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ITERATIVE ALGORITHM FOR APPROXIMATING SOLUTIONS OF SPLIT MONOTONE VARIATIONAL INCLUSION, VARIATIONAL INEQUALITY AND FIXED POINT PROBLEMS IN REAL HILBERT SPACES

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Abstract. The goal of this paper is to introduce a modified Halpern iterative algorithm for approximating solutions of split monotone variational inclusion, variational inequality and fixed point problems of an infinite families of multi-valued type-one demicontractive mappings in the framework of real Hilbert spaces. Using our iterative algorithm, we state and prove a strong convergence result for approximating a common solution of split monotone variational inclusion, variational inequality problems and fixed point problem for countable family of multi-valued type-one demicontractive mappings. The iterative algorithm employed in this paper is designed in such a way that it does not require the knowledge of operator norm. Lastly, we give some consequences of our main result and give application of one of the consequences to split minimization problem. The result presented in this paper extends and generalizes some related results in literature.

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1. Introduction

Let C be a nonempty, closed and convex subset of a real Hilbert space H and $A : C \to H$ be a nonlinear mapping. Then, the variational inequality problem (VIP) is to find $u \in C$ such that

$$\langle Au, v-u \rangle \geq 0, \ \forall \ v \in C. \tag{1.1}$$

We denote by VI(C, A) the solution set of VIP (1.1). Let $CB(C)$ and $K(C)$ denote the families of nonempty closed bounded subsets and nonempty compact subsets of C, respectively. The Pompeiu-Hausdorff metric on $CB(C)$ is defined by

$$\mathcal{H}(A,B) = \max\big\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\big\} \text{ for } A,B \in CB(C),$$

where $d(x,C) = \inf\{||x-y|| : y \in C\}$.

Let $T : C \to CB(C)$ be a multi-valued mapping. Then $P_T x = \{u \in Tx : ||x-u|| = d(x,Tx)\}$. A point $x \in C$ is called a fixed point of T if $x \in Tx$. However, if $Tx = \{x\}$, then x is called a strict point of T. We denote the set of fixed point if T by $F(T)$. A multi-valued mapping $T : C \to CB(C)$ is said to be λ - hybrid if there exists $\lambda \in \mathbb{R}$ such that

$$\begin{aligned}&(1+\lambda)\mathcal{H}(Tx,Ty)^2\\ &\leq (1-\lambda)||x-y||^2 + \lambda d(y,Tx)^2 + \lambda d(x,Ty)^2, \ \ \forall \ x,y \in C.\end{aligned} \tag{1.2}$$

Note that if $\lambda = 0$ in (1.2), then we have the following nonexpansive mapping:

$$\mathcal{H}(Tx,Ty) \leq ||x-y||, \ \ \forall \ x,y \in C.$$

T is said to be

(i) of type-one if

$$||u-v|| \leq \mathcal{H}(Tx,Ty), \ \forall \ x,y \in C, u \in P_T x, v \in P_T y,$$

(ii) demicontractive-type in the sense of [13] if $F(T) \neq \phi$ and

$$\mathcal{H}^2(Tx,Ty) \leq ||x-y||^2 + kd^2(x,Tx), \ \ x \in C, y \in F(T) \text{ and } k \in (0,1).$$

Definition 1.1. A multi-valued mapping $B : H \to 2^H$ with nonempty values is said to be monotone, if $\langle u-v, x-y \rangle \geq 0$, for all $u \in Bx$ and $v \in By$.

A monotone mapping M is said to be maximal if the graph of M, denoted by $G(M)$ is not properly contained in the graph of any other monotone mapping and for multi-valued mapping M,

$$G(M) = \{(x,y) : y \in M(x)\}.$$

It is well known that M is maximal if and only if for $(x,y) \in H \times H$, $\langle x-y, u-v \rangle \geq 0$ for all $(y,v) \in G(M)$ implies $u \in M(x)$.

The split feasibility problem (SFP) introduced in 1994 by Censor and Elfving [6] is to find a point

$$x \in C \quad \text{such that} \quad Ax \in Q, \tag{1.3}$$

where C and Q are nonempty closed convex sets in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively and A is an $m \times n$ real matrix. The SFP has wide applications in many fields such as phase retrieval, medical image reconstruction, signal processing and radiation therapy treatment planning (for example see [1, 4] and the references therein).

Based on SFP (1.3), Censor et al. [5] introduced the following split variational inclusion problem (SVIP) which is to find $x^* \in C$ such that

$$\langle f(x^*), x - x^* \rangle \geq 0, \ \forall \ x \in C, \tag{1.4}$$

and such that

$$y^* = Ax^* \in Q \quad \text{solves} \quad \langle g(y^*), y - y^* \rangle \geq 0, \ \forall \ y \in Q; \tag{1.5}$$

where f and g are given mappings.

Recently, Moudafi [15] introduced the following split monotone variational inclusion problem (SMVIP) which is to find

$$x^* \in H_1 \quad \text{such that} \quad 0 \in f(x^*) + B_1(x^*), \tag{1.6}$$

and

$$y^* = Ax^* \in H_2 \quad \text{such that} \quad 0 \in g(y^*) + B_2(y^*); \tag{1.7}$$

where $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are multi-valued maximal monotone mappings, $f : H_1 \to H_1$ and $g : H_2 \to H_2$ are two given mappings.

In 2017, Deepho et al. [11] considered the viscosity iterative algorithm to approximate a common element of the set of solutions of SVIP of a finite family of k-strictly pseudo-contractive nonself mappings. They proved a strong convergence result under suitable conditions, which also solves some variational inequality problem. The following iteration process was used to approximate the aforementioned problems.

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2})Ax_n), \\ y_n = \beta_n u_n + (1 - \beta_n) \sum_{i=1}^N \eta_{i=1}^n T_i u_n, \\ x_{n+1} = \alpha_n \tau g(x_n) + (I - \alpha_n D) y_n, \ n \geq 1, \end{cases}$$

where $\alpha_n, \beta_n \in (0,1), \lambda > 0$, g a contraction mapping with coefficient $\rho \in (0,1)$, $\sum_{i=1}^N \eta_{i=1}^n = 1, \{T_i\}_{i=1}^N$ a finite family of k_i-strictly pseudo-contraction mappings and $J_\lambda^{B_i} (i = 1, 2)$ is the resolvent of the maximal monotone mappings.

Also, recently Shehu and Agbebaku [18] introduced an iterative algorithm for solving split variational inclusion and fixed point problems for multi-valued quasi-nonexpansive mapping. They employed the following iterative algorithm to prove a strong convergence result:

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n + \gamma_n A^*(J_\lambda^{B_2} - I)Ax_n), \\ x_{n+1} = \alpha_n f_n(x_n) + \beta_n x_n + \delta_n(\sigma w_n + (1-\sigma)u_n), \end{cases}$$

$w_n \in Sx_n$ for each $n \geq 1$, where $\{\alpha_n\}, \{\beta_n\}$ and $\{\delta_n\}$ are the real sequences in $(0,1)$ such that $\alpha_n + \beta_n + \delta_n = 1$, $\sigma \in (0,1)$, $\gamma_n := \tau_n \frac{||(J_\lambda^{B_2} - I)Ax_n||^2}{||A^*(J_\lambda^{B_2} - I)Ax_n||^2}$, where $0 < a \leq \tau_n < b < 1$, and $\{f_n(x_n)\}$ is the uniform convergence sequence for any x_n in a bounded subset D of real Hilbert space H.

Motivated by the aforementioned results discussed above, we introduce a modified Halpern iteration process which does not require the knowledge of operator norm to approximate a common solution of split monotone variational inclusion, variational inequality and fixed point problems for countable families of type-one demicontractive multi-valued mappings in real Hilbert spaces. Futhermore, we prove a strong convergence result and state some consequences of our main result. An application of our consequence to split minimization problem was displayed. The result presented in this paper extends and complements the result of Deepho et al. [11] and other related results in literature [3].

2. Preliminaries

We denote the weak and the strong convergence of a sequence $\{x_n\}$ to a point x by $x_n \rightharpoonup x$ and $x_n \to x$, respectively.

Let C be a nonempty, closed and convex subset of a real Hilbert space H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that

$$||x - P_C x|| \leq ||x - y||, \ \forall \ y \in C.$$

P_C is called the metric projection of H onto C and it is well known that P_C is a nonexpansive mapping of H onto C and also satisfies

$$||P_C x - P_C y|| \leq \langle x - y, P_C x - P_C y \rangle.$$

Moreover, $P_C x$ is characterized by the following properties:

$$\langle x - P_C x, y - P_C x \rangle \leq 0$$

and

$$||x - y||^2 \geq ||x - P_C x||^2 + ||y - P_C x||^2, \ \forall \ x \in H, \ y \in C.$$

Definition 2.1. Let H be a real Hilbert space and $T : H \to CB(H)$ a multi-valued mapping. Then, T is said to be demiclosed at the origin if for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x^*$, and $||x_n - T(x_n)|| \to 0$, we have $x^* \in Tx^*$.

Lemma 2.2. ([10]) *Let H be a real Hilbert space and $T : H \to H$ be a nonexpansive mapping. Then for all $x, y \in H$,*

$$\langle (x - Tx) - (y - Ty), Ty - Tx \rangle \leq \frac{1}{2}||(Tx - x) - (Ty - y)||^2 \tag{2.1}$$

and consequently, if $y \in F(T)$, then

$$\langle x - Tx, Ty - Tx \rangle \leq \frac{1}{2}||Tx - x||^2. \tag{2.2}$$

Lemma 2.3. ([7]) *Let H be a real Hilbert space. Then for all $x, y \in H$ and $\alpha \in (0, 1)$, we have*

(i) $2\langle x, y\rangle = ||x||^2 + ||y||^2 - ||x - y||^2 = ||x + y||^2 - ||x||^2 - ||y||^2$,
(ii) $||\alpha x + (1 - \alpha)y||^2 = \alpha||x||^2 + (1 - \alpha)||y||^2 - \alpha(1 - \alpha)||x - y||^2$.

Lemma 2.4. ([9]) *Let H be a real Hilbert space and $\{x_i\}_{i\geq 1}$ be a bounded sequence in H. For $\alpha_i \in (0, 1)$ such that $\sum_{i=1}^{\infty} \alpha_i = 1$, the following identity holds*

$$||\sum_{i=1}^{\infty} \alpha_i x_i||^2 = \sum_{i=1}^{\infty} \alpha_i ||x||^2 - \sum_{1\leq i<j<\infty} \alpha_i \alpha_j ||x_i - x_j||^2.$$

Lemma 2.5. ([8]) *Let H be a real Hilbert space $T : H \to CB(H)$ be a multi-valued k-demicontractive mapping. Assume that for every $p \in F(T)$, $Tp = \{p\}$. Then*

$$\mathcal{H}(Tx, Tp) \leq \frac{1 + \sqrt{k}}{1 - \sqrt{k}}||x - p||, \ \ \forall \ x \in C, p \in F(T).$$

Lemma 2.6. ([14]) *Let $A : H \to 2^H$ be a maximal monotone mapping and $g : H \to H$ be a Lipschitz continuous mapping. Then the mapping $G = A + g : H \to 2^H$ is also a maximal monotone mapping.*

Proposition 2.7. *Let $D : C \to H$ be an inverse strongly monotone(ism) mapping. Then,*

$$u \in VI(C, D) \Longleftrightarrow u = P_C(u - \lambda Du), \ \ \lambda > 0.$$

Proposition 2.8. *Let D be an ism mapping of C into H. Let $N_C v$ be the normal cone to C at $v \in C$, i.e.*

$$N_C v = \{w \in H \mid \langle v - u, w\rangle \geq 0, \ \forall \ u \in C\},$$

and define

$$Tv = \begin{cases} Dv + N_C v, \ v \in C \\ \emptyset, \ v \in H \backslash C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, D)$.

Lemma 2.9. ([19]) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \sigma_n)a_n + \sigma_n \delta_n, \ n > 0,$$

where $\{\sigma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a real sequence such that
(i) $\sum_{n=1}^{\infty} \sigma_n = \infty$,
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\sigma_n \delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

In this section, we state and prove a strong convergence result.

Theorem 3.1. *Let H_1 and H_2 be real Hilbert spaces and C be a nonempty, closed and convex subset of H_1. Let $A : H_1 \to H_2$ be a bounded linear operator with A^* its adjoint. Let $f : H_1 \to H_1$ be σ-ism mapping and $g : H_2 \to H_2$ be ρ-ism mapping. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be multi-valued maximal monotone mappings, and $T_i : H_1 \to CB(H_1), \ i = 1, 2, ...$ be an infinite family of multi-valued type-one demicontractive type mappings with constant k_i such that $k = \sup_{n\geq 1}\{k_i\} \in (0, 1)$. Let $D : C \to H_1$ be a δ-ism mapping and P_C a metric projection of H_1 onto C. Assume that*

$$\Gamma := \bigcap_{i=1}^{\infty} F(T_i) \cap VI(C, D) \cap \Omega \neq \emptyset,$$

and γ_n is chosen in such a way that for some $\varepsilon > 0$,

$$\gamma_n \in \left(\varepsilon, \frac{||(J_\lambda^{B_2}(I - \lambda g) - I)Ax_n||^2}{||A^*(J_\lambda^{B_2}(I - \lambda g) - I)Ax_n||^2} - \varepsilon\right), \tag{3.1}$$

for $J_\lambda^{B_2}(I - \lambda g)Ax_n \neq Ax_n$ and $\gamma_n = \gamma$, otherwise (γ being any nonnegative real number). The sequences $\{u_n\}$, $\{w_n\}$ and $\{x_n\}$ generated iteratively for *an arbitrary* $x_1 \in C$ *and a fixed* $u \in C$ *by*

$$\begin{cases} u_n = J_\lambda^{B_1}(I-\lambda f)(x_n + \gamma_n A^*(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n), \\ w_n = P_C(u_n - \xi D u_n), \\ x_{n+1} = \alpha_n u + (\beta_{n,0} - \alpha_n) w_n + \sum_{i=1}^\infty \beta_{n,i} z_n^i, \end{cases} \tag{3.2}$$

where $z_n^i \in P_{T_i} w_n$ *and* $P_{T_i} w_n := \{z_n^i \in T_i w_n : ||z_n^i - w_n|| = d(w_n, T_i w_n)\}$, $\lambda > 0$ *with conditions:*

(i) $\beta_{n,0} \in (k,1)$, $\beta_{n,i}\beta_{n,j} \in (0,1)$, $i,j = 1,2,...$ *such that* $\sum_{i=0}^\infty \beta_{n,i} = 1$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^\infty \alpha_n = \infty$,
(iii) *for each* $i \geq 1$, $\liminf_{n\to\infty} \beta_{n,0}\beta_{n,i} > 0$,
(iv) $\alpha_n < \beta_{n,0}$ *for each* $n \geq 1$,
(v) *for each* $p \in \cap_{i=1}^\infty F(T_i)$ *and* $T_i p = \{p\}$.

Then the sequence $\{x_n\}$ *converges strongly to* $z \in \Gamma$.

Proof. Let $p \in \Gamma$. Then we have from (3.2) that

$$\begin{aligned} ||u_n - p||^2 &= ||J_\lambda^{B_1}(I-\lambda f)(x_n + \gamma_n A^*(J_\lambda^{B_2}(I-\lambda g)Ax_n)) - p||^2 \\ &\leq ||x_n + \gamma_n A^*(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n - p||^2 \\ &= ||x_n - p||^2 + \gamma_n^2 ||A^*(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n||^2 \\ &\quad + 2\gamma_n \langle x_n - p, A^*(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n\rangle. \end{aligned} \tag{3.3}$$

But from Lemma 2.2, we have

$$\begin{aligned} &2\gamma_n\langle x_n - p,\ A^*(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n\rangle \\ &= 2\gamma_n\langle A(x_n - p),\ (J_\lambda^{B_2}(I-\lambda g) - I)Ax_n\rangle \\ &= 2\gamma_n\big[\langle J_\lambda^{B_2}(I-\lambda g)Ax_n - Ap, (J_\lambda^{B_2}(I-\lambda g) - I)Ax_n\rangle \\ &\quad - ||(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n||^2\big] \\ &\leq 2\gamma_n\big[\frac{1}{2}||(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n||^2 - ||(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n||^2\big] \\ &= -\gamma_n ||(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n||^2. \end{aligned} \tag{3.4}$$

Thus from (3.3) and (3.4), we obtain

$$\begin{aligned} ||u_n - p||^2 &\leq ||x_n - p||^2 + \gamma_n^2 ||A^*(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n||^2 \\ &\quad - \gamma_n ||(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n||^2 \\ &= ||x_n - p||^2 + \gamma_n\big[\gamma_n ||A^*(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n||^2 \\ &\quad - ||(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n||^2\big]. \end{aligned} \tag{3.5}$$

Using condition on γ_n in (3.1), we obtain that

$$||u_n - p||^2 \leq ||x_n - p||^2. \tag{3.6}$$

Since D is δ-ism and $0 < 2\xi < 2\delta$, we estimate

$$\begin{aligned} ||w_n - p||^2 &= ||P_C(I - \xi D)u_n - P_C(I - \xi D)p||^2 \\ &= ||(I - \xi D)u_n - (I - \xi D)p||^2 \\ &= ||(u_n - p) - \xi(Du_n - Dp)||^2 \\ &= ||u_n - p||^2 - 2\xi\langle Du_n - Dp, u_n - p\rangle + \xi^2||Du_n - Dp||^2 \\ &\leq ||u_n - p||^2 - 2\xi\delta||Du_n - Dp||^2 + \xi^2||Du_n - Dp||^2 \\ &= ||u_n - p||^2 + \xi(\xi - 2\delta)||Du_n - Dp||^2 \\ &\leq ||u_n - p||^2. \end{aligned} \tag{3.7}$$

Using Lemma 2.4, (3.7) and the convexity of $||.||^2$, we have

$$\begin{aligned} &||x_{n+1} - p||^2 \\ &= ||\alpha_n u + (\beta_{n,0} - \alpha_n)w_n + \sum_{i=1}^{\infty}\beta_{n,i}z_n^i - p||^2 \\ &= ||\alpha_n(u - p) + (\beta_{n,0} - \alpha_n)(w_n - p) + \sum_{i=1}^{\infty}\beta_{n,i}z_n^i(z_n^i - p)||^2 \\ &\leq \alpha_n||u - p||^2 + (\beta_{n,0} - \alpha_n)||w_n - p||^2 + \sum_{i=1}^{\infty}\beta_{n,i}||z_n^i - p||^2 \\ &\leq \alpha_n||u - p||^2 + (\beta_{n,0} - \alpha_n)||u_n - p||^2 + \sum_{i=1}^{\infty}(\mathcal{H}(T_iw_n, T_ip))^2 \\ &\leq \alpha_n||u - p||^2 + (\beta_{n,0} - \alpha_n)||x_n - p||^2 \\ &\quad + \sum_{i=1}^{\infty}\beta_{n,i}\big[||w_n - p||^2 + k(d(w_n, T_iw_n))^2\big] \\ &\leq \alpha_n||u - p||^2 + (\beta_{n,0} - \alpha_n)||x_n - p||^2 + \sum_{i=1}^{\infty}\big[||w_n - p||^2 + k||w_n - z_n^i||^2\big] \\ &= (1 - \alpha_n)||x_n - p||^2 + \alpha_n||u - p||^2 + (k - \beta_{n,0})\sum_{i=1}^{\infty}\beta_{n,i}||w_n - z_n^i||^2 \\ &\leq (1 - \alpha_n)||x_n - p||^2 + \alpha_n||u - p||^2 \\ &\quad \vdots \\ &\leq \max\{||x_1 - p||^2, ||u - p||^2\}. \end{aligned} \tag{3.8}$$

Therefore, $\{||x_1 - p||^2\}$ is bounded. Hence, $\{x_n\}$ is bounded. Consequently, $\{u_n\}$, $\{w_n\}$ and $\{z_n^i\}$ are all bounded.

From (3.2), (3.5) and (3.7), we have that

$$\begin{aligned}||x_{n+1} - p||^2 \leq (1-\alpha_n)||x_n - p||^2 &+ \alpha_n ||u-p||^2 \\ &+ (1-\alpha_n)\gamma_n\big[\gamma_n ||A^*(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n||^2 \\ &- ||(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n||^2\big] \\ &+ (1-\alpha_n)\xi(\xi - 2\delta)||Du_n - Dp||^2. \end{aligned} \tag{3.9}$$

By condition on γ_n in (3.1), we have from (3.9) that

$$\begin{aligned}||x_{n+1} - p||^2 \ &\leq (1-\alpha_n)||x_n - p||^2 + \alpha_n||u-p||^2 \\ &\quad - (1-\alpha_n)\varepsilon^2 ||A^*(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n||^2 \\ &\quad - (1-\alpha_n)\xi(2\delta - \xi)||Du_n - Dp||^2. \end{aligned} \tag{3.10}$$

CASE 1: Assume that $\{||x_n - p||\}$ is a monotonically decreasing sequence. Then, $\{x_n\}$ is convergent and thus

$$\lim_{n\to\infty} ||x_n - p|| = \lim_{n\to\infty} ||x_{n+1} - p||.$$

From (3.10), we have that

$$\lim_{n\to\infty} ||A^*(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n|| = 0. \tag{3.11}$$

Also from (3.9), we have that

$$\lim_{n\to\infty} ||(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n|| = 0. \tag{3.12}$$

Using (3.10) and condition (ii), we obtain that

$$\lim_{n\to\infty} ||Du_n - Dp|| = 0. \tag{3.13}$$

From (3.8), we have that

$$\begin{aligned}(\beta_{n,0} - k)\sum_{i=1}^{\infty}\beta_{n,i}||w_n - z_n^i||^2 &\leq (1-\alpha_n)||x_n - p||^2 \\ &\quad - ||x_{n+1} - p||^2 + \alpha_n||u-p||^2.\end{aligned}$$

Hence, from condition (ii),

$$\lim_{n\to\infty}(\beta_{n,0} - k)\sum_{i=1}^{\infty}\beta_{n,i}||w_n - z_n^i||^2 = 0. \tag{3.14}$$

Now, for each $i = 1, 2, ...$ and condition (i), we obtain

$$\lim_{n\to\infty} ||w_n - z_n^i||^2 = \lim_{n\to\infty} d(w_n, T_i w_n) = 0. \tag{3.15}$$

Using (3.2), we have

$$\begin{aligned}
||u_n - p||^2 &= ||J_\lambda^{B_1}(I-\lambda f)(x_n + \gamma_n A^*(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n) - p||^2 \\
&\le \langle x_n - p,\ x_n + \gamma_n A^*(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n - p\rangle \\
&= \frac{1}{2}\big[||u_n - p||^2 + ||x_n + \gamma_n A^*(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n - p||^2 \\
&\quad - ||u_n - p - (x_n + \gamma_n A^*(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n) - p||^2\big] \\
&\le \frac{1}{2}\big[||u_n - p||^2 + ||x_n - p||^2 \\
&\quad + \gamma_n(\gamma_n ||A^*(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n||^2 \\
&\quad - ||(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n||^2) \\
&\quad - ||u_n - p - (x_n + \gamma_n A^*(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n - p)||^2\big] \\
&\le \frac{1}{2}\big[||u_n - p||^2 + ||x_n - p||^2 \\
&\quad - (||u_n - x_n||^2 + \gamma_n^2 ||A^*(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n||^2 \\
&\quad - 2\gamma_n\langle u_n - x_n,\ A^*(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n\rangle)\big] \\
&\le \frac{1}{2}\big[||u_n - p||^2 + ||x_n - p||^2 - ||u_n - x_n||^2 \\
&\quad + 2\gamma_n ||u_n - x_n||\ ||A^*(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n||\big]. \qquad (3.16)
\end{aligned}$$

Thus,

$$\begin{aligned}
||u_n - p||^2 \le &||x_n - p||^2 - ||u_n - x_n||^2 \\
&+ 2\gamma_n ||u_n - x_n||\ ||A^*(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n||. \qquad (3.17)
\end{aligned}$$

From (3.2) and (3.17), we have that

$$\begin{aligned}
||x_{n+1} - p||^2 &\le (1-\alpha_n)||u_n - p||^2 + \alpha_n ||u - p||^2 \\
&\quad + (k - \beta_{n,0}) \sum_{i=1}^{\infty} \beta_{n,i} ||w_n - z_n^i||^2 \\
&\le (1-\alpha_n)\big[||x_n - p||^2 - ||u_n - x_n||^2 \\
&\quad + 2\gamma_n ||u_n - x_n||\ ||A^*(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n||\big] \\
&\quad + (k - \beta_{n,0}) \sum_{i=1}^{\infty} \beta_{n,i} ||w_n - z_n^i||^2
\end{aligned}$$

$$
\begin{aligned}
&= (1-\alpha_n)||x_n - p||^2 - ||u_n - x_n||^2 + \alpha_n||u_n - x_n||^2 \\
&\quad + (1-\alpha_n)2\gamma_n||u_n - x_n||\ ||A^*(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n|| \\
&\quad + (k-\beta_{n,0})\sum_{i=1}^{\infty}\beta_{n,i}||w_n - z_n^i||^2, \qquad (3.18)
\end{aligned}
$$

this implies that

$$
\begin{aligned}
||u_n - x_n||^2 &\le (1-\alpha_n)||x_n - p||^2 - ||x_{n+1} - p||^2 + \alpha_n||u_n - x_n||^2 \\
&\quad + (1-\alpha_n)2\gamma_n||u_n - x_n||\ ||A^*(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n|| \\
&\quad + (k-\beta_{n,0})\sum_{i=1}^{\infty}\beta_{n,i}||w_n - z_n^i||^2.
\end{aligned}
$$

From condition (ii), (3.11) and (3.14), we obtain that

$$
\lim_{n\to\infty}||u_n - x_n|| = 0. \qquad (3.19)
$$

Using (3.2), we have

$$
\begin{aligned}
&||w_n - p||^2 \\
&= ||P_C(I-\xi D)u_n - P_C(I-\xi D)p||^2 \\
&\le \langle P_C(I-\xi D)u_n - P_C(I-\xi D)p, (I-\xi D)u_n - (I-\xi D)p\rangle \\
&= \langle w_n - p, (I-\xi D)u_n - (I-\xi D)p\rangle \\
&= \frac{1}{2}\big[||w_n - p||^2 + ||u_n - p - \xi(Du_n - Dp)||^2 \\
&\quad - ||(w_n - p) - ((1-\xi D)u_n - (1-\xi D)p)||^2\big] \\
&\le \frac{1}{2}\big[||w_n - p||^2 + ||u_n - p||^2 + \xi(\xi - 2\delta)||Du_n - Dp||^2 \\
&\quad - ||(w_n - u_n) + \xi(Du_n - Dp)||^2\big] \\
&\le \frac{1}{2}\big[||w_n - p||^2 + ||u_n - p||^2 + \xi(\xi - 2\delta)||Du_n - Dp||^2\big] \\
&\quad - ||w_n - u_n||^2 - \xi^2||Du_n - Dp||^2 - 2\xi\langle w_n - u_n, Du_n - Dp\rangle \\
&= \frac{1}{2}\big[||w_n - p||^2 + ||u_n - p||^2 - 2\xi\delta||Du_n - Dp||^2 \\
&\quad - ||w_n - u_n||^2 + 2\xi\langle u_n - w_n, Du_n - Dp\rangle \\
&\le \frac{1}{2}\big[||w_n - p||^2 + ||x_n - p||^2 - ||w_n - u_n||^2 \\
&\quad + 2\xi||u_n - w_n||\ ||Du_n - Dp||\big]. \qquad (3.20)
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
||w_n - p||^2 \leq ||x_n - p||^2 - ||w_n - u_n||^2 \\
+ 2\xi||u_n - w_n||\,||Du_n - Dp||.
\end{aligned} \tag{3.21}
$$

Using (3.2), (3.21) and following the same process as in (3.18), we have that

$$
\begin{aligned}
||x_{n+1} - p||^2 &\leq (1-\alpha_n)||w_n - p||^2 + \alpha_n||u - p||^2 \\
&\quad + (k - \beta_{n,0})\sum_{i=1}^{\infty}\beta_{n,i}||w_n - z_n^i||^2 \\
&\leq (1-\alpha_n)\big[||x_n - p||^2 - ||w_n - u_n||^2 \\
&\quad + 2\xi||u_n - w_n||\,||Du_n - Dp||\big] \\
&\quad + 2\gamma_n||u_n - x_n||\,||A^*(J_\lambda^{B_2}(I - \lambda g) - I)Ax_n|| \\
&\quad + (k - \beta_{n,0})\sum_{i=1}^{\infty}\beta_{n,i}||w_n - z_n^i||^2 \\
&= (1-\alpha_n)||x_n - p||^2 - ||w_n - u_n||^2 + \alpha_n||w_n - u_n||^2 \\
&\quad + 2(1-\alpha_n)\xi||u_n - w_n||\,||Du_n - Dp|| \\
&\quad + 2\gamma_n||u_n - x_n||\,||A^*(J_\lambda^{B_2}(I - \lambda g) - I)Ax_n|| \\
&\quad + (k - \beta_{n,0})\sum_{i=1}^{\infty}\beta_{n,i}||w_n - z_n^i||^2.
\end{aligned} \tag{3.22}
$$

Hence, we have from (3.22) that

$$
\begin{aligned}
||w_n - u_n||^2 &\leq (1-\alpha_n)||x_n - p||^2 - ||x_{n+1} - p||^2 + \alpha_n||w_n - u_n||^2 \\
&\quad + 2(1-\alpha_n)\xi||u_n - w_n||\,||Du_n - Dp|| \\
&\quad + 2\gamma_n||u_n - x_n||\,||A^*(J_\lambda^{B_2}(I - \lambda g) - I)Ax_n|| \\
&\quad + (k - \beta_{n,0})\sum_{i=1}^{\infty}\beta_{n,i}||w_n - z_n^i||^2.
\end{aligned} \tag{3.23}
$$

Thus, using condition (ii), (3.13) and (3.14), we obtain

$$
\lim_{n\to\infty}||w_n - u_n|| = 0. \tag{3.24}
$$

From (3.19) and (3.24), we have that

$$
\lim_{n\to\infty}||w_n - x_n|| = 0. \tag{3.25}
$$

From (3.2) and (3.15), we obtain that

$$||x_{n+1} - w_n|| \le \alpha_n ||u - w_n|| + \sum_{i=1}^{\infty} \beta_{n,i} ||z_n^i - w_n|| \to 0, \ n \to \infty. \tag{3.26}$$

Using (3.25) and (3.26), we have that

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{3.27}$$

Let $t_n = x_n + \gamma_n A^*(J_\lambda^{B_2}(I - \lambda g) - I)Ax_n$, then, we have from (3.11) that

$$||t_n - x_n|| \le \gamma_n A^* ||J_\lambda^{B_2}(I - \lambda g) - I)Ax_n|| \to 0, \ n \to \infty. \tag{3.28}$$

Also, we have from (3.19) and (3.28) that

$$||u_n - t_n|| \le ||u_n - x_n|| + ||t_n - x_n|| \to 0, \ n \to \infty. \tag{3.29}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ which converges weakly to $z \in H$ and consequently, we have $\{u_n\}$, $\{w_n\}$ and $\{t_n\}$ with subsequences $\{u_{n_j}\}$, $\{w_{n_j}\}$ and $\{t_{n_j}\}$ which converges weakly to z. Using the demiclosedness principle and (3.15), we have that $z \in \cap_{i=1}^{\infty} F(T_i), \ i = 1, 2, ...$

We now show that $z \in I(f, B_1)$. Let $(a, b) \in G(B_1 + f)$ which implies that $b - fa \in B_1(a)$. Since $w_{n_j} = J_\lambda^{B_1}(I - \lambda f)t_{n_j}$, we have that $(I - \lambda f)t_{n_j} \in (I + \lambda B_1)u_{n_j}$, that is $\frac{1}{\lambda}(t_{n_j} - \lambda f t_{n_j} - u_{n_j}) \in B_1(u_{n_j})$. Using the maximal monotonicity of $(B_1 + f)$, we have

$$\langle a - u_{n_j}, b - fa - \frac{1}{\lambda}(t_{n_j} - \lambda f t_{n_j} - u_{n_j})\rangle \ge 0.$$

Hence, we have

$$\begin{aligned}
\langle a - u_{n_j}, b\rangle &\ge \langle a - u_{n_j}, fa + \frac{1}{\lambda}(t_{n_j} - \lambda f t_{n_j} - u_{n_j})\rangle \\
&= \langle a - u_{n_j}, fa - fu_{n_j} + fu_{n_j} - ft_{n_j} + \frac{1}{\lambda}(t_{n_j} - u_{n_j})\rangle \\
&\ge 0 + \langle a - u_{n_j}, fu_{n_j} - ft_{n_j}\rangle + \langle a - u_{n_j}, \frac{1}{\lambda}(t_{n_j} - u_{n_j})\rangle.
\end{aligned} \tag{3.30}$$

Using (3.29), we have that

$$||fu_{n_j} - ft_{n_j}|| = 0. \tag{3.31}$$

Since $u_{n_j} \rightharpoonup z$, we have

$$\lim_{j\to\infty} \langle a - u_{n_j}, b\rangle = \langle a - z, b\rangle. \tag{3.32}$$

Thus, from (3.30), we obtain that

$$\langle a - z, b\rangle \ge 0.$$

From the fact that $B_1 + f$ is maximal monotone, we conclude that $0 \in (B_1 + f)z$ which implies that $z \in I(f, B_1)$.

Consequently, since $Ax_{n_j} \rightharpoonup Az$, we have from (3.12) and Lemma 2.6 that

$$0 \in gAz + B_2(Az).$$

Hence, we have $Az \in I(g, B_2)$.

Moreover, it follows from (3.24) that $w_{n_j} \rightharpoonup z$. Define

$$\mathcal{H}a = \begin{cases} Da + N_C a, \ a \in C, \\ \emptyset, \ a \in H_1 \backslash C. \end{cases}$$

By Proposition 2.8, we have that $\mathcal{H}$ is maximal monotone. Take $(a, b) \in G(\mathcal{H})$, it is easy to see that $b - Da \in N_C a$. Since $w_n \in C$, we have

$$\langle a - w_n, b - Da\rangle \geq 0. \tag{3.33}$$

Since $w_n = P_C(u_n - \xi Du_n)$, we have that

$$\langle u_n - \xi Du_n - w_n, w_n - a\rangle \geq 0 \tag{3.34}$$

and hence

$$\langle a - w_n, \frac{w_n - u_n}{\xi} + Du_n\rangle \geq 0. \tag{3.35}$$

Thus, from (3.33) and (3.35), we obtain that

$$\begin{aligned}
\langle a - w_{n_j},\ b\rangle &\geq \langle a - w_{n_j}, Da\rangle \\
&\geq \langle a - w_{n_j}, Da\rangle - \langle a - w_{n_j}, Du_{n_j} + \frac{w_{n_j} - u_{n_j}}{\xi}\rangle \\
&= \langle a - w_{n_j},\ Da - Dw_{n_j}\rangle + \langle a - w_{n_j},\ Dw_{n_j} - Du_{n_j}\rangle \\
&\quad - \langle a - w_{n_j}, Du_{n_j} + \frac{w_{n_j} - u_{n_j}}{\xi}\rangle \\
&\geq \delta||Da - Dw_{n_j}||^2 + \langle a - w_{n_j},\ Dw_{n_j} - Du_{n_j}\rangle \\
&\quad - \langle a - w_{n_j}, Du_{n_j} + \frac{w_{n_j} - u_{n_j}}{\xi}\rangle \\
&\geq \langle a - w_{n_j},\ Dw_{n_j} - Du_{n_j}\rangle - \langle a - w_{n_j}, Du_{n_j} + \frac{w_{n_j} - u_{n_j}}{\xi}\rangle.
\end{aligned}$$

By letting $j \to \infty$ and (3.24), we have that $\langle a - z, b\rangle \geq 0$. Since $\mathcal{H}$ is maximal monotone, we have that $z \in \mathcal{H}^{-1}0$. So it follows from Proposition 2.8 that $z \in VI(C, D)$. Hence, we conclude that

$$z \in \Gamma = \cap_{i=1}^{\infty} F(T_i) \cap VI(C, D) \cap \Omega \neq \emptyset.$$

Next, we prove that $\{x_n\}$ converges strongly to z. Now, applying Lemma 2.3, we have that

$$\begin{aligned}
||x_{n+1}-z||^2 &= ||\alpha_n u + (\beta_{n,0}-\alpha_n)w_n + \sum_{i=1}^{\infty}\beta_{n,i}z_n^i - z||^2 \\
&= ||(\beta_{n,0}-\alpha_n)(w_n-z) + \sum_{i=1}^{\infty}\beta_{n,i}(z_n^i - z) + \alpha_n(u-z)||^2 \\
&\le ||(\beta_{n,0}-\alpha_n)(w_n-z) + \sum_{i=1}^{\infty}\beta_{n,i}(z_n^i - z)||^2 \\
&\quad + 2\alpha_n\langle x_{n+1}-z, u-z\rangle \\
&\le \big[(\beta_{n,0}-\alpha_n)||w_n-z|| + (k-\beta_{n,0})\sum_{i=1}^{\infty}\beta_{n,i}||w_n - z_n^i||\big]^2 \\
&\quad + 2\alpha_n\langle x_{n+1}-z, u-z\rangle \\
&= (1-\alpha_n)||w_n-z||^2 + 2\alpha_n\langle x_{n+1}-z, u-z\rangle \\
&\le (1-\alpha_n)||x_n-z||^2 + 2\alpha_n\langle x_{n+1}-z, u-z\rangle. \qquad (3.36)
\end{aligned}$$

Since $x_n \rightharpoonup z$, using Lemma 2.9 and condition (ii), we obtain that $||x_n - z|| \to 0$, as $n \to \infty$, which implies that $\{x_n\}$ converges strongly to $z \in \Gamma$.

CASE 2: Assume that $\{||x_n - p||\}$ is not a monotonically decreasing sequence. Set $\Upsilon_n = ||x_n - p||^2$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \ge n_0$ (for some n_0 large enough) defined by

$$\tau(n) := \max\{k \in \mathbb{N} : k \ge n, \psi_k \le \psi_{k+1}\}.$$

Clearly, τ is a nondecreasing sequence such that $\tau(n) \to \infty$ and $\psi_{\tau(n)} \le \psi_{\tau(n)+1}$, for $n \ge n_0$. It follows from (3.9) and (3.10) that

$$\lim_{\tau(n)\to\infty} ||A^*(J_\lambda^{B_2}(I-\lambda g) - I)Ax_{\tau(n)}|| = 0$$

and

$$\lim_{\tau(n)\to\infty} ||(J_\lambda^{B_2}(I-\lambda g) - I)Ax_{\tau(n)}|| = 0.$$

Also, using (3.10) and condition (ii), we obtain that

$$\lim_{\tau(n)\to\infty} ||Du_{\tau(n)} - Dp|| = 0.$$

By following the same argument as in Case 1, we can show that

$$d(w_{\tau(n)}, z_{\tau(n)}^i) = d(w_{\tau(n)}, T_i w_{\tau(n)}) = 0.$$

Now, for all $n \geq n_0$, we have from (3.36) that

$$\begin{aligned}0 &\leq ||x_{\tau(n)+1} - z||^2 - ||x_{\tau(n)} - z||^2\\ &\leq (1-\alpha_{\tau(n)})||x_{\tau(n)} - z||^2 + 2\alpha_{\tau(n)}\langle x_{\tau(n)+1} - z, u - z\rangle - ||x_{\tau(n)} - z||^2.\end{aligned}$$

Thus,

$$||x_{\tau(n)} - z||^2 \leq 2\langle x_{\tau(n)} - z, u - z\rangle \to 0.$$

Hence,

$$\lim_{\tau(n)\to\infty} ||x_{\tau(n)} - z||^2 = 0. \tag{3.37}$$

Therefore,

$$\lim_{\tau(n)\to\infty} \psi_{\tau(n)} = \lim_{\tau(n)\to\infty} \psi_{\tau(n)+1} = 0.$$

Moreover for $n \geq n_0$, it is easily observed that $\psi_{\tau(n)} \leq \psi_{\tau(n)}$ if $n \neq \tau(n)$ (that is $\tau(n) < n$) because $\psi_j > \psi_{j+1}$ for $\tau(n) + 1 \leq j \leq n$. Consequently,

$$0 \leq \psi_n \leq \max\{\psi_{\tau(n)},\ \psi_{\tau(n)+1}\} = \psi_{\tau(n)+1}.$$

Hence, $\lim_{n\to\infty} \psi_n = 0$, which implies that $\{x_n\}$ converges strongly to $z \in \Gamma$. This completes the proof. $\square$

Remark 3.2. In this article, we considered a split monotone variational inclusion problem which generalizes the problems considered in [11] and [18]. Also, the mappings considered is a countable family of multi-valued type one demicontractive mappings which generalizes the ones considered in [11], [18] and some other related results in literature.

Here, we consider the class of quasi-nonexpansive multi-valued mappings which is a subclass of demicontractive mappings, see [8].

Corollary 3.3. *Let H_1 and H_2 be real Hilbert spaces and C be a nonempty, closed and convex subset of H_1. Let $A : H_1 \to H_2$ be a bounded linear operator with A^* its adjoint. Let $f : H_1 \to H_1$ be σ-ism mapping and $g : H_2 \to H_2$ be ρ-ism mapping. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be multi-valued maximal monotone mappings, and $T_i : H_1 \to CB(H_1),\ i = 1, 2, ...$ be an infinite family of multi-valued quasi-nonexpansive mappings. Let $D : C \to H_1$ be a δ-ism mappings and P_C a metric projection of H_1 onto C. Assume that*

$$\Gamma := \bigcap_{i=1}^{\infty} F(T_i) \cap VI(C, D) \cap \Omega \neq \emptyset$$

and γ_n is chosen in such a way that for some $\varepsilon > 0$,

$$\gamma_n \in \left(\varepsilon, \frac{||(J_\lambda^{B_2}(I - \lambda g) - I)Ax_n||^2}{||A^*(J_\lambda^{B_2}(I - \lambda g) - I)Ax_n||^2} - \varepsilon\right), \tag{3.38}$$

for $J_\lambda^{B_2}(I-\lambda g)Ax_n \neq Ax_n$ *and* $\gamma_n = \gamma$*, otherwise* (γ *being any nonnegative real number*)*. The sequences* $\{u_n\}$, $\{w_n\}$ *and* $\{x_n\}$ *generated iteratively for an arbitrary* $x_1 \in C$ *and a fixed* $u \in C$ *by*

$$\begin{cases} u_n = J_\lambda^{B_1}(I-\lambda f)(x_n + \gamma_n A^*(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n), \\ w_n = P_C(u_n - \xi D u_n), \\ x_{n+1} = \alpha_n u + (\beta_{n,0} - \alpha_n) w_n + \sum_{i=1}^{\infty} \beta_{n,i} z_n^i, \end{cases} \tag{3.39}$$

where $z_n^i \in P_{T_i} w_n$ *and* $P_{T_i} w_n := \{z_n^i \in T_i w_n : ||z_n^i - w_n|| = d(w_n, T_i w_n)\}$, $\lambda > 0$ *with conditions:*

(i) $\beta_{n,0} \in (0,1)$, $\beta_{n,i}\beta_{n,j} \in (0,1)$, $i, j = 1, 2, \ldots$ *such that* $\sum_{i=0}^{\infty} \beta_{n,i} = 1$,
(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$,
(iii) *for each* $i \geq 1$, $\liminf_{n\to\infty} \beta_{n,0}\beta_{n,i} > 0$,
(iv) $\alpha_n < \beta_{n,0}$ *for each* $n \geq 1$,
(v) *for each* $p \in \cap_{i=1}^{\infty} F(T_i)$ *and* $T_i p = \{p\}$.

Then, the sequence $\{x_n\}$ *converges strongly to* $z \in \Gamma$.

In the following result, we considered an infinite family of multi-valued type one demicontractive type mappings and a split monotone variational inclusion problem.

Corollary 3.4. *Let* H_1 *and* H_2 *be real Hilbert spaces and* C *be a nonempty, closed and convex subset of* H_1*. Let* $A : H_1 \to H_2$ *be a bounded linear operator with* A^* *its adjoint. Let* $f : H_1 \to H_1$ *be* σ*-ism mapping and* $g : H_2 \to H_2$ *be* ρ*-ism mapping. Let* $B_1 : H_1 \to 2^{H_1}$ *and* $B_2 : H_2 \to 2^{H_2}$ *be multi-valued maximal monotone mappings, and* $T_i : H_1 \to CB(H_1)$, $i = 1, 2, \ldots$ *be an infinite family of multi-valued type-one demicontractive type mappings with constant* k_i *such that* $k = \sup_{n\geq 1}\{k_i\} \in (0,1)$*. Assume that*

$$\Gamma := \bigcap_{i=1}^{\infty} F(T_i) \cap \Omega \neq \emptyset$$

and γ_n *is chosen in such a way that for some* $\varepsilon > 0$,

$$\gamma_n \in \left(\varepsilon, \frac{||(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n||^2}{||A^*(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n||^2} - \varepsilon\right), \tag{3.40}$$

for $J_\lambda^{B_2}(I-\lambda g)Ax_n \neq Ax_n$ *and* $\gamma_n = \gamma$*, otherwise* (γ *being any nonnegative real number*)*. The sequences* $\{u_n\}$, $\{w_n\}$ *and* $\{x_n\}$ *generated iteratively for an arbitrary* $x_1 \in C$ *and a fixed* $u \in C$ *by*

$$\begin{cases} u_n = J_\lambda^{B_1}(I-\lambda f)(x_n + \gamma_n A^*(J_\lambda^{B_2}(I-\lambda g) - I)Ax_n), \\ x_{n+1} = \alpha_n u + (\beta_{n,0} - \alpha_n) u_n + \sum_{i=1}^{\infty} \beta_{n,i} z_n^i, \end{cases} \tag{3.41}$$

where $z_n^i \in P_{T_i}u_n$ *and* $P_{T_i}w_n := \{z_n^i \in T_i w_n : ||z_n^i - u_n|| = d(u_n, T_i u_n)\}$, $\lambda > 0$ *with conditions:*

(i) $\beta_{n,0} \in (k,1)$, $\beta_{n,i}\beta_{n,j} \in (0,1)$, $i,j = 1,2,...$ *such that* $\sum_{i=0}^{\infty}\beta_{n,i} = 1$,

(ii) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$,

(iii) *for each* $i \geq 1$, $\liminf_{n\to\infty}\beta_{n,0}\beta_{n,i} > 0$,

(iv) $\alpha_n < \beta_{n,0}$ *for each* $n \geq 1$,

(v) *for each* $p \in \cap_{i=1}^{\infty}F(T_i)$ *and* $T_i p = \{p\}$.

Then, the sequence $\{x_n\}$ *converges strongly to* $z \in \Gamma$.

4. Application

In this section, we give an application of Corollary 3.4 to the split minimization problem (SMP).

Let $\varphi : H \to \mathbb{R}$ be a proper convex lower semi-continuous function and $\phi : H \to \mathbb{R}$ be a convex and differentiable function. Consider the minimization problem:

$$\text{minimize}\{\varphi(x) + \phi(x) : x \in H\}. \tag{4.1}$$

Problem (4.1) is equivalent to finding $x^* \in H$ such that

$$0 \in \partial\varphi(x^*) + \nabla\phi(x^*),$$

where $\partial\varphi$ is the subdifferential of φ and $\nabla\phi$ is the gradient of ϕ. It is well known that $\nabla\phi$ is $\frac{1}{\alpha}$-Lipschitz continuous if and only if it is α-inverse strongly monotone. Also $\partial\varphi$ is maximal monotone. The proximal operator associated with $\partial\varphi$ is defined by

$$prox_{\partial\varphi}(x) = \arg\min\{\varphi(x) + \frac{1}{2}\|x-u\|^2 : u \in H\} \quad \text{for each } x \in H.$$

Consequently, $prox_{\partial\varphi}(I - \nabla\phi)$ is nonexpansive. In addition, $F(prox_{\partial\varphi}(I - \nabla\phi)) = (\partial\varphi + \nabla\phi)^{-1}(0)$. Hence, setting $f = \nabla\phi_1$, $g = \nabla\phi_2$, $B_1 = \partial\varphi_1$ and $B_2 = \partial\varphi_2$ in SMVIP, we obtain the following SMP:

$$\text{find } x^* \in C \quad \text{such that} \quad x^* = \arg\min\{\varphi_1(x) + \phi_1(x) : x \in H_1\} \tag{4.2}$$

and

$$y^* = Ax^* \in Q \quad \text{solves} \quad y^* = \arg\min\{\varphi_2(y) + \phi_2(y) : y \in H_2\}. \tag{4.3}$$

Thus, we present the following algorithm for solving the SMP (4.2)-(4.3). We denote the set of solution of the SMP by Δ.

Theorem 4.1. *Let* C *and* Q *be nonempty, closed and convex subset of real Hilbert spaces* H_1 *and* H_2*, respectively. Let* $A : H_1 \to H_2$ *be a bounded linear operator with* A^* *its adjoint. Suppose* $\varphi_1 : H_1 \to \mathbb{R}$ *and* $\varphi_2 : H_2 \to \mathbb{R}$ *be two proper, convex and lower semi-continuous functions,* $\phi_1 : H_1 \to \mathbb{R}$ *and* $\phi_2 : H_2 \to \mathbb{R}$ *be two convex and differentiable functions such that their gradients*

$\nabla\phi_i$ is $\frac{1}{\alpha_i}$-Lipschitz continuous, $i = 1, 2.$ and $T_i : H_1 \to CB(H_1),\ i = 1, 2, ...$ be an infinite family of multi-valued type-one demicontractive type mappings with constant k_i such that $k = \sup_{n\geq 1}\{k_i\} \in (0, 1)$. Assume that

$$\Gamma := \bigcap_{i=1}^{\infty} F(T_i) \cap \Delta \neq \emptyset$$

and γ_n is chosen in such a way that for some $\varepsilon > 0$,

$$\gamma_n \in \left(\varepsilon, \frac{||prox_{\partial\varphi_2}(I - \nabla\phi_2)Ax_n||^2}{||A^*prox_{\partial\varphi_2}(I - \nabla\phi_2) - I)Ax_n||^2} - \varepsilon\right), \tag{4.4}$$

for $prox_{\partial\varphi_2}(I - \nabla\phi_2)Ax_n \neq Ax_n$ and $\gamma_n = \gamma$, otherwise (γ being any nonnegative real number). The sequences $\{u_n\}$, $\{w_n\}$ and $\{x_n\}$ generated iteratively for an arbitrary $x_1 \in C$ and a fixed $u \in C$ by

$$\begin{cases} u_n = prox_{\partial\varphi_1}(I - \nabla\phi_1)(x_n + \gamma_n A^*(prox_{\partial\varphi_2}(I - \nabla\phi_2)Ax_n), \\ x_{n+1} = \alpha_n u + (\beta_{n,0} - \alpha_n)u_n + \sum_{i=1}^{\infty} \beta_{n,i} z_n^i, \end{cases} \tag{4.5}$$

where $z_n^i \in P_{T_i}u_n$ and $P_{T_i}w_n := \{z_n^i \in T_iw_n : ||z_n^i - u_n|| = d(u_n, T_iu_n)\},\ \lambda > 0$ with conditions:

(i) $\beta_{n,0} \in (k, 1),\ \beta_{n,i}\beta_{n,j} \in (0, 1),\ i, j = 1, 2, ...$ *such that* $\sum_{i=0}^{\infty} \beta_{n,i} = 1$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(iii) *for each* $i \geq 1,\ \liminf_{n\to\infty} \beta_{n,0}\beta_{n,i} > 0$,

(iv) $\alpha_n < \beta_{n,0}$ *for each* $n \geq 1$,

(v) *for each* $p \in \cap_{i=1}^{\infty} F(T_i)$ *and* $T_ip = \{p\}$.

Then, the sequence $\{x_n\}$ converges strongly to $z \in \Gamma$.

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