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# ON THE HYPERSTABILITY OF p-RADICAL FUNCTIONAL EQUATION RELATED TO CUBIC MAPPING IN ULTRAMETRIQUE BANACH SPACE

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**Abstract.** The aim of this paper is to introduce and solve the p-radical functional equation related to cubic functional equation

$$f\left(\sqrt[p]{2x^p+y^p}\right)+f\left(\sqrt[p]{2x^p-y^p}\right)=2f\left(\sqrt[p]{x^p+y^p}\right)+2f\left(\sqrt[p]{x^p-y^p}\right)+12f(x),\quad x,y\in\mathbb{R},$$

where $f$ is a mapping from $\mathbb{R}$ into a vector space and $p$ is an odd integer. We also establish the hyperstability results in the senses of Gâvruta-Ulam -Rassiass for the considered equation in non-Archimedean Banach spaces, by using an analogue version of theorem of J. Brzdęk in [12].

## 1. Introduction

A classical question in the theory of functional equation is the following: "When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of equation". If the answer is affirmative, we say that the equation is stable.

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In 1940, Ulam, when he discussed a number of important unsolved problems in his talk at the university of wisconsin(see [33]), he asked the following question concerning the stability of group homomorphisms, this question, seem to be the starting point of studying the stability of functional equations, as follows:

*Let* $(G_1, *_1)$ *be a group and let* $(G_2, *_2)$ *be a metric group with a metric* $d(.,.)$. *Given* $\varepsilon > 0$, *does there exists a* $\delta > 0$ *such that if a mapping* $h : G_1 \to G_2$ *satisfies the inequality*

$$d\big(h(x *_1 y), h(x) *_2 h(y)\big) < \delta$$

*for all* $x, y \in G_1$, *then there exists a homomorphism* $H : G_1 \to G_2$ *with*

$$d\big(h(x), H(x)\big) < \varepsilon$$

*for all* $x \in G_1$*?*

Since then, this question has attracted the attention of many researchers. The first partial answer was raised by Hyers [24] in 1941 under the assumption that $G_1$ and $G_1$ are Banach spaces for the additive functional equation as follows:

**Theorem 1.1.** ([24]) *Let* $E_1$ *and* $E_2$ *be two Banach spaces and* $f : E_1 \to E_2$ *be a function such that*

$$\|f(x+y) - f(x) - f(y)\| \le \delta$$

*for some* $\delta > 0$ *and for all* $x, y \in E_1$. *Then the limit*

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

*exists for each* $x \in E_1$, *and* $A : E_1 \to E_2$ *is the unique additive function such that*

$$\|f(x) - A(x)\| \le \delta$$

*for all* $x \in E_1$. *Moreover, if* $f(tx)$ *is continuous in* $t$ *for each fixed* $x \in E_1$, *then the function* $A$ *is linear.*

Later, Aoki [9] and Bourgin [10] considered the problem of stability with unbounded Cauchy differences. In 1978, Rassias [30] attempted to weaken the condition for the bound of the norm of Cauchy difference

$$\|f(x+y) - f(x) - f(y)\|$$

and proved a generalization of Theorem 1.1 by using a direct method(cf. Theorem 1.2):

**Theorem 1.2.** ([30]) *Let $E_1$ and $E_2$ be two Banach spaces. If $f : E_1 \to E_2$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta\big(\|x\|^p + \|y\|^p\big)$$

*for some $\theta \geq 0$, for some $p \in \mathbb{R}$ with $0 \leq p < 1$, and for all $x, y \in E_1$, then there exists a unique additive function $A : E_1 \to E_2$ such that*

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2 - 2^p}\|x\|^p$$

*for each $x \in E_1$. If, in addition, $f(tx)$ is continuous in $t$ for each fixed $x \in E_1$, then the function $A$ is linear.*

After then, Rassias ([31], [32]) motivated Theorem 1.2 as follows:

**Theorem 1.3.** ([31], [32]) *Let $E_1$ be a normed space, $E_2$ be a Banach space, and $f : E_1 \to E_2$ be a function. If $f$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta\big(\|x\|^p + \|y\|^p\big) \tag{1.1}$$

*for some $\theta \geq 0$, for some $p \in \mathbb{R}$ with $p \neq 1$, and for all $x, y \in E_1 - \{0_{E_1}\}$, then there exists a unique additive function $A : E_1 \to E_2$ such that*

$$\|f(x) - A(x)\| \leq \frac{2\theta}{|2 - 2^p|}\|x\|^p \tag{1.2}$$

*for each $x \in E_1 - \{0_{E_1}\}$.*

Note that Theorem 1.3 reduces to Theorem 1.1 when $p = 0$. For $p = 1$, the analogous result is not valid. Also, Brzdęk [11] showed that estimation (1.2) is optimal for $p \geq 0$ in the general case.

In 1994, Gâvruta [23] provided a further generalization of Rassias theorem in which he replaced the bound $\theta\left(\|x\|^p + \|y\|^p\right)$ in (1.1) by a general control function $\phi(x, y)$ for the existence of a unique linear mapping.

Recently, Brzdęk [15] showed that Theorem 1.3 can be significantly improved; namely, in the case $p < 0$, each $f : E_1 \to E_2$ satisfying (1.1) must actually be additive, this result is called the hyperstability of Cauchy functional equation. However, the term of hyperstability was introduced for the first time probably in [28], and it was developed with fixed point theorem of Brzdęk in [12] and there after, the hyperstability of a several functional equation have been studied by many authors (for example see [2, 3, 7, 15, 28]).

In 2013, Brzdęk [14] improved, extended and complemented several earlier classical stability results concerning the additive Cauchy equation (in particular Theorem 1.3). Over the last few years, many mathematicians have investigated various generalizations, extensions and applications of the Hyers-Ulam stability of a number of functional equations (see, for instance, [16],

[17] and references therein); in particular, the stability problem of the radical functional equations in various spaces was proved in [6, 7, 8, 19, 20, 25, 27].

Jun and Kim [26] introduced the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x), \quad x,y\in\mathbb{R}, \tag{1.3}$$

where $f:\mathbb{R}\to X$, and they established the general solution and the generalized Hyers-Ulam-Rassias stability for the functional equation (1.3). It is easy to see that the function $f(x)=cx^3$ is a solution of the above functional equation. Thus, it is natural that equation (1.3) is called a cubic functional equation and every solution of the cubic functional equation is said to be a cubic function.

Now we introduce the following functional equation

$$\begin{aligned} &f\left(\sqrt[p]{2x^p+y^p}\right)+f\left(\sqrt[p]{2x^p-y^p}\right)\\ &=2f\left(\sqrt[p]{x^p+y^p}\right)+2f\left(\sqrt[p]{x^p-y^p}\right)+12f(x), \end{aligned} \tag{1.4}$$

for all $x,y\in\mathbb{R}$ where $p$ is an odd integer and $f:\mathbb{R}\to X$ which is the p-radical cubic functional equation related to equation (1.3).

The main purpose of this article is to achieve the general solution of the functional equation (1.4) and establish some hyperstability in the spirit of Găvruta-Ulam -Rassiass of the considered equation in non-Archimedean Banach space. We also provide some corollaries and outcomes concerning the hyperstability results for the inhomogeneous of p-radical functional equation.

Throughout this paper, we will denote the set of natural numbers by $\mathbb{N}$, the set of real numbers by $\mathbb{R}$, $\mathbb{R}_+=[0,\infty)$ the set of non negative real numbers and $\mathbb{R}_0=\mathbb{R}\backslash\{0\}$. By $\mathbb{N}_{m_0}$, $m_0\in\mathbb{N}$, we will denote the set of all natural numbers greater than or equal to $m_0$.

Let us recall(see, for instance, [25]) some basic definitions and facts concerning non-Archimedean normed spaces.

**Definition 1.4.** By a *non-Archimedean* field we mean a field $\mathbb{K}$ equipped with a function (*valuation*) $|\cdot|:\mathbb{K}\to[0,\infty)$ such that for all $r,s\in\mathbb{K}$, the following conditions hold:

(1) $|r|=0$ if and only if $r=0$,
(2) $|rs|=|r||s|$,
(3) $|r+s|\le\max\{|r|,|s|\}$.

The pair $(\mathbb{K},|.|)$ is called a *valued field.*

In any non-Archimedean field we have $|1|=|-1|=1$ and $|n|\le 1$ for $n\in\mathbb{N}_0$. In any field $\mathbb{K}$ the function $|\cdot|:\mathbb{K}\to\mathbb{R}_+$ given by

$$|x| := \begin{cases} 0, & x = 0, \\ 1, & x \neq 0, \end{cases}$$

is a valuation which is called *trivial*, but the most important examples of non-Archimedean fields are $p$-adic numbers which have gained the interest of physicists for their research in some problems coming from quantum physics, $p$-adic strings and superstrings.

**Definition 1.5.** Let $X$ be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $||\cdot||_* : X \to \mathbb{R}$ is a *non-Archimedean norm (valuation)* if it satisfies the following conditions:

(1) $\|x\|_* = 0$ if and only if $x = 0$,
(2) $\|rx\|_* = |r| \, \|x\|_*$ $(r \in \mathbb{K}, x \in X)$,
(3) The strong triangle inequality (ultrametric); namely :
$\|x + y\|_* \leq \max\left\{\|x\|_*, \|y\|_*\right\}$ $x, y \in X$.

Then $(X, \|\cdot\|_*)$ is called a *non-Archimedean normed space* or an *ultrametric normed space.*

**Definition 1.6.** Let $\{x_n\}$ be a sequence in a non-Archimedean normed space $X$.

(1) A sequence $\{x_n\}_{n=1}^{\infty}$ in a non-Archimedean space is a *Cauchy sequence* if the sequence $\{x_{n+1} - x_n\}_{n=1}^{\infty}$ converges to zero.
(2) The sequence $\{x_n\}$ is said to be *convergent* if, there exists $x \in X$ such that, for any $\varepsilon > 0$, there is a positive integer $N$ such that $\|x_n - x\|_* \leq \varepsilon$, for all $n \geq N$. Then the point $x \in X$ is called the *limit* of the sequence $\{x_n\}$, which is denoted by $\lim_{n\to\infty} x_n = x$.
(3) If every Cauchy sequence in $X$ converges, then the non-Archimedean normed space $X$ is called a *non-Archimedean Banach space* or an *utrametric Banach space.*

In this paper, we make non-Archimedean versions of results in [16]. Indeed, by using the fixed point method derived from [11] and [16], we present some hyperstability results for the p-radical functional equation related to cubic mapping in non-Archimedean Banach spaces. Before proceeding to the main results, we state Theorem 1.7 which is useful for our purpose. To present it, we introduce the following three hypotheses:

(**H1**) $X$ is a nonempty set, $Y$ is an non-Archimedean Banach space over a non-Archimedean field, $f_1, ..., f_k : X \longrightarrow X$ and $L_1, ..., L_k : X \longrightarrow \mathbb{R}_+$ are given.

(**H2**) $\mathcal{T} : Y^X \longrightarrow Y^X$ is an operator satisfying the inequality

$$\left\|\mathcal{T}\xi(x) - \mathcal{T}\mu(x)\right\|_* \leq \max_{1\leq i\leq k} \left\{L_i(x)\left\|\xi\Big(f_i(x)\Big) - \mu\Big(f_i(x)\Big)\right\|_*\right\},$$

for $\xi, \mu \in Y^X$, $x \in X$.

(**H3**) $\Lambda : \mathbb{R}_+^X \longrightarrow \mathbb{R}_+^X$ is a linear operator defined by

$$\Lambda\delta(x) := \max_{1\le i\le k} \left\{ L_i(x)\delta\Big(f_i(x)\Big)\right\}, \qquad \delta \in \mathbb{R}_+^X, \quad x \in X.$$

Thanks to a result due to Brzdęk and Ciepliñski [13, Remark 2], we state an analogue of the fixed point theorem [13, Theorem 1] in non-Archimedean Banach space. We use it to assert the existence of a unique fixed point of operator $\mathcal{T} : Y^X \longrightarrow Y^X$.

**Theorem 1.7.** *Let hypotheses* (**H1**)-(**H3**) *be valid and functions* $\varepsilon : X \longrightarrow \mathbb{R}_+$ *and* $\varphi : X \longrightarrow Y$ *fulfil the following two conditions*

$$\|\mathcal{T}\varphi(x) - \varphi(x)\|_* \le \varepsilon(x), \;\; x \in X,$$

$$\lim_{n\to\infty} \Lambda^n \varepsilon(x) = 0, \;\; x \in X.$$

*Then there exists a unique fixed point* $\psi \in Y^X$ *of* $\mathcal{T}$ *with*

$$\|\varphi(x) - \psi(x)\|_* \le \sup_{n\in\mathbb{N}_0} \Lambda^n\varepsilon(x), \;\; x \in X.$$

*Moreover*

$$\psi(x) := \lim_{n\to\infty} \mathcal{T}^n\varphi(x), \;\; x \in X.$$

## 2. Solution and hyperstability of equation (1.4)

Note that Jun and Kim in [26] are solved the cubic functional equation (1.3) and every solution of (1.3) is given as follows : $f(x) = B(x, x, x)$ where $B : E_1 \times E_1 \times E_1 \to E_2$ defined by

$$B(x,y,z) = \frac{1}{24}\{f(x+y+z)+f(x+y+z)+f(x-y-z)-f(x+y-z)-f(x-y+z)\}$$

and $B$ is symmetric for fixed one variable and $B$ is additive for fixed two variables, where $E_1$, $E_2$ are two vector spaces.

The following theorem give the general solution of Eq. (1.4).

**Theorem 2.1.** *Let* $X$ *be a linear space and* $p$ *be an odd integer. A function* $f : \mathbb{R} \to X$ *is solution of the functional equation* (1.4) *if and only if* $f(x) = Q(x^p)$ *for all* $x \in \mathbb{R}$*, such that* $Q$ *is a solution of a cubic functional equation* (1.3)*.*

*Proof.* It's not hard to see that if $f(x) = B(x^p, x^p, x^p)$, then $f$ is solution of the equation (1.4).
On the other hand, if $f$ is solution of Eq. (1.4), then for all $x, y \in \mathbb{R}$ :

$$\begin{aligned} Q(2x+y) + Q(2x-y) =& f\left(\sqrt[p]{2\sqrt[p]{x^p} + \sqrt[p]{y^p}}\right) + f\left(\sqrt[p]{2\sqrt[p]{x^p} - \sqrt[p]{y^p}}\right) \\ =& 2f\left(\sqrt[p]{\sqrt[p]{x^p} + \sqrt[p]{y^p}}\right) + 2f\left(\sqrt[p]{\sqrt[p]{x^p} - \sqrt[p]{y^p}}\right) \\ &+ 12f(\sqrt[p]{x}) \\ =& 2Q(x+y) + 2Q(x-y) + 12Q(x). \end{aligned}$$

□

Next, we examine the hyperstability of Eq. (1.4) in non-Archimedean Banach space, by using as a basic tool, Theorem 1.7.

**Theorem 2.2.** *Let $p$ be an odd integer, $X$ be a non-Archimedean Banach space and $h_1, h_2 : \mathbb{R}_0 \to \mathbb{R}_+$ be two functions such that*

$$\begin{aligned} \mathcal{U} := \big\{n \in \mathbb{N} : \alpha_n = \max\{&\lambda_1(n+1)\lambda_2(n+1), \lambda_1(3n+2)\lambda_2(3n+2), \\ &\lambda_1(-n)\lambda_2(-n), \lambda_1(4n+3)\lambda_2(4n+3)\} < 1\big\} \neq \phi, \end{aligned}$$

*is an infinite set, where*

$$\lambda_i(m) = \inf\left\{t \in \mathbb{R}_+ : h_i(mx) \leq t\, h_i(x), \quad x \in \mathbb{R}_0\right\},$$

*for all $m \in \mathbb{N}$, where $i = 1, 2$ such that*

$$\lim_{m\to\infty} \lambda_1(m+1)\lambda_2(2m+1) = 0. \tag{2.1}$$

*Assume that $f : \mathbb{R} \to X$ satisfies the inequality*

$$\begin{aligned} &\left\| f\left(\sqrt[p]{2x^p + y^p}\right) + f\left(\sqrt[p]{2x^p - y^p}\right) f\left(\sqrt[p]{x^p + y^p}\right) - 2f\left(\sqrt[p]{x^p - y^p}\right) - 12f(x) \right\|_* \\ &\leq h_1(x^p)h_2(y^p), \end{aligned} \tag{2.2}$$

*for all $x, y \in \mathbb{R}_0$ such that $x \neq y, x \neq -y,\ y \neq \sqrt[p]{2}\, x$ and $y \neq -\sqrt[p]{2}x$. Then $f$ is a solution of Eq. (1.4) on $\mathbb{R}_0$.*

*Proof.* Replacing $x$ by $\sqrt[p]{m+1}\, x$ and $y$ by $\sqrt[p]{2m+1}\, x$ in inequality (2.2), we get

$$\begin{aligned} &\left\| 12f\left(\sqrt[p]{m+1}x\right) + 2f\left(\sqrt[p]{3m+2}x\right) + 2f\left(\sqrt[p]{-m}x\right) - f\left(\sqrt[p]{4m+3}x\right) - f(x) \right\|_* \\ &\leq h_1((m+1)x^p)h_2((2m+1)x^p), \end{aligned} \tag{2.3}$$

for all $x \in \mathbb{R}_0$. For each $m \in \mathbb{N}$, we define operators $\mathcal{T}_m : X^{\mathbb{R}_0} \to X^{\mathbb{R}_0}$ by

$$\mathcal{T}_m\xi(x) = 12\xi\left(\sqrt[p]{m+1}\, x\right) + 2\xi\left(\sqrt[p]{3m+2}\, x\right) + 2\xi\left(\sqrt[p]{-m}\, x\right) - \xi\left(\sqrt[p]{4m+3}\, x\right),$$

for all $\xi \in X^{\mathbb{R}_0}$, $x \in \mathbb{R}_0$ and $\varepsilon_m : \mathbb{R}_0 \to \mathbb{R}_+$ by

$$\varepsilon_m(x) = h_1((m+1)x^p)h_2((2m+1)x^p), \quad m \in \mathbb{N}, \quad x \in \mathbb{R}_0,$$

we observe that

$$\varepsilon_m(x) \leq \lambda_1(m+1)\lambda_2(2m+1)h_1(x^p)h_2(x^p), \tag{2.4}$$

for all $x \in \mathbb{R}_0$ and all $m \in \mathcal{U}$. Then the inequality (2.3) become as

$$\left\|\mathcal{T}_m f(x) - f(x)\right\|_* \leq \varepsilon_m(x), \quad x \in \mathbb{R}_0.$$

Furthermore, the operator $\Lambda_m : \mathbb{R}_+^{\mathbb{R}_0} \to \mathbb{R}_+^{\mathbb{R}_0}$ defined by

$$\begin{aligned}\Lambda_m\delta(x) &= \max\{\delta\left(\sqrt[p]{m+1}\,x\right), \delta\left(\sqrt[p]{3m+2}\,x\right), \delta\left(\sqrt[p]{-m}\,x\right), \delta\left(\sqrt[p]{4m+3}\,x\right)\}\\ &= \max_{1\leq i\leq 4}\{L_i(x)\delta\left(f_i(x)\right)\},\end{aligned}$$

for all $x \in \mathbb{R}_0$ and $\delta \in \mathbb{R}_+^{\mathbb{R}_0}$ has the form described in (**H3**) with $k = 4$ and $f_1(x) = \sqrt[p]{m+1}\ x$, $f_2(x) = \sqrt[p]{3m+2}\ x$, $f_3(x) = \sqrt[p]{-m}\ x$, $f_4(x) = \sqrt[p]{4m+3}\,x$, and $L_1(x) = L_2(x) = L_3(x) = L_4(x) = 1$, for all $x \in \mathbb{R}_0$.

Moreover, for every $x \in \mathbb{R}_0$, $\xi, \mu \in X^{\mathbb{R}_0}$, we obtain

$$\begin{aligned}&\left\|\mathcal{T}_m\xi(x) - \mathcal{T}_m\mu(x)\right\|_*\\ &= \left\|12\left(\xi\left(\sqrt[p]{m+1}x\right) - \mu\left(\sqrt[p]{m+1}x\right)\right) + 2\left(\xi\left(\sqrt[p]{3m+2}x\right) - \mu\left(\sqrt[p]{3m+2}x\right)\right)\right.\\ &\quad\left. + 2\left(\xi\left(\sqrt[p]{-m}x\right) - \mu\left(\sqrt[p]{-m}x\right)\right) - \left(\xi\left(\sqrt[p]{4m+3}x\right) - \mu\left(\sqrt[p]{4m+3}x\right)\right)\right\|_*\\ &\leq \max\Big\{12\|\xi\left(\sqrt[p]{m+1}x\right) - \mu\left(\sqrt[p]{m+1}x\right)\|_*,\\ &\qquad 2\|\xi\left(\sqrt[p]{3m+2}x\right) - \mu\left(\sqrt[p]{3m+2}x\right)\|_*,\\ &\qquad 2\|\xi\left(\sqrt[p]{-m}x\right) - \mu\left(\sqrt[p]{-m}x\right)\|_*, \|\xi\left(\sqrt[p]{4m+3}x\right) - \mu\left(\sqrt[p]{4m+3}x\right)\|_*\Big\}\\ &\leq \max\{\|\xi\left(\sqrt[p]{m+1}x\right) - \mu\left(\sqrt[p]{m+1}x\right)\|_*,\\ &\qquad \|\xi\left(\sqrt[p]{3m+2}x\right) - \mu\left(\sqrt[p]{3m+2}x\right)\|_*,\\ &\qquad \|\xi\left(\sqrt[p]{-m}x\right) - \mu\left(\sqrt[p]{-m}x\right)\|_*, \|\xi\left(\sqrt[p]{4m+3}x\right) - \mu\left(\sqrt[p]{4m+3}x\right)\|_*\}\\ &= \max_{1\leq i\leq 4}\{L_i(x)\|\xi(f_i(x)) - \mu(f_i(x))\|_*\},\end{aligned}$$

so (**H2**) is valid.

Now we will show, by induction, that for all $x \in \mathbb{R}_0$, $m \in \mathcal{U}$ and $n \in \mathbb{N}$

$$\Lambda^n\varepsilon_m(x) \leq \lambda_1(m+1)\lambda_2(2m+1)\alpha_m^n h_1(x^p)h_2(x^p). \tag{2.5}$$

For $n = 0$, inequality (2.5) is exactly (2.4). Next we will assume that (2.5) holds for $n = k$, where $k \in \mathbb{N}$. Then

$$\begin{aligned}\Lambda_m^{k+1}\varepsilon_m(x) &= \Lambda_m\left(\Lambda_m^k\varepsilon_m(x)\right)\\ &= \max\left\{\Lambda_m^k\varepsilon_m\left(\sqrt[p]{m+1}\,x\right),\ \Lambda_m^k\varepsilon_m\left(\sqrt[p]{3m+2}\,x\right), \Lambda_m^k\varepsilon_m\left(\sqrt[p]{-m}\,x\right),\right.\\ &\qquad \left.\Lambda_m^k\varepsilon_m\left(\sqrt[p]{4m+3}\,x\right)\right\}\\ &\le \lambda_1(m+1)\lambda_2(2m+1)\alpha_m^k\max\left\{h_1((m+1)x^p)h_2((m+1)x^p),\right.\\ &\qquad h_1((3m+2)x^p)h_2((3m+2)x^p),\ h_1((-m)x^p)h_2((-m)x^p),\\ &\qquad \left.h_1((4m+3)x^p)h_2((4m+3)x^p)\right\}\\ &\le \lambda_1(m+1)\lambda_2(2m+1)\alpha_m^k\max\left\{\lambda_1(m+1)\lambda_2(m+1),\right.\\ &\qquad \lambda_1(3m+2)\lambda_2(3m+2), \lambda_1(-m)\lambda_2(-m),\\ &\qquad \left.\lambda_1(4m+3)\lambda_2(4m+3)\right\}h_1(x^p)h_2(x^p)\\ &= \lambda_1(m+1)\lambda_2(2m+1)\alpha_m^{k+1}h_1(x^p)h_2(x^p),\end{aligned}$$

for all $x \in \mathbb{R}_0$, $m \in \mathcal{U}$. This show that (2.5) holds for $n = k+1$. We conclude that the inequality (2.5) holds for all $n \in \mathbb{N}$.

Since, for each $m \in \mathcal{U}, \alpha_m < 1$, hence, we get

$$\lim_{n\to\infty}\Lambda^n\varepsilon_m(x) = 0,$$

for all $x \in \mathbb{R}_0$. Therefore, according to Theorem 1.7, there exists, for each $m \in \mathcal{U}$, a fixed point $\mathcal{F}_m : \mathbb{R}_0 \to X$ of the operator $\mathcal{T}_m$ such that

$$\begin{aligned}\left\|f(x) - \mathcal{F}_m(x)\right\|_* &\le \sup_{n\in\mathbb{N}}\left\{\Lambda_m^n\varepsilon_m(x)\right\} \qquad (2.6)\\ &\le \sup_{n\in\mathbb{N}}\left\{\lambda_1(m+1)\lambda_2(2m+1)\alpha_m^n h_1(x^p)h_2(x^p)\right\},\quad x \in \mathbb{R}_0.\end{aligned}$$

Moreover $\mathcal{F}_m(x) = \lim_{n\to\infty}\left(\mathcal{T}_m^n f\right)(x),\quad x \in \mathbb{R}_0.$

Next, we should prove the following inequality

$$\begin{aligned}&\left\|\mathcal{T}_m^n f\left(\sqrt[p]{2x^p+y^p}\right) + \mathcal{T}_m^n f\left(\sqrt[p]{2x^p-y^p}\right) - 2\mathcal{T}_m^n f\left(\sqrt[p]{x^p+y^p}\right)\right. \qquad (2.7)\\ &\quad \left.- 2\mathcal{T}_m^n f\left(\sqrt[p]{x^p-y^p}\right) - 12\mathcal{T}_m^n f(x)\right\|_* \le \alpha_m^n h_1(x^p)h_2(y^p),\end{aligned}$$

for every $x, y \in \mathbb{R}_0$ such that $x \neq y$, $x \neq -y$, $y \neq \sqrt[p]{2}x$, $y \neq -\sqrt[p]{2}x$, $n \in \mathbb{N}$ and $m \in \mathcal{U}$.

We proceed by induction, since the case $n = 0$ is just (2.2), we take $k \in \mathbb{N}$ and assume that (2.7) holds for $n = k$ and every $x, y \in \mathbb{R}_0$ such that $x \neq y$, $x \neq -y$, $y \neq \sqrt[p]{2}x$, $y \neq -\sqrt[p]{2}x$ and $m \in \mathcal{U}$. Then, for each $x, y \in \mathbb{R}_0$ such that $x \neq y$, $x \neq -y$, $y \neq \sqrt[p]{2}x$, $y \neq -\sqrt[p]{2}x$ and $m \in \mathcal{U}$, we get

$$\begin{aligned}
&\Big\|\mathcal{T}_m^{k+1} f\left(\sqrt[p]{2x^p+y^p}\right)+\mathcal{T}_m^{k+1} f\left(\sqrt[p]{2x^p-y^p}\right)-2\mathcal{T}_m^{k+1} f\left(\sqrt[p]{x^p+y^p}\right)\\
&\quad-2\mathcal{T}_m^{k+1} f\left(\sqrt[p]{x^p-y^p}\right)-12\mathcal{T}_m^{k+1} f(x)\Big\|_*\\
=&\Big\|12\mathcal{T}_m^k f\left(\sqrt[p]{m+1}\sqrt[p]{2x^p+y^p}\right)+2\mathcal{T}_m^k f\left(\sqrt[p]{3m+2}\sqrt[p]{2x^p+y^p}\right)\\
&\quad+2\mathcal{T}_m^k f\left(\sqrt[p]{-m}\sqrt[p]{2x^p+y^p}\right)-\mathcal{T}_m^k f\left(\sqrt[p]{4m+3}\sqrt[p]{2x^p+y^p}\right)\\
&\quad+12\mathcal{T}_m^k f\left(\sqrt[p]{m+1}\sqrt[p]{2x^p-y^p}\right)+2\mathcal{T}_m^k f\left(\sqrt[p]{3m+2}\sqrt[p]{2x^p-y^p}\right)\\
&\quad+2\mathcal{T}_m^k f\left(\sqrt[p]{-m}\sqrt[p]{2x^p-y^p}\right)-\mathcal{T}_m^k f\left(\sqrt[p]{4m+3}\sqrt[p]{2x^p-y^p}\right)\\
&\quad-24\mathcal{T}_m^k f\left(\sqrt[p]{m+1}\sqrt[p]{x^p+y^p}\right)-4\mathcal{T}_m^k f\left(\sqrt[p]{3m+2}\sqrt[p]{x^p+y^p}\right)\\
&\quad-4\mathcal{T}_m^k f\left(\sqrt[p]{-m}\sqrt[p]{x^p+y^p}\right)+2\mathcal{T}_m^k f\left(\sqrt[p]{4m+3}\sqrt[p]{x^p+y^p}\right)\\
&\quad-24\mathcal{T}_m^k f\left(\sqrt[p]{m+1}\sqrt[p]{x^p-y^p}\right)-4\mathcal{T}_m^k f\left(\sqrt[p]{3m+2}\sqrt[p]{x^p-y^p}\right)\\
&\quad-4\mathcal{T}_m^k f\left(\sqrt[p]{-m}\sqrt[p]{x^p-y^p}\right)+2\mathcal{T}_m^k f\left(\sqrt[p]{4m+3}\sqrt[p]{x^p-y^p}\right)\\
&\quad-144\mathcal{T}_m^k f\left(\sqrt[p]{m+1}\,x\right)-24\mathcal{T}_m^k f\left(\sqrt[p]{3m+2}\,x\right)\\
&\quad-24\mathcal{T}_m^k f\left(\sqrt[p]{-m}\,x\right)+12\mathcal{T}_m^k f\left(\sqrt[p]{4m+3}\,x\right)\Big\|_*\\
\leq&\max\Big\{12\Big\|\mathcal{T}_m^k f\left(\sqrt[p]{m+1}\sqrt[p]{2x^p+y^p}\right)+\mathcal{T}_m^k f\left(\sqrt[p]{m+1}\sqrt[p]{2x^p-y^p}\right)\\
&\quad-2\mathcal{T}_m^k f\left(\sqrt[p]{m+1}\sqrt[p]{x^p+y^p}\right)-2\mathcal{T}_m^k f\left(\sqrt[p]{m+1}\sqrt[p]{x^p-y^p}\right)\\
&\quad-12\mathcal{T}_m^k f\left(\sqrt[p]{m+1}\,x\right)\Big\|_*,\\
&\quad 2\Big\|\mathcal{T}_m^k f\left(\sqrt[p]{3m+2}\sqrt[p]{2x^p+y^p}\right)+\mathcal{T}_m^k f\left(\sqrt[p]{3m+2}\sqrt[p]{2x^p-y^p}\right)\\
&\quad-2\mathcal{T}_m^k f\left(\sqrt[p]{3m+2}\sqrt[p]{x^p+y^p}\right)-2\mathcal{T}_m^k f\left(\sqrt[p]{3m+2}\sqrt[p]{x^p-y^p}\right)\\
&\quad-12\mathcal{T}_m^k f\left(\sqrt[p]{3m+2}\,x\right)\Big\|_*,\\
&\quad 2\Big\|\mathcal{T}_m^k f\left(\sqrt[p]{-m}\sqrt[p]{2x^p+y^p}\right)+\mathcal{T}_m^k f\left(\sqrt[p]{-m}\sqrt[p]{2x^p-y^p}\right)\\
&\quad-2\mathcal{T}_m^k f\left(\sqrt[p]{-m}\sqrt[p]{x^p+y^p}\right)-2\mathcal{T}_m^k f\left(\sqrt[p]{-m}\sqrt[p]{x^p-y^p}\right)\\
&\quad-12\mathcal{T}_m^k f\left(\sqrt[p]{-m}\,x\right)\Big\|_*,\\
&\quad \Big\|\mathcal{T}_m^k f\left(\sqrt[p]{4m+3}\sqrt[p]{2x^p+y^p}\right)+\mathcal{T}_m^k f\left(\sqrt[p]{4m+3}\sqrt[p]{2x^p-y^p}\right)\\
&\quad-2\mathcal{T}_m^k f\left(\sqrt[p]{4m+3}\sqrt[p]{x^p+y^p}\right)-2\mathcal{T}_m^k f\left(\sqrt[p]{4m+3}\sqrt[p]{x^p-y^p}\right)\\
&\quad-12\mathcal{T}_m^k f\left(\sqrt[p]{4m+3}\,x\right)\Big\|_*\Big\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \max\Big\{\big\|\mathcal{T}_m^k f\left(\sqrt[p]{m+1}\sqrt[p]{2x^p+y^p}\right)+\mathcal{T}_m^k f\left(\sqrt[p]{m+1}\sqrt[p]{2x^p-y^p}\right)\\
&\quad -2\mathcal{T}_m^k f\left(\sqrt[p]{m+1}\sqrt[p]{x^p+y^p}\right)-2\mathcal{T}_m^k f\left(\sqrt[p]{m+1}\sqrt[p]{x^p-y^p}\right)\\
&\quad -12\mathcal{T}_m^k f\left(\sqrt[p]{m+1}\,x\right)\big\|_*,\\
&\quad \big\|\mathcal{T}_m^k f\left(\sqrt[p]{3m+2}\sqrt[p]{2x^p+y^p}\right)+\mathcal{T}_m^k f\left(\sqrt[p]{3m+2}\sqrt[p]{2x^p-y^p}\right)\\
&\quad -2\mathcal{T}_m^k f\left(\sqrt[p]{3m+2}\sqrt[p]{x^p+y^p}\right)-2\mathcal{T}_m^k f\left(\sqrt[p]{3m+2}\sqrt[p]{x^p-y^p}\right)\\
&\quad -12\mathcal{T}_m^k f\left(\sqrt[p]{3m+2}\,x\right)\big\|_*,\\
&\quad \big\|\mathcal{T}_m^k f\left(\sqrt[p]{-m}\sqrt[p]{2x^p+y^p}\right)+\mathcal{T}_m^k f\left(\sqrt[p]{-m}\sqrt[p]{2x^p-y^p}\right)\\
&\quad -2\mathcal{T}_m^k f\left(\sqrt[p]{-m}\sqrt[p]{x^p+y^p}\right)-2\mathcal{T}_m^k f\left(\sqrt[p]{-m}\sqrt[p]{x^p-y^p}\right)\\
&\quad -12\mathcal{T}_m^k f\left(\sqrt[p]{-m}\,x\right)\big\|_*\big\|\mathcal{T}_m^k f\left(\sqrt[p]{4m+3}\sqrt[p]{2x^p+y^p}\right)\\
&\quad +\mathcal{T}_m^k f\left(\sqrt[p]{4m+3}\sqrt[p]{2x^p-y^p}\right)-2\mathcal{T}_m^k f\left(\sqrt[p]{4m+3}\sqrt[p]{x^p+y^p}\right)\\
&\quad -2\mathcal{T}_m^k f\left(\sqrt[p]{4m+3}\sqrt[p]{x^p-y^p}\right)-12\mathcal{T}_m^k f\left(\sqrt[p]{4m+3}\,x\right)\big\|_*\Big\}\\
&\leq \max\left\{\alpha_m^k h_1((m+1)x^p)h_2((m+1)y^p), \alpha_m^k h_1((3m+2)x^p)h_2((3m+2)y^p),\right.\\
&\quad \left.\alpha_m^k h_1((-m)x^p)h_2((-m)y^p),\ \alpha_m^k h_1((4m+3)x^p)h_2((4m+3)y^p)\right\}\\
&\leq \alpha_m^k h_1(x^p)h_2(y^p)\max\left\{\lambda_1(m+1)\lambda_2(m+1),\ \lambda_1(3m+2)\lambda_2(3m+2)\right.\\
&\quad \left.\lambda_1(-m)\lambda_2(-m),\ \lambda_1(4m+3)\lambda_2(4m+3)\right\}\\
&\leq \alpha_m^{k+1}h_1(x^p)h_2(y^p).
\end{aligned}
$$

Thus, by induction, we have shown that (2.7) holds for $n \in \mathbb{N}$, and $m \in \mathcal{U}$. Letting $n \to \infty$ in (2.7), we obtain

$$
\begin{aligned}
\mathcal{F}_m\left(\sqrt[p]{2x^p+y^p}\right)+\mathcal{F}_m\left(\sqrt[p]{2x^p-y^p}\right) &= 2\mathcal{F}_m\left(\sqrt[p]{x^p+y^p}\right)+2\mathcal{F}_m\left(\sqrt[p]{x^p-y^p}\right)\\
&\quad -12\mathcal{F}_m(x),
\end{aligned}
$$

for all $x, y \in \mathbb{R}_0$ such that $x \neq y$ , $x \neq -y$ , $y \neq \sqrt[p]{2}x$ , $y \neq -\sqrt[p]{2}x$ and $m \in \mathcal{U}$. This implies that $\mathcal{F}_m : \mathbb{R} \to X$ is a solution of the Eq. (1.4). Therefore, we construct a sequence $\{\mathcal{F}_m\}_{m\in\ \mathcal{N}}$ of solutions of Eq (1.4) on $X^{R_0}$ such that

$$
\begin{aligned}
\big\|\mathcal{F}_m(x)-f(x)\|_* &\leq \sup_{n\in\mathbb{N}} \Lambda_m^n \varepsilon_m(x)\\
&\leq \sup_{n\in\mathbb{N}}\{\lambda_1(m+1)\lambda_2(2m+1)\alpha_m^n h_1(x^p)h_2(x^p)\},
\end{aligned}
$$

for all $x \in \mathbb{R}_0$ and $m \in \mathcal{U}$. letting $n \to \infty$ and using (2.1) we deduce that $f$ is solution of Eq. (1.4) on $\mathbb{R}_0$. □

With an analogous proof of the above theorem, we can prove the following theorem.

**Theorem 2.3.** *Let $p$ be an odd integer, $X$ be a non-Archimedean Banach space and $h:\mathbb{R}_0 \to \mathbb{R}_+$ be a mapping such that*

$$\mathcal{U} := \left\{ n \in \mathbb{N} : \alpha_n = \max\{\lambda(n+1)\,;\ \lambda(3n+2),\ \lambda(-n),\ \lambda(4n+3)\} < 1 \right\} \neq \phi,$$

*is an infinite set, where*

$$\lambda(n) = \inf\left\{ t \in \mathbb{R}_+ : h(nx) \le t\, h(x),\ \ x \in \mathbb{R}_0 \right\},$$

*for all $a \in \mathbb{N}$, such that*

$$\lim_{n\to\infty} \left(\lambda(n+1) + \lambda(2n+1)\right) = 0.$$

*Assume that $f:\mathbb{R} \to X$ satisfies the inequality*

$$\begin{aligned} &\Big\| f\left(\sqrt[p]{2x^p + y^p}\right) + f\left(\sqrt[p]{2x^p - y^p}\right) - 2f\left(\sqrt[p]{x^p + y^p}\right) - 2f\left(\sqrt[p]{x^p - y^p}\right) \\ &\quad - 12f(x)\Big\|_* \le h(x^p) + h(y^p), \end{aligned} \tag{2.8}$$

*for all $x, y \in \mathbb{R}_0$ such that $x \neq y$, $x \neq -y$, $y \neq \sqrt[p]{2}\,x$ and $y \neq -\sqrt[p]{2}\,x$. Then $f$ is a solution of Eq.* (1.4) *on $\mathbb{R}_0$.*

*Proof.* Replacing in (2.8) $x$ by $\sqrt[p]{m+1}\ x$ and $y$ by $\sqrt[p]{2m+1}\ x$ where $x \in \mathbb{R}_0$, $m \in \mathcal{U}$, then we get

$$\begin{aligned} &\left\| 12f\left(\sqrt[p]{m+1}\ x\right) + 2f\left(\sqrt[p]{3m+2}x\right) + 2f\left(\sqrt[p]{-m}x\right) - f\left(\sqrt[p]{4m+3}x\right) - f(x) \right\|_* \\ &\le h((m+1)x^p) + h((2m+1)x^p) \\ &\le \left(\lambda(m+1) + \lambda(2m+1)\right) h(x^p), \end{aligned}$$

for all $m \in \mathcal{U}$ and all $x \in \mathbb{R}_0$.

We define operators $\mathcal{T}_m : X^{\mathbb{R}_0} \to X^{\mathbb{R}_0}$ by

$$\mathcal{T}_m\xi(x) = 12\xi\left(\sqrt[p]{m+1}x\right) + 2\xi\left(\sqrt[p]{3m+2}x\right) + 2\xi\left(\sqrt[p]{-m}x\right) - \xi\left(\sqrt[p]{4m+3}\ x\right),$$

for all $\xi \in X^{\mathbb{R}_0}$, $x \in \mathbb{R}_0$. We also define the operator $\varepsilon_m : \mathbb{R}_0 \to \mathbb{R}_+$ by

$$\varepsilon_m(x) = h((m+1)x^p) + h((2m+1)x^p) \le \left(\lambda(m+1) + \lambda(2m+1)\right) h(x^p),\ x \in \mathbb{R}_0,$$

and the operator $\Lambda_m : \mathbb{R}_+^{\mathbb{R}_0} \to \mathbb{R}_+^{\mathbb{R}_0}$ defined by

$$\Lambda_m\delta(x) = \max\{\delta\left(\sqrt[p]{m+1}\ x\right), \delta\left(\sqrt[p]{3m+2}\ x\right), \delta\left(\sqrt[p]{-m}\ x\right), \delta\left(\sqrt[p]{4m+3}\ x\right)\}.$$

As in theorem 2.2, we observe that inequality (2.8) take the form

$$\left\| f(x) - \mathcal{T}_m f(x) \right\|_* \le \varepsilon_m(x),\ \ x \in \mathbb{R}_0,\ m \in \mathcal{U}.$$

This completes the proof. □

Next, as a particular cases of those above theorems, when we change the control $h_1(x)h_2(y)$ (or $h(x)+h(y)$) by $c|x^r||y^s|$ (or $c|x^r|+|y^s|$) we will fall into stability in the sense of Hyers-Ulam-Rassiass. In this subject, by according the previous theorems we define $h_1, h_2, h : \mathbb{R} \to \mathbb{R}_+$ as follows:
$h_1(x^p) = c_1|Q_1(x^p)|^r$ , $h_2(x^p) = c_2|Q_2(x^p)|^s$ and $h(x^p) = c|Q(x^p)|^r$ for all $x \in \mathbb{R}_0$, where $c_1, c_2, c \geq 0$, $r, s \in \mathbb{R}$, $p$ is an odd integer and $Q_1, Q_2, Q$ are Cubic mappings, we derive some particular cases.

**Corollary 2.4.** *Let $X$ be a non-Archimedean Banach space. Assume that a function $f : \mathbb{R} \to X$ verify the inequality*

$$\begin{aligned}\Big\|f\left(\sqrt[p]{2x^p+y^p}\right)+f\left(\sqrt[p]{2x^p-y^p}\right)-2f\left(\sqrt[p]{x^p+y^p}\right)-2f\left(\sqrt[p]{x^p-y^p}\right)\\ -12f(x)\Big\|_* \leq c|Q_1(x^p)|^r|Q_2(y^p)|^s,\end{aligned} \tag{2.9}$$

*for all $x, y \in \mathbb{R}_0$, where $c = c_1 \times c_2 \geq 0$, $r + s < 0$. Then $f$ is a solution of the equation* (1.4) *on $\mathbb{R}_0$.*

*Proof.* For each $m \in \mathbb{N}$, we define $\lambda_1(m)$ as in Theorem 2.2:

$$\begin{aligned}\lambda_1(m+1) &= \inf\{t \in \mathbb{R}_+ : h_1((m+1)x^p) \leq t h_1(x^p)\}\\ &= \inf\left\{t \in \mathbb{R}_+ : c_1\Big|Q_1((m+1)x^p)\Big|^r \leq t c_1|Q_1(x^p)|^r\right\}\\ &= \inf\left\{t \in \mathbb{R}_+ : (m+1)^{3r}|Q_1(x^p)|^r \leq t|Q_1(x^p)|^r\right\}\\ &= (m+1)^{3r},\end{aligned}$$

for all $x \in \mathbb{R}_0$. Also, for each $m \in \mathbb{N}$ we have $\lambda_2(2m+1) = (2m+1)^{3s}$. It's clear that there exists $m_0 \in \mathbb{N}$ such that, for each $m \geq m_0$ we get

$$\begin{aligned}\alpha_m &= \max\big\{\lambda_1(m+1)\lambda_2(m+1),\ \lambda_1(3m+2)\lambda_2(3m+2),\ \lambda_1(-m)\lambda_2(-m),\\ &\qquad \lambda_1(4m+3)\lambda_2(4m+3)\big\}\\ &= \max\big\{(m+1)^{3(r+s)},\ (3m+2)^{3(r+s)},\ (-m)^{3(r+s)},\ (4m+3)^{3(r+s)}\big\}\\ &< 1.\end{aligned}$$

According to Theorem 2.2, there exists a unique p-radical function
$\mathcal{F}_m : \mathbb{R}_0 \to X$ such that

$$\begin{aligned}\|\mathcal{F}_m(x) - f(x)\|_* &\leq c\sup_{n\in\mathbb{N}}\left\{\lambda_1(m+1)\lambda_2(2m+1)\alpha_m^n|Q_1(x^p)|^r|Q_2(x^p)|^s\right\}\\ &= c(m+1)^{3r}(2m+1)^{3s}|Q_1(x^p)|^r|Q_2(x^p)|^s\sup_{n\in\mathbb{N}}\left\{\alpha_m^n\right\},\end{aligned}$$

for all $x \in \mathbb{R}_0$.

On the other hand, since $r + s < 0$, one of $r$, $s$ must be negative. Assume that $r < 0$. Then

$$\lim_{m\to\infty}\lambda_1(m+1)\lambda_2(2m+1) = \lim_{m\to\infty}(m+1)^{3(r+s)} = 0. \tag{2.10}$$

We get the desired results. □

**Corollary 2.5.** *Let $X$ be a non-Archimedean Banach space. Assume that a function $f : \mathbb{R} \to X$ verify the inequality*

$$\begin{aligned}&\left\| f\left(\sqrt[p]{2x^p + y^p}\right) + f\left(\sqrt[p]{2x^p - y^p}\right) - 2f\left(\sqrt[p]{x^p + y^p}\right) - 2f\left(\sqrt[p]{x^p - y^p}\right)\right.\\&\quad \left. - 12f(x)\right\|_* \leq c\left(|Q(x^p)|^r + |Q(y^p)|^r\right),\end{aligned} \tag{2.11}$$

*for all $x, y \in \mathbb{R}_0$, where $c \geq 0$, $r < 0$. Then $f$ is a solution of the equation* (1.4) *on $\mathbb{R}_0$.*

*Proof.* The proof is similar to the proof of Corollary 2.4. □

In the following corollaries, we get the hyperstability results for the inhomogeneous general p-radical functional equation.

**Corollary 2.6.** *Let $X$ be a non-Archimedean Banach space, $p$ be an odd integer, $G : \mathbb{R} \times \mathbb{R} \to X$ be a function such that $G(0, 0) = 0$ and $f : \mathbb{R} \to X$ be a function such that $f(0) = 0$. Assume that $f, G$ satisfy the inequality:*

$$\begin{aligned}&\left\| f\left(\sqrt[p]{2x^p + y^p}\right) + f\left(\sqrt[p]{2x^p - y^p}\right) - 2f\left(\sqrt[p]{x^p + y^p}\right) - 2f\left(\sqrt[p]{x^p - y^p}\right)\right.\\&\quad \left. - 12f(x) - G(x, y)\right\|_* \leq c|Q_1(x^p)|^r |Q_2(y^p)|^s,\end{aligned} \tag{2.12}$$

*for all $x, y \in \mathbb{R}_0$, where $c, r, s \in \mathbb{R}$ such that $c \geq 0$, $r + s < 0$. If the functional equation;*

$$\begin{aligned}&f\left(\sqrt[p]{2x^p + y^p}\right) + f\left(\sqrt[p]{2x^p - y^p}\right) - 2f\left(\sqrt[p]{x^p + y^p}\right) - 2f\left(\sqrt[p]{x^p - y^p}\right)\\&\quad - 12f(x) - G(x, y) = 0,\end{aligned} \tag{2.13}$$

*has a solution $f_0 : \mathbb{R} \to X$, then $f$ is a solution of Eq.* (2.13)*.*

*Proof.* Let $\psi : \mathbb{R} \to X$ be a function defined by $\psi(x) = f(x) - f_0(x)$ for all $x \in \mathbb{R}$. Then,

$$\begin{aligned}&\left\| \psi\left(\sqrt[p]{2x^p + y^p}\right) + \psi\left(\sqrt[p]{2x^p - y^p}\right) - 2\psi\left(\sqrt[p]{x^p + y^p}\right)\right.\\&\quad \left. - 2\psi\left(\sqrt[p]{x^p - y^p}\right) - 12\psi(x)\right\|_*\\&= \left\| f\left(\sqrt[p]{2x^p + y^p}\right) + f\left(\sqrt[p]{2x^p - y^p}\right) - 2f\left(\sqrt[p]{x^p + y^p}\right)\right.\\&\quad - 2f\left(\sqrt[p]{x^p - y^p}\right) - 12f(x)\\&\quad - G(x, y) - f_0\left(\sqrt[p]{2x^p + y^p}\right) - f_0\left(\sqrt[p]{2x^p - y^p}\right) + 2f_0\left(\sqrt[p]{x^p + y^p}\right)\\&\quad \left. + 2f_0\left(\sqrt[p]{x^p - y^p}\right) + 12f(x) + G(x, y)\right\|_*\end{aligned}$$

$$\begin{aligned}
&\le \max\Big\{\Big\|f\left(\sqrt[p]{2x^p+y^p}\right)+f\left(\sqrt[p]{2x^p-y^p}\right)\\
&\qquad -2f\left(\sqrt[p]{x^p+y^p}\right)-2f\left(\sqrt[p]{x^p-y^p}\right)\\
&\qquad -12f(x)-G(x,y)\Big\|_*,\ \Big\|f_0\left(\sqrt[p]{2x^p+y^p}\right)+f_0\left(\sqrt[p]{2x^p-y^p}\right)\\
&\qquad -2f_0\left(\sqrt[p]{x^p+y^p}\right)-2f_0\left(\sqrt[p]{x^p-y^p}\right)-12f_0(x)-G(x,y)\Big\|_*\Big\}\\
&\le \Big\|f\left(\sqrt[p]{2x^p+y^p}\right)+f\left(\sqrt[p]{2x^p-y^p}\right)-2f\left(\sqrt[p]{x^p+y^p}\right)\\
&\quad -2f\left(\sqrt[p]{x^p-y^p}\right)-12f(x)-G(x,y)\Big\|_*\\
&\le c|Q_1(x^p)|^r|Q_2(y^p)|^s,
\end{aligned}$$

for all $x, y \in \mathbb{R}_0$. By using Corollary 2.4, we deduce that $\psi$ is a solution of equation (1.4). Moreover, for all $x, y \in \mathbb{R}_0$, we have

$$\begin{aligned}
&f\left(\sqrt[p]{2x^p+y^p}\right)+f\left(\sqrt[p]{2x^p-y^p}\right)-2f\left(\sqrt[p]{x^p+y^p}\right)-2f\left(\sqrt[p]{x^p-y^p}\right)\\
&\quad -12f(x)-G(x,y)\\
&=\psi\left(\sqrt[p]{2x^p+y^p}\right)+\psi\left(\sqrt[p]{2x^p-y^p}\right)-2\psi\left(\sqrt[p]{x^p+y^p}\right)-2\psi\left(\sqrt[p]{x^p-y^p}\right)\\
&\quad -12\psi(x)+f_0\left(\sqrt[p]{2x^p+y^p}\right)+f_0\left(\sqrt[p]{2x^p-y^p}\right)-2f_0\left(\sqrt[p]{x^p+y^p}\right)\\
&\quad -2f_0\left(\sqrt[p]{x^p-y^p}\right)-12f_0(x)-G(x,y)=0,
\end{aligned}$$

which means that $f$ is a solution of (2.13). □

With an analogous proof of Corollary 2.6, we find the following corollary.

**Corollary 2.7.** *Let $X$ be a non-Archimedean Banach space, $p$ be an odd integer and $G : \mathbb{R}\times\mathbb{R} \to X$ be a function such that $G(0, 0) = 0$ and $f : \mathbb{R} \to X$ is a function such that $f(0) = 0$. Assume that $f, G$ satisfy the inequality*

$$\begin{aligned}
&\Big\|f\left(\sqrt[p]{2x^p+y^p}\right)+f\left(\sqrt[p]{2x^p-y^p}\right)-2f\left(\sqrt[p]{x^p+y^p}\right)-2f\left(\sqrt[p]{x^p-y^p}\right)\\
&\quad -12f(x)-G(x,y)\Big\|_*\le c\left(|Q(x^p)|^r+|Q(y^p)|^s\right),
\end{aligned}\tag{2.14}$$

*for all $x, y \in \mathbb{R}_0$, where $c, r, s \in \mathbb{R}$ such that $c \ge 0$, $r + s < 0$. If the functional equation*

$$\begin{aligned}
&f\left(\sqrt[p]{2x^p+y^p}\right)+f\left(\sqrt[p]{2x^p-y^p}\right)-2f\left(\sqrt[p]{x^p+y^p}\right)-2f\left(\sqrt[p]{x^p-y^p}\right)\\
&\quad -12f(x)-G(x,y)=0,
\end{aligned}\tag{2.15}$$

*has a solution $f_0 : \mathbb{R} \to X$, then $f$ is a solution of Eq. (2.15).*

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