ is defined by

$$\partial g(x) := \{w_x \in \mathcal{R}^s : g(y) - g(x) \geq \langle w_x, y - x\rangle \ \forall y \in \mathcal{R}^s\}.$$

Algorithm 3.1. *Now we construct an algorithm as follows:*

Step 0. *Let the initial point $x^0 \in \mathcal{R}^s$. Parameters $\bar{L}, \lambda_k$ and γ_k satisfy the following restrictions:*

$$\begin{cases} a \in (0,1),\ \bar{L} > L,\ 0 < \lambda_k < \frac{1}{\bar{L}},\ \sum\limits_{k=0}^{\infty} \lambda_k < \infty, \\ 0 < \gamma_k < 2,\ 0 < \liminf\limits_{k\to\infty} \gamma_k \leq \limsup\limits_{k\to\infty} \gamma_k < 2. \end{cases} \tag{3.1}$$

Step 1. $(k = 0, 1, ...)$ *Take $u^k \in F(x^k)$. Compute*

$$y^k = prox_{\lambda_k g}(x^k - \lambda_k u^k).$$

If $y^k = x^k$ then stop. Otherwise, go to Step 2.

Step 2. *Choose $v^k \in F(y^k)$ such that $\|u^k - v^k\| \leq \bar{L}\|x^k - y^k\|$.*

$$x^{k+1} = x^k - \gamma_k \rho_k d^k,$$

where $d^k = x^k - y^k - \lambda_k(u^k - v^k)$ and $\rho_k = \frac{1}{\|d^k\|^2}\langle x^k - y^k, d^k\rangle$. Set $k := k+1$ and return to Step 1.

Remark 3.2. (i) If $d^k = 0$ then

$$\|x^k - y^k\| = \lambda_k \|u^k - v^k\| \leq \lambda_k \bar{L}\|x^k - y^k\|.$$

Using this and $\lambda_k < \frac{1}{\bar{L}}$, we have

$$(1 - \lambda_k \bar{L})\|x^k - y^k\| \leq 0 \Rightarrow x^k = y^k.$$

Thus, we also observe that $d^k \neq 0$ and parameter ρ_k of Step 2 is defined.

(ii) For each $u^k \in F(x^k)$. Set $\bar{u}^k = Pr_{F(y^k)}(u^k)$. By the Lipschitz continuity of F,

$$\|u^k - \bar{u}^k\| = d[u^k, F(y^k)] \leq \rho[F(x^k), F(y^k)] \leq L\|x^k - y^k\| \leq \bar{L}\|x^k - y^k\|.$$

Thus, after taking $u^k \in F(x^k)$ there exists $v^k \in F(y^k)$ of Step 2 such that $v^k := \bar{u}^k$.

Lemma 3.3. *From the Algorithm* 3.1*, we have the following inequality.*

$$\rho_k \geq \frac{1 - \lambda_k \bar{L}}{1 + \lambda_k^2 \bar{L}^2}, \quad \forall k \geq 0.$$

Proof. Since the assumption F is $L-$Lipschitz continuous, $L < \bar{L}$ and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\phi_k :=&\langle x^k - y^k, d^k\rangle\\ =&\|x^k - y^k\|^2 - \lambda_k\langle x^k - y^k, u^k - v^k\rangle\\ \geq&\|x^k - y^k\|^2 - \lambda_k\|x^k - y^k\|\|u^k - v^k\|\\ \geq&(1 - \lambda_k\bar{L})\|x^k - y^k\|^2. \end{aligned} \tag{3.2}$$

From the monotonicity of F, it follows that

$$\langle v^k - u^k, y^k - x^k\rangle \geq 0$$

and hence

$$\begin{aligned}\|d^k\|^2 =&\|x^k - y^k - \lambda_k(u^k - v^k)\|^2\\ =&\|x^k - y^k\|^2 + \lambda_k^2\|u^k - v^k\|^2 - 2\lambda_k\langle x^k - y^k, u^k - v^k\rangle\\ \leq&\|x^k - y^k\|^2 + \lambda_k^2\|u^k - v^k\|^2\\ \leq&(1 + \lambda_k^2\bar{L}^2)\|x^k - y^k\|^2.\end{aligned}$$

Combining this and (3.2), we get the claim of Lemma 3.3 that

$$\begin{aligned}\rho_k &= \frac{1}{\|d^k\|^2}\phi_k\\ &\geq \frac{1}{\|d^k\|^2}(1 - \lambda_k\bar{L})\|x^k - y^k\|^2\\ &\geq \frac{1}{\|d^k\|^2}(1 - \lambda_k\bar{L})\frac{\|d^k\|^2}{(1 + \lambda_k^2\bar{L}^2)}\\ &= \frac{1 - \lambda_k\bar{L}}{1 + \lambda_k^2\bar{L}^2}.\end{aligned}$$

□

Lemma 3.4. *Let* (x^*, w^*) *be a solution of Problem* (1.1)*. Then,*

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - 2\rho_k\gamma_k(2 - \gamma_k)\phi_k$$

and there exists the limit $\lim_{k\to\infty}\|x^k - y^k\| = 0$.

Proof. Definition of y^k yields

$$\begin{aligned}y^k &= prox_{\lambda_k g}(x^k - \lambda_k u^k)\\ &= \operatorname{argmin}\left\{\lambda_k\langle u^k, y\rangle + \lambda_k g(y) + \frac{1}{2}\|y - x^k\|^2 : y \in C\right\}.\end{aligned}$$

There exists $\xi^k \in \partial g(y^k)$ such that

$$0 \in \lambda_k u^k + \lambda_k \xi^k + y^k - x^k + N_C(y^k),$$

where $N_C(y^k) = \{w \in \mathcal{R}^s : \langle w, y - y^k\rangle \leq 0, \quad \forall y \in C\}$ is the outer normal cone of C at $y^k \in C$. Then,

$$\langle \lambda_k u^k + \lambda_k \xi^k + y^k - x^k, y - y^k\rangle \geq 0, \quad \forall y \in C.$$

Taking $x^* \in C$, we get

$$\lambda_k \langle \xi^k, x^* - y^k\rangle \geq \langle \lambda_k u^k + y^k - x^k, y^k - x^*\rangle.$$

From $\xi^k \in \partial g(y^k)$, it follows that

$$\lambda_k [g(x^*) - g(y^k)] \geq \lambda_k \langle \xi^k, x^* - y^k\rangle.$$

Therefore,

$$\lambda_k [g(x^*) - g(y^k)] \geq \langle \lambda_k u^k + y^k - x^k, y^k - x^*\rangle. \tag{3.3}$$

By the monotonicity of F, $w^* \in F(x^*)$ and $v^k \in F(y^k)$, we have

$$\langle \lambda_k v^k - \lambda_k w^*, y^k - x^*\rangle \geq 0. \tag{3.4}$$

Since (x^*, w^*) is a solution of Problem (1.1) and $y^k \in C$, we obtain

$$\lambda_k [\langle w^*, y^k - x^*\rangle + g(y^k) - g(x^*)] \geq 0. \tag{3.5}$$

Adding three inequalities (3.3)-(3.5), we have

$$\langle d^k, y^k - x^*\rangle = \langle x^k - y^k - \lambda_k u^k + \lambda_k v^k, y^k - x^*\rangle \geq 0. \tag{3.6}$$

This implies that

$$\begin{aligned}
\|x^{k+1} - x^*\|^2 &= \|x^k - x^* - \gamma_k \rho_k d^k\|^2 \\
&= \|x^k - x^*\|^2 - 2\gamma_k \rho_k \langle x^k - x^*, d^k\rangle + \gamma_k^2 \rho_k^2 \|d^k\|^2 \\
&= \|x^k - x^*\|^2 - 2\gamma_k \rho_k \langle x^k - y^k, d^k\rangle \\
&\quad - 2\gamma_k \rho_k \langle y^k - x^*, d^k\rangle + \gamma_k^2 \rho_k^2 \|d^k\|^2 \\
&\leq \|x^k - x^*\|^2 - 2\gamma_k \rho_k \phi_k + \gamma_k^2 \rho_k^2 \|d^k\|^2 \\
&= \|x^k - x^*\|^2 - 2\gamma_k (2 - \gamma_k) \rho_k \phi_k.
\end{aligned}$$

So, $\|x^{k+1} - x^*\| \leq \|x^k - x^*\|$ for all $k \geq 1$ and there exists the limit

$$\lim_{k\to\infty} \|x^k - x^*\| = c < \infty.$$

Using the conditions $\lambda_k \in (0, \frac{1}{L})$, Lemma 3.3 and $\gamma_k \in (0, 2)$, we obtain

$$\begin{aligned}\phi_k \leq & \frac{\|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2}{2\gamma_k(2-\gamma_k)\rho_k} \\ \leq & \frac{1+\lambda_k^2 \bar{L}^2}{2\gamma_k(2-\gamma_k)\rho_k}(\|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2). \end{aligned} \tag{3.7}$$

Combining (3.2) and(3.7), we obtain

$$\begin{aligned}\|x^k - y^k\|^2 \leq & \frac{\phi_k}{1-\lambda_k \bar{L}} \\ \leq & \frac{1+\lambda_k^2 \bar{L}^2}{2(1-\lambda_k \bar{L})\gamma_k(2-\gamma_k)\rho_k}(\|x^k - x^*\|^2 - \|x^{k+1} - x^*\|) \\ \leq & M(\|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2), \end{aligned} \tag{3.8}$$

where $M = \sup_k \left\{\frac{1+\lambda_k^2 \bar{L}^2}{2(1-\lambda_k \bar{L})\gamma_k(2-\gamma_k)\rho_k}\right\} \in (0, \infty)$. Then, we have

$$\sum_{k=1}^{\infty} \|x^k - y^k\|^2 \leq M(\|x^1 - x^*\|^2 - c^2) < \infty,$$

and hence $\lim_{k\to\infty} \|x^k - y^k\| = 0$. This completes the proof. □

Lemma 3.5. *If the sequence $\{x^k\}$ is bounded, then every its convergent point belongs to Ω, where Ω denotes the solution set of Problem* (1.1).

Proof. Since $\{x^k\}$ is bounded, there exists a subsequence $\{x^{k_i}\}$ of $\{x^k\}$ which converges to $\bar{x}$. Let $Q : \mathcal{R}^s \to 2^{\mathcal{R}^s}$ be defined by

$$Q(x) = \begin{cases} F(x) + \partial g(x) + N_C(x), & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

Next, we show that $Q^{-1}(0) \subseteq \Omega$. Indeed, let $\hat{x} \in Q^{-1}(0) \subseteq C$, i.e.,

$$0 \in F(\hat{x}) + \partial g(\hat{x}) + N_C(\hat{x})$$

or equivalently there exist $w_1 \in F(\hat{x})$ and $w_2 \in \partial g(\hat{x})$ such that

$$\langle w_1 + w_2, x - \hat{x}\rangle \geq 0, \quad \forall x \in C.$$

Using the definition of $w_2 \in \partial g(\hat{x})$ that $g(x) - g(\hat{x}) \geq \langle w_2, x - \hat{x}\rangle$, we get

$$\langle w_1, x - \hat{x}\rangle + g(x) - g(\hat{x}) \geq 0, \quad \forall x \in C.$$

It means that $\hat{x} \in \Omega$.

On the other hand, it is well known to see that Q is a maximal monotone operator. Taking an arbitrary point $(x, w_x) \in graph(Q) := \{(x, y) \in \mathcal{R}^s \times \mathcal{R}^s :$

$y \in Q(x)\}$. Then, we get $w_x - w_F - w_g \in N_C(x)$, where $w_F \in F(x), w_g \in \partial g(x)$ and hence

$$\langle w_x - w_F - w_g, x - y^k \rangle \geq 0. \tag{3.9}$$

By the argument as (3.3) with $\lambda_k > 0$, we also have

$$g(x) - g(y^k) + \left\langle u^k + \frac{y^k - x^k}{\lambda_k}, x - y^k \right\rangle \geq 0. \tag{3.10}$$

Replacing k in (3.9) and (3.10) by k_i, and using the monotonicity of F, we obtain

$$\begin{aligned}
\langle w_x, x - y^{k_i} \rangle &\geq \langle w_F + w_{g_i}, x - y^{k_i} \rangle \\
&\geq \langle w_F + w_{g_i}, x - y^{k_i} \rangle - g(x) + g(y^{k_i}) \\
&\quad - \left\langle u^{k_i} + \frac{y^{k_i} - x^{k_i}}{\lambda_{k_i}}, x - y^{k_i} \right\rangle \\
&= \langle w_{g_i}, x - y^{k_i} \rangle + \langle w_F - u^{k_i}, x - y^{k_i} \rangle - g(x) + g(y^{k_i}) \\
&\quad - \left\langle \frac{y^{k_i} - x^{k_i}}{\lambda_{k_i}}, x - y^{k_i} \right\rangle \\
&\geq \langle w_{g_i}, x - y^{k_i} \rangle - g(x) + g(y^{k_i}) - \left\langle \frac{y^{k_i} - x^{k_i}}{\lambda_{k_i}}, x - y^{k_i} \right\rangle \\
&\geq - \left\langle \frac{y^{k_i} - x^{k_i}}{\lambda_{k_i}}, x - y^{k_i} \right\rangle,
\end{aligned}$$

where $w_{g_i} \in \partial g(y^{k_i})$ and the last inequality follows from the definition of w_{g_i}. Let the limit as $i \to \infty$, we have

$$\langle w_x - 0, x - \bar{x} \rangle \geq 0, \quad \forall x \in C.$$

Since Q is maximal monotone, so $\bar{x} \in Q^{-1}(0)$. The proof is complete. □

Theorem 3.6. *Let $\{x^k\}$ and $\{y^k\}$ be the sequences generated by Algorithm 3.1. Let $F : \mathcal{R}^s \to 2^{\mathcal{R}^s}$ be monotone and Lipschitz continuous, and $g : C \to \mathcal{R}$ be convex. Then, the sequences $\{(x^k, u^k)\}$ and $\{(y^k, v^k)\}$ converge to a solution (x^*, u^*) of Problem (1.1). Moreover,*

$$x^* = \lim_{k \to \infty} Pr_\Omega(x^k).$$

Proof. Since Lemmas 2.2 and 3.5, we can suppose that $\{x^k\}$ converges to $\bar{x} \in \Omega$. Set

$$\bar{x}^k = Pr_\Omega(x^k).$$

Then it follows from (3.8) that

$$\|x^{k+1} - \hat{x}\| \leq \|x^k - \hat{x}\|, \quad \forall \hat{x} \in \Omega.$$

Thus, $\{x^k\}$ is Fejér monotone with respect to Ω and hence the sequence $\{\bar{x}^k\}$ converges to some $x^* \in \Omega$. Note that Ω is convex when F is monotone. Otherwise, using the definition of Pr_Ω and $\bar{x} \in \Omega$, we have

$$\langle \bar{x} - \bar{x}^k, \bar{x}^k - x^k \rangle \geq 0.$$

Letting the limit as $k \to \infty$, we get

$$\langle \bar{x} - x^*, x^* - \bar{x} \rangle \geq 0$$

and $\bar{x} = x^*$. By Lemma 3.4, we have $\|x^k - y^k\| \to 0$ and $y^k \rightharpoonup x^*$. In Algorithm 3.1, $\|u^k - v^k\| \leq \bar{L}\|x^k - y^k\| \to 0$. Using the Lipschitz continuity of F, we get that u^k and v^k converge to some point $u^* \in F(x^*)$. □

4. Rate of convergence

Now we consider the ergodic rate of convergence for the sequence $\{y^k\}$ for Algorithm 3.1 via using the gap functions of Problem (1.1). There exist various gap functions for variational inequalities that have been proposed and their properties have been studied [10, 17]. Among them, the gap function

$$h(x) := \max\{\langle F(x), x - y \rangle + g(x) - g(y) : y \in dom(g)\}$$

first introduced by Auslender [9], has the property that its minimum on C coincides with a solution of Problem $VI(C, F, g)$, and hence the problem can be reformulated as the optimization problem

$$\min\{h(x) : x \in C\}.$$

Based on this idea, Marcotte [27] proposed a descent algorithm for monotone variational inequalities, Taji and Fukushima [32] have proposed a new regularized gap function and a descent method for solving Problem $VI(C, F, g)$ using the regularized gap function.

In the sequel, we suppose that $C \subseteq dom(g)$. Using the following error gap function which also is said to be the *dual gap function* for Problem (1.1):

$$h(x, w_x) = \max\left\{\Phi(x, y) := \langle w_x, y - x \rangle + g(y) - g(x) : x \in C\right\},$$

where $w_x \in F(x)$ and $x \in C$. It is clear that for each $x \in C$, $\Phi(x, x) = 0$, so $h(x, w_x) \geq 0$ for all $w_x \in F(x)$.

Lemma 4.1. ([18]) *Point $x^* \in C$ is a solution of Problem* (1.1) *if and only if $x^* \in dom(g)$ and $h(x^*, w_{x^*}) = 0$.*

To show the rate of convergence of $\{y^k\}$, we recall the following classical lemma for its proofs we refer to [10].

Lemma 4.2. *Let $g : \mathcal{R}^s \to (-\infty, +\infty]$ be a convex function, $x \in \mathcal{R}^s$. Then, $z = prox_g(x)$ if and only if*

$$\langle z - x, y - z\rangle \geq g(z) - g(y), \quad \forall y \in \mathcal{R}^s.$$

Theorem 4.3. *Let $\{y^k\}$ be the sequence generated by Algorithm 3.1. For any $N \geq 1$, define σ_N and $\bar{y}^N$ as*

$$\sigma_N = \sum_{k=1}^{N} \lambda_k, \quad \bar{y}^N = \frac{1}{\sigma_N} \sum_{k=1}^{N} \lambda_k y^k.$$

Then,

$$\Phi(x, \bar{y}^N) \leq \frac{K\sqrt{M}\|x^1 - x^*\|}{\lambda\sqrt{N}},$$

where $K := \sup_k \left\{\sqrt{1 + \lambda_k^2 \bar{L}^2}\|y^k - x\|\right\}$ and $M = \sup_k \left\{\frac{1+\lambda_k^2\bar{L}^2}{(1-\lambda_k\bar{L})^2\gamma_k(2-\gamma_k)\rho_k}\right\} \in (0, \infty)$.

Proof. Applying Lemma 4.2 for $y^k = prox_{\lambda_k g}(x^k - \lambda_k w^k)$,

$$\langle y^k - x^k + \lambda_k u^k, x - y^k\rangle \geq \lambda_k g(y^k) - \lambda_k g(x), \quad \forall x \in \mathcal{R}^s.$$

Hence we have

$$\langle y^k - x^k, x - y^k\rangle \geq \lambda_k[\langle u^k, y^k - x\rangle + g(y^k) - g(x)], \quad \forall x \in \mathcal{R}^s. \tag{4.1}$$

Using (4.1), the convexity of g and the monotonicity of F, we have

$$\begin{aligned}
\Phi(x, \bar{y}^N) =& \left\langle w_x, \bar{y}^N - x\right\rangle + g\left(\bar{y}^N\right) - g(x) \\
=& \frac{1}{\sigma_N}\sum_{k=1}^{N}\lambda_k\langle w_x, y^k - x\rangle + g\left(\sum_{k=1}^{N}\frac{\lambda_k}{\sigma_N}y^k\right) - g(x) \\
\leq& \frac{1}{\sigma_N}\sum_{k=1}^{N}\lambda_k\langle w_x, y^k - x\rangle + \frac{1}{\sigma_N}\sum_{k=1}^{N}\lambda_k g(y^k) - g(x) \\
=& \frac{1}{\sigma_N}\sum_{k=1}^{N}\lambda_k[\langle w_x, y^k - x\rangle + g(y^k) - g(x)] \\
=& \frac{1}{\sigma_N}\sum_{k=1}^{N}\lambda_k\langle w_x - v^k, y^k - x\rangle + \frac{1}{\sigma_N}\sum_{k=1}^{N}\lambda_k\langle v^k - u^k, y^k - x\rangle \\
&+ \frac{1}{\sigma_N}\sum_{k=1}^{N}\lambda_k[\langle u^k, y^k - x\rangle + g(y^k) - g(x)]
\end{aligned}$$

$$
\begin{aligned}
\leq & \frac{1}{\sigma_N}\sum_{k=1}^{N}\lambda_k\langle w_x - v^k, y^k - x\rangle + \frac{1}{\sigma_N}\sum_{k=1}^{N}\lambda_k\langle v^k - u^k, y^k - x\rangle \\
& + \frac{1}{\sigma_N}\sum_{k=1}^{N}\langle y^k - x^k, x - y^k\rangle \\
\leq & \frac{1}{\sigma_N}\sum_{k=1}^{N}\lambda_k\langle v^k - u^k, y^k - x\rangle + \frac{1}{\sigma_N}\sum_{k=1}^{N}\langle y^k - x^k, x - y^k\rangle \\
= & \frac{1}{\sigma_N}\sum_{k=1}^{N}\langle -\lambda_k(u^k - v^k) + x^k - y^k, y^k - x\rangle \\
= & \frac{1}{\sigma_N}\sum_{k=1}^{N}\langle d^k, y^k - x\rangle \\
\leq & \frac{1}{\sigma_N}\sum_{k=1}^{N}\|d^k\|\|y^k - x\| \\
\leq & \frac{K}{\sigma_N}\sum_{k=1}^{N}\|x^k - y^k\| \\
\leq & \frac{K}{\sigma_N}\sqrt{N\sum_{k=1}^{N}\|x^k - y^k\|^2} \\
\leq & \frac{K}{\sigma_N}\sqrt{NM}\|x^1 - x^*\| \\
\leq & \frac{K\sqrt{M}\|x^1 - x^*\|}{\lambda\sqrt{N}},
\end{aligned}
$$

where $K := \sup_k\left\{\sqrt{1+\lambda_k^2\bar{L}^2}\|y^k - x\|\right\}$ and $M = \sup_k\left\{\frac{1+\lambda_k^2\bar{L}^2}{(1-\lambda_k\bar{L})^2\gamma_k(2-\gamma_k)\rho_k}\right\} \in (0, \infty)$. □

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