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VISCOSITY APPROXIMATION FOR FIXED POINTS OF NONEXPANSIVE SEMIGROUP IN BANACH SPACES

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Abstract. In this paper, in a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, we obtain the convergence theorems of the implicit iteration process and the explicit iteration process for the nonexpansive semigroup. Our results improve and generalize some previous results.

1. Introduction

It is well known that the theory of fixed points is an important and widely used branch of nonlinear analysis. Perhaps the most well known result in the theory of fixed points is Banach's contraction mapping principle. One classical method to study fixed points of nonexpansive mappings is to use contractions to approximate directly or approximate by iterations nonexpansive mappings.

Let $T : K \to K$ be a nonexpansive mapping, for a fixed $u \in K$ and for each $t \in (0, 1)$, we define a contraction $T_t : K \to K$ by

$$T_t x = tu + (1 - t)Tx.$$

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Browder[2] proved that as $t \to 0$ the fixed point x_t of T_t converges strongly to a fixed point of T in a Hilbert space. Reich[6] extended Browde's theorem to a uniformly smooth Banach space. Halpern[5] firstly introduced the explicit iterative scheme

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n$$

in a Hilbert space and pointed out that the control conditions $\alpha_n \to 0$ and $\sum \alpha_n = \infty$ are necessary for the convergence of $\{x_n\}$ to a fixed point of T. Wittmann[10], still in Hilbert space, obtained a strong convergence result for the above iteration $\{x_n\}$ under an additional condition $\sum |\alpha_n - \alpha_{n+1}| < \infty$. Shioji and Takahashi[9] extended Wittmann's results to a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Xu Hongkun[12] proposed the following viscosity iterative process

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n$$

and proved that $\{x_n\}$ converges to a fixed point of T, where $f : K \to K$ is a contractive mapping. In 2002, Suzuki[8] first introduced in Hilbert space the implicit iteration process

$$x_n = \alpha_n u + (1 - \alpha_n)Tx_n$$

for the nonexpansive semigroup case and gived the strong convergence theorem. Benavides, Acedo and Xu[1], in a uniformly smooth Banach space, showed that both the implicit iteration process and the explicit iteration process

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n$$

are strongly convergent under some conditions such as asymptotic regularity.

In 2005, Xu Hongkun[11] studied the strong convergence of the implicit iteration process in a uniformly convex Banach space which admits a weakly sequentially continuous duality mapping.

Recently, in a real reflexive Banach space with a weakly sequentially continuous duality mapping, Chen Rudong and He Huimin[4] investigated the strong convergence of the implicit iteration process

$$x_n = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n$$

and the explicit iteration process

$$y_{n+1} = \beta_n f(y_n) + (1 - \beta_n)Ty_n$$

for the nonexpansive semigroup $\Im$, and proved that both the implicit iteration process and the explicit iteration process converge strongly to a fixed point $p \in F(\Im)$, which is the unique solution in $F(\Im)$ to the following variational inequality $\langle (f - I)q, j(x - q)\rangle \leq 0$.

In this paper, we obtain the convergence theorems of the implicit iteration process and the explicit iteration process for the nonexpansive semigroupa in a reflexive and strictly convex Banach space with a uniformly differentiable norm, which improves and generalizes some previous results.

2. Preliminaries

Let X be a reflexive Banach space, and K a closed convex subset of X. Let $T : K \to K$ be a nonexpansive mapping. We denote by $F(T)$ the set of fixed point of T, i.e.,

$$F(T) = \{x \in K : x = Tx\}.$$

Definition 2.1. *The set $\Im = \{T(t) : t \in [0, +\infty)\}$ is called a nonexpansive semigroup on K, if the following condition one satisfied:*
(1) $\forall t \in [0, +\infty)$, $T(t)$ is a nonexpansive mapping of K into itself;
(2) $T(0)x = x, \forall x \in K$;
(3) $T(s + t) = T(s) \circ T(t), \forall s, t \in R^+$;
(4) The mapping $T(\cdot)x : R^+ \to K$ is continuous, for all $x \in X$.

We denote by $F(\Im)$ the set of common fixed point of $\Im$, i.e.,

$$F(\Im) = \bigcap_{t \geq 0} F(T(t)).$$

A selfmapping $f : K \to K$ is called a contraction on K if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x - y\|, \forall x, y \in K,$$

where α is a contraction constant.

Let X be a real Banach space with dual X^*. Let $J : X \to 2^{X^*}$ denote the mormalized duality mapping defined by

$$J(x) = \{f \in X^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\},\ x \in X.$$

Lemma 2.2. [9] *Let X be a Banach space. Then*

$$\|x\|^2 + 2\langle y, j(x)\rangle \leq \|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \forall x, y \in X, \qquad (2.1)$$

where $j(x) \in J(x)$, $j(x + y) \in J(x + y)$.

Let $S = \{x \in X : \|x\| = 1\}$ denote the unit sphere of the Banach space X. The space X is said to have a Gâteaux differentiable norm if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \qquad (2.2)$$

exists for all $x, y \in S$, and X is said to be smooth; X is said to have a uniformly Gâteaux differentiable norm if the limit (2.2) is attained uniformly for $x \in S$. Further, X is said to be uniformly smooth if the limit (2.2) exists uniformly for $(x, y) \in S \times S$, and we call X have a uniformly Fréchet differentiable norm. The space X is said to be strictly convex if

$$\|\frac{x + y}{2}\| \leq 1, \forall x, y \in S,\ x \neq y.$$

The space X is said to be uniformly convex if there is $\delta_\varepsilon > 0$, such that

$$\|\frac{x+y}{2}\| < 1 - \delta_\varepsilon$$

for each $\varepsilon \in (0, 2]$ and $x, y \in S$, $\|x - y\| > \varepsilon$. We know that if X is a reflexive and strictly convex Banach space then it is uniformly convex, and if X is an uniformly smooth Banach space then it is a reflexive Banach space with a uniformly Gâteaux differentiable norm. If X has a uniformly Gâteaux differentiable norm then the duality mapping J is strong-to-weak* uniformly continuous on bounded sets, and if X is uniformly smooth and only if the duality mapping is strong-to-strong uniformly continuous.

We denote by LIM the Banach limit, where $LIM \in (l^\infty)^*$, $\|LIM\| = 1$, and $\liminf_{n\to\infty} a_n \le LIM_n a_n \le \limsup_{n\to\infty} a_n$, $LIM_n a_n = LIM_n a_{n+1}, \{a_n\}_n \in l^\infty$.

Let $\{x_n\} \subset K$ be a bounded sequence, and $g(x) = LIM_n\|x_n - x\|$, $x \in K$. Then we have the following conclusion.

Lemma 2.3. *There exists $x_0 \in K$ such that $g(x_0) = \min_{x\in K} g(x)$.*

Proof. Since $g(x)$ is bounded below, $d = \inf_{x\in K} g(x)$ exists, and there is $\{x'_n\} \subset K$ such that $g(x'_n) \to d, (n \to \infty)$. We denote $A_\varepsilon = \{x \in X : g(x) \le d + \varepsilon\}$ for each $\varepsilon > 0$. It is easy to see that A_ε is a nonempty closed convex set , moreover, $x'_n \in A_\varepsilon$ if n large enough. Since X be reflexive, there exists $x_0 \in X$ and subset $\{x'_{n_k}\}$ such that x'_{n_k} converges weakly to x_0. As A_ε is closed convex, $x_0 \in A_\varepsilon$, $d \le g(x_0) \le d + \varepsilon$. Since ε is arbitrary, $g(x_0) = d$. That is,

$$g(x_0) = \min_{x\in K} g(x).$$

□

Lemma 2.4. *Let X be a Banach space with a uniformly Gâteaux differentiable norm, $K \subset X$ be nonempty closed and convex, and $g(z) = \min_{x\in K} g(x)$. Then*

$$LIM_n\langle u - z, j(x_n - z)\rangle \le 0, \text{ for all } u \in K. \tag{2.3}$$

Proof. Let $C = \{z : g(z) = \min_{x\in K} g(x)\}$. From Lemma 2.2, we get

$$\|x_n - y\|^2 = \|x_n - z + z - y\|^2 \le \|x_n - z\|^2 + 2\langle z - y, j(x_n - y)\rangle,$$

for $z \in C$, $y \in K$. Then

$$LIM_n\|x_n - y\|^2 \le LIM_n\|x_n - z\|^2 + 2LIM_n\langle z - y, j(x_n - y)\rangle,$$

hence

$$LIM_n\langle z - y, j(x_n - y)\rangle \ge 0, \forall y \in K. \tag{2.4}$$

We take $y = tu + (1 - t)z$, $0 < t < 1$, then

$$z - y = t(z - u). \tag{2.5}$$

From (2.4) and (2.5), we have

$$LIM_n\langle t(z - u), j(x_n - tu - (1 - t)z)\rangle \ge 0,$$

that is

$$LIM_n\langle z-u, j(x_n - tu - (1-t)z)\rangle \geq 0. \tag{2.6}$$

Let $t \to 0$. From (2.4) and J is strong-weak* continuous, we get

$$LIM_n\langle z-u, j(x_n - z)\rangle \geq 0, \forall u \in K,$$

that is

$$LIM_n\langle u-z, j(x_n - z)\rangle \leq 0, \forall u \in K.$$

This completes the proof. □

Lemma 2.5. [9] *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n, n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in R such that

(i) $\lim_{n\to\infty} \gamma_n = 0$; (ii) $\sum_{n=1}^{\infty} \gamma_n = \infty$; (iii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

Firstly, we give the convergence theorems of the implicit iteration process for the nonexpansive semigroup.

Theorem 3.1. *Let X be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, and K be a nonempty closed convex subset of X. Let $\Im = \{T(t) : t \geq 0\}$ be a nonexpansive semigroup on K such that $F(\Im) \neq \emptyset$, and $f : K \to K$ be a contraction with a contraction constants $\alpha \in (0,1)$. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying $0 < \alpha_n < 1$, $t_n > 0$, and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{\alpha_n}{t_n} = 0$. Then the implicit iteration process*

$$x_n = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n, \ n \in N, \tag{3.1}$$

converges strongly to a fixed point $p \in F(\Im)$, which is also the unique solution to the variational inequality:

$$\langle (f-I)q, j(x-q)\rangle \leq 0, \forall x \subset F(\Im). \tag{3.2}$$

Proof. We define the mapping $S_n : K \to K$ by

$$S_n(x) = \alpha_n f(x) + (1-\alpha_n)T(t_n)x, \tag{3.3}$$

for each fixed $n \geq 0$. Then, S_n is a contraction mapping on K.

In fact, for all $x, y \in K$,

$$\begin{aligned}\|S_n(x) - S_n(y)\| &\leq \alpha_n\|f(x) - f(y)\| + (1-\alpha_n)\|T(t_n)x - T(t_n)y\| \\ &\leq (1-\alpha_n + \alpha\alpha_n)\|x-y\|.\end{aligned}$$

From the Banach fixed-point principle and $1-\alpha_n+\alpha\alpha_n < 1$, we get that there is a unique point $x_n \in K$ such that (3.3) is satisfied. This shows the implicit iteration process (3.1) is well defined.

Step 1. The solution to (3.2) is unique.

Let $p, q \subset F(\Im)$ are solutions of (3.2). Then

$$\langle (f-I)q, j(p-q)\rangle \le 0, \tag{3.4}$$

and

$$\langle (f-I)p, j(q-p)\rangle \le 0. \tag{3.5}$$

Adding up (3.4) and (3.5) yields

$$\langle p-q-(f(p)-f(q)), j(p-q)\rangle \le 0,$$

as a result,

$$\|p-q\|^2 \le \alpha\|p-q\|^2.$$

Since $0<\alpha<1$, we get $p=q$.

Step 2. The sequence $\{x_n\}$ defined by (3.1) is bounded.

For $x \in F(\Im)$,

$$\begin{aligned}\|x_n - x\|^2 &= \langle x_n - x, j(x_n - x)\rangle\\ &= \alpha_n\langle f(x_n)-x, j(x_n-x)\rangle + (1-\alpha_n)\langle T(t_n)x_n - x, j(x_n-x)\rangle\\ &\le \alpha_n\|f(x_n)-f(x)\|\cdot\|x_n-x\| + \alpha_n\langle f(x)-x, j(x_n-x)\rangle\\ &\quad +(1-\alpha_n)\|T(t_n)x_n - T(t_n)x\|\cdot\|x_n-x\|\\ &\le (1-(1-\alpha)\alpha_n)\|x_n-x\|^2 + \alpha_n\langle f(x)-x, j(x_n-x)\rangle,\end{aligned}$$

which implies that

$$\begin{aligned}\|x_n-x\|^2 &\le \frac{1}{1-\alpha}\langle f(x)-x, j(x_n-x)\rangle\\ &\le \frac{1}{1-\alpha}\|f(x)-x\|\cdot\|x_n-x\|.\end{aligned}$$

Consequently,

$$\|x_n-x\| \le \frac{1}{1-\alpha}\|f(x)-x\|.$$

Then $\{x_n\}$ is bounded, so are $\{T(t_n)x_n\}$ and $\{f(x_n)\}$.

Step 3. We show $F(\Im)\bigcap C \ne \emptyset$.

Let $g(x) = LIM_n\|x_n - x\|$, $C = \{z : g(z) = \min_{x\in K} g(x)\}$. It is easy to see that C is a nonempty closed bounded subset of X. Let $\sigma_n = \dfrac{\alpha_n}{t_n}$. We have

$$\alpha_n = \sigma_n t_n, x_n = \sigma_n t_n f(x_n) + (1-\sigma_n t_n)T(t_n)x_n.$$

Let $t = p_n t_n + q_n$ for each $t>0$, where $0 \le q_n < t_n$ and p_n is a integral. We denote $d = 2\sup\{\|T(t_n)x_n\| + \|f(x_n)\|\}$, $\varepsilon_n = \|T(t_n)x_n - x_n\|$. Since $\{T(t_n)x_n\}$ and $\{f(x_n)\}$ are bounded, we have

$$p_n\varepsilon_n = p_n\sigma_n t_n\|f(x_n) - T(t_n)x_n\| \le \sigma_n t d \to 0 \text{ as } n\to\infty.$$

Hence for each $p \in C$, it follows that

$$\begin{aligned}\|x_n - T(t)p\| &= \|x_n - T(t_n)x_n + T(t_n)x_n - T(2t_n)x_n + T(2t_n)x_n\\ &\qquad - \cdots - T(p_nt_n)x_n + T(p_nt_n)x_n - T(t)p\|\\ &\le \sum_{k=0}^{p_n-1}\|T((k+1)t_n)x_n - T(kt_n)x_n\| + \|T(p_nt_n)x_n - T(t)p\|\\ &\le p_n\|T(t_n)x_n - x_n\| + \|x_n - T(q_n)x_n\|\end{aligned}$$

$$\leq p_n\varepsilon_n + \|x_n - p\| + \|T(q_n)p - p\|.$$

Taking the Banach limit, we get

$$\begin{aligned} LIM_n\|x_n - p\| &\leq LIM_n\|x_n - T(t)p\| \\ &\leq LIM_n p_n\varepsilon_n + LIM_n\|x_n - p\| + LIM_n\|T(q_n)p - p\| \\ &= LIM_n\|x_n - p\|. \end{aligned}$$

Then $g(T(t)p) = g(p)$ and thus $T(t)C \subset C$. For a fixed $p_0 \in F(\Im)$, since X a reflexive and strictly convex Banach space and C a nonempty closed convex subset of X, there is unique $p \in C$ such that

$$\|p - p_0\| = \inf_{x\in C} \|x - p_0\|.$$

Hence for arbitrary $t \geq 0$,

$$\|p_0 - T(t)p\| = \|T(t)p_0 - T(t)p\| \leq \|p_0 - p\| = \inf_{x\in C} \|p_0 - x\|.$$

Since $T(t)p \subset C$ and $p \in C$ is unique, we obtain $T(t)p = p$, $\forall t \geq 0$. That is, p is a fixed point of $\Im$.

Step 4. For the sequence $\{x_n\}$ defined by (3.1), there is a subset $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \to p \in F(\Im)\bigcap C$.

Since $x_n - f(x_n) = \dfrac{1-\alpha_n}{\alpha_n}(T(t_n)x_n - x_n)$, for arbitrary $q \in F(\Im)$, it follows that

$$\begin{aligned} \langle x_n - f(x_n), j(x_n - q)\rangle &= \frac{1-\alpha_n}{\alpha_n}\langle T(t_n)x_n - x_n, j(x_n - q)\rangle \\ &= \frac{1-\alpha_n}{\alpha_n}\langle T(t_n)x_n - q + q - x_n, j(x_n - q)\rangle \\ &\leq \frac{1-\alpha_n}{\alpha_n}(\|T(t_n)x_n - q\|\cdot\|x_n - q\| - \|x_n - q\|^2) \leq 0, \end{aligned}$$

that is,

$$\langle x_n - f(x_n), j(x_n - q)\rangle \leq 0. \tag{3.6}$$

Hence

$$\begin{aligned} \|x_n - p\|^2 &= \langle x_n - p, j(x_n - q)\rangle \\ &= \langle x_n - f(x_n) + f(x_n) - f(p) + f(p) - p, j(x_n - p)\rangle \\ &= \langle x_n - f(x_n), j(x_n - p)\rangle + \langle f(x_n) - f(p), j(x_n - p)\rangle. \\ &\quad + \langle f(p) - p, j(x_n - p)\rangle \\ &\leq 0 + \alpha\|x_n - p\|^2 + \langle f(p) - p, j(x_n - p)\rangle. \end{aligned}$$

By Lemma 2.4, we have $\mathrm{LIM}_n\|x_n - p\|^2 = 0$. Therefore $\liminf_n \|x_n - p\| = 0$. Then there is a subset $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \to p$.

Step 5. The sequence $\{x_n\}$ is compact.

Take arbitrary $\{x_{n_k}\} \subset \{x_n\}$. Let $\phi(x) = \|x_{n_k} - x\|, x \in K$. Set $\phi(x)$ instead of $g(x)$ in step 3 and step 4, we conclude that there is a subset of $\{x_{n_k}\}$ converges strongly to $p \in F(\Im) \cap \widetilde{C}$, where $\widetilde{C} = \{z : \phi(z) = \min_n \phi(x)\}$. So, the sequence $\{x_n\}$ is compact.

Step 6. We show that $\{x_n\}$ converges strongly to $p \in F(\Im)$.

Let $\{x_{n_i}\} \subset \{x_n\}$, $\{x_{m_i}\} \subset \{x_n\}$, $x_{n_i} \to p,\ x_{m_i} \to q$. From (3.6), we have

$$\langle p-f(p), j(p-q)\rangle = \lim_{i\to\infty}\langle x_{n_i}-f(x_{n_i}), j(x_{n_i}-q)\rangle \le 0, \tag{3.7}$$

$$\langle q-f(q), j(q-p)\rangle = \lim_{i\to\infty}\langle x_{m_i}-f(x_{m_i}), j(x_{m_i}-p)\rangle \le 0. \tag{3.8}$$

Adding up (3.7) and (3.8) yields

$$\langle p-q-f(p)+f(q), j(p-q)\rangle \le 0.$$

As a result,

$$\|p-q\|^2 \le \|f(p)-f(q)\|\cdot\|p-q\| \le \alpha\|p-q\|^2.$$

This implies that $p=q$. Therefore $x_n \to p$.

Step 7. We show that p is a solution to (3.2).

In (3.6), since $x_n \to p$, we have $\langle f(p)-p, j(x-p)\rangle = \lim_{n\to\infty}\langle f(x_n)-x_n, j(x-x_n)\rangle \le 0$. That is, $p \in F(\Im)\bigcap C$ is the unique solution to the variational inequality (3.2). This completes the proof. □

Theorem 3.2. *Let X be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, and K be a nonempty closed convex subset of X. Let $\Im = \{T(t) : t \ge 0\}$ be a nonexpansive semigroup on K such that $F(\Im) \ne \emptyset$, and $f : K \to K$ be a contraction with a contraction constants $\alpha \in (0,1)$. Let $\{\beta_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying $0<\beta_n<1$, $\sum_{n=1}^{\infty}\beta_n=\infty, t_n>0$, and $\lim_{n\to\infty} t_n = \lim_{n\to\infty}\frac{\beta_n}{t_n}=0$. Then the explicit iteration process: for $y_0 \in K$,*

$$y_{n+1} = \beta_n f(y_n) + (1-\beta_n)T(t_n)y_n,\ n \in N \tag{3.9}$$

converges strongly to a fixed point $p \in F(\Im)$, which is also the unique solution to the variational inequality:

$$\langle (f-I)q, j(x-q)\rangle \le 0, \forall x \subset F(\Im). \tag{3.10}$$

Proof. Proceeding as the proof of step 1 of Theorem 3.1, we have that the solution to (3.10) is unique.

Step 1. The sequence $\{y_n\}$ defined by (3.1) is bounded.

For $y \in F(\Im)$, we have

$$\begin{aligned}\|y_{n+1}-y\| &\le (1-\beta_n)\|T(t_n)y_n-y\| + \beta_n\|f(y_n)-y\| \\ &\le (1-\beta_n)\|y_n-y\| + \beta_n\|f(y_n)-f(y)\| + \beta_n\|f(y)-y\| \\ &\le (1-(1-\alpha)\beta_n)\|y_n-y\| + \beta_n\|f(y)-y\| \\ &\le \max\{\|y_n-y\|, \tfrac{1}{1-\alpha}\|f(y)-y\|\}.\end{aligned}$$

Then $\{y_n\}$ is bounded, so are $\{T(t_n)y_n\}$ and $\{f(y_n)\}$.

Step 2. Let $q \in F(\Im)$ be the solution of (3.10). We show

$$\limsup_{n\to\infty}\langle (f-I)q, j(y_{n+1}-q)\rangle \le 0.$$

Let $n, m \in N,\ n > m$, and $\{x_m\}$ be a sequence in the Theorem 3.1. From Theorem 3.1, we get $x_m \to q$ and

$$\begin{aligned}x_m - y_n &= \alpha_m f(x_m) + (1-\alpha_m)T(t_m)x_m - y_n \\ &= \alpha_m(f(x_m)-y_n) + (1-\alpha_m)(T(t_m)x_m - y_n).\end{aligned}$$

Hence

$$\begin{aligned}\|x_m - y_n\|^2 &= \|\alpha_m(f(x_m) - y_n) + (1 - \alpha_m)(T(t_m)x_m - y_n)\|^2 \\ &\le (1-\alpha_m)^2\|(T(t_m)x_m - y_n)\|^2 + 2\alpha_m\langle f(x_m) - y_n, j(x_m - y_n)\rangle \\ &\le (1 - \alpha_m)^2(\|(T(t_m)x_m - T(t_m)y_n)\| + \|T(t_m)y_n - y_n\|)^2 \\ &\quad +2\alpha_m\langle f(x_m) - y_n, j(x_m - y_n)\rangle \\ &\le (1 - \alpha_m)^2(\|x_m - y_n\| + \|T(t_m)y_n - y_n\|)^2 \\ &\quad +2\alpha_m\langle f(x_m) - y_n, j(x_m - y_n)\rangle \\ &= (1 - \alpha_m)^2(\|x_m - y_n\|^2 + \|T(t_m)y_n - y_n\|^2 \\ &\quad +2\|x_m - y_n\| \cdot \|T(t_m)y_n - y_n\|) \\ &\quad +2\alpha_m\langle f(x_m) - x_m + x_m - y_n, j(x_m - y_n)\rangle \\ &\le (1 - \alpha_m)^2(\|x_m - y_n\|^2 + \|T(t_m)y_n - y_n\|^2 \\ &\quad +2\|x_m - y_n\| \cdot \|T(t_m)y_n - y_n\|) \\ &\quad +2\alpha_m\langle f(x_m) - x_m, j(x_m - y_n)\rangle + 2\alpha_m\|x_m - y_n\|^2 \\ &= (1+\alpha_m^2)\|x_m - y_n\|^2 + (1-\alpha)^2\|T(t_m)y_n - y_n\|(2\|x_m - y_n\|) \\ &\quad +\|T(t_m)y_n - y_n\|) + 2\alpha_m\langle f(x_m) - x_m, j(x_m - y_n)\rangle,\end{aligned}$$

which implies that

$$\begin{aligned}\langle f(x_m) - x_m, j(y_n - x_m)\rangle \le \frac{\alpha_m^2}{2}\|x_m - y_n\|^2 + \frac{(1 - \alpha_m)^2}{2\alpha_m}\|T(t_m)y_n \\ -y_n\|(2\|x_m - y_n\| + \|T(t_m)y_n - y_n\|).\end{aligned}$$

Since $\{x_m\}, \{y_n\}$ are bounded, $\{x_m - y_n\}$ and $\{T(t_m)y_n - y_n\}$ are bounded. Then there is $M > 0$, such that

$$M = \sup_{m,n}\{\frac{1}{2}\|x_m - y_n\|^2, \frac{1}{2}(2\|x_m - y_n\| + \|T(t_m)y_n - y_n\|)\}.$$

Hence

$$\langle f(x_m) - x_m, j(x_m - y_n)\rangle \le \alpha_m^2 M + \frac{(1 - \alpha_m)^2}{\alpha_m}\|T(t_m)y_n - y_n\|M.$$

Taking the upper limit, we get

$$\limsup_{n\to\infty}\langle f(x_m) - x_m, j(x_m - y_n)\rangle \le \alpha_m^2 M.$$

Put $m \to \infty$, we obtain $\limsup_{n\to\infty}\langle f(q) - q, j(y_n - q)\rangle \le 0$,

that is,

$$\limsup_{n\to\infty}\langle f(q) - q, j(y_{n+1} - q)\rangle \le 0.$$

Step 3. We show that $\{y_n\}$ converges strongly to q.

By Lemma 2.2, we obtain

$$\begin{aligned}\|y_{n+1} - q\|^2 &= \|(1 - \beta_n)(T(t_n)y_n - q) + \beta_n(f(y_n) - q)\|^2 \\ &\le (1 - \beta_n)^2\|T(t_n)y_n - q\|^2 + 2\beta_n\langle f(y_n) - q, j(y_{n+1} - q)\rangle \\ &\le (1 - \beta_n)^2\|y_n - q\|^2 + 2\beta_n\langle f(y_n) - f(q), j(y_{n+1} - q)\rangle \\ &\quad +2\beta_n\langle f(q) - q, j(y_{n+1} - q)\rangle \\ &\le (1 - \beta_n)^2\|y_n - q\|^2 + \alpha\beta_n(\|y_n - q\|^2 + \|y_{n+1} - q\|^2) \\ &\quad +2\beta_n\langle f(q) - q, j(y_{n+1} - q)\rangle.\end{aligned}$$

Since $\{\beta_n\}$ and $\{y_n\}$ are bounded, there exists $\widetilde{M} > 0$ such that

$$\frac{1}{1-\alpha\beta_n}\|y_n - q\|^2 \le \widetilde{M}.$$

It follows that

$$\begin{aligned}
\|y_{n+1} - q\|^2 &\le \frac{1-(2-\alpha)\beta_n + \beta_n^2}{1-\alpha\beta_n}\|y_n - q\|^2 + \frac{2\beta_n}{1-\alpha\beta_n}\langle f(q) - q, j(y_{n+1} - q)\rangle \\
&\le \frac{1-\alpha\beta_n - 2(1-\alpha)\beta_n}{1-\alpha\beta_n}\|y_n - q\|^2 + \frac{2\beta_n}{1-\alpha\beta_n}\langle f(q) - q, j(y_{n+1} - q)\rangle + \beta_n^2\widetilde{M} \\
&\le (1 - \frac{2(1-\alpha)\beta_n}{1-\alpha\beta_n})\|y_n - q\|^2 + \frac{2\beta_n}{1-\alpha\beta_n}\langle f(q) - q, j(y_{n+1} - q)\rangle + \beta_n^2\widetilde{M} \\
&\le (1 - \frac{2(1-\alpha)\beta_n}{1-\alpha\beta_n})\|y_n - q\|^2 + \beta_n(\frac{2}{1-\alpha\beta_n}\langle f(q) - q, j(y_{n+1} - q)\rangle + \beta_n\widetilde{M}).
\end{aligned}$$

Putting

$$\gamma_n = \frac{2(1-\alpha)\beta_n}{1-\alpha\beta_n}, \delta_n = \beta_n(\frac{2}{1-\alpha\beta_n}\langle f(q) - q, j(y_{n+1} - q)\rangle + \beta_n\widetilde{M}),$$

we get

$$\gamma_n \to 0, \sum_{n=1}^{\infty}\gamma_n = \infty, \ \|y_{n+1} - q\|^2 \le (1-\gamma_n)\|y_n - q\|^2 + \delta_n$$

and

$$\limsup_{n\to\infty}\frac{\delta_n}{\gamma_n} = \limsup_{n\to\infty}\tfrac{1}{(1-\alpha)}\langle (f-I)q, j(y_{n+1} - q)\rangle \le 0.$$

From Lemma 2.5 we have $y_n \to q$. This completes the proof. □

Remark. Theorem 3.1 and 3.2 of [4] are obtained in a real reflexive Banach space with a weakly sequentially continuous duality mapping. Comparatively, in a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, we obtain the convergence theorems. The main results generalize or generalize partly the corresponding results of Suzuki[8], Xu Hongkun[12], Chen Rudong[3], Chen Rudong and He Huimin[4].

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