Nonlinear Functional Analysis and Applications
Vol. 25, No. 3 (2020), pp. 511-523
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2020.25.03.07
http://nfaa.kyungnam.ac.kr/journal-nfaa

BERINDE TYPE RESULTS VIA SIMULATION FUNCTIONS IN METRIC SPACES

Anantachai Padcharoen[1] and Jong Kyu Kim[2]

[1]Department of Mathematics, Faculty of Science and Technology
Rambhai Barni Rajabhat University, Chanthaburi 22000, Thailand
e-mail: anantachai.p@rbru.ac.th

[2]Department of Mathematics Education
Kyungnam University, Changwon, Gyeongnam, 51767, Korea
e-mail: jongkyuk@kyungnam.ac.kr

Abstract. In this paper, we introduce coincidence point theorems for Beride type contraction mappings via simulation functions and obtain some sufficient axioms for the existence and uniqueness of coincidence point for such class of mappings in the setting of metric spaces.

1. Introduction

Some real world problems can be created as mathematical models. The existence of solutions for these problems has been investigated in various mathematics for example, functional analysis, differential equations, integral equations. Some methods via fixed point theory can show the solution of these problems. Fixed point theory gains very large impetus due to its wide range of applications in various fields such as economics, computer science, engineering, biology, physics, etc.

In addition, Banach's contraction principle [1] is crucial to present the existence of solutions for some nonlinear equations, differential and integral equations, and other nonlinear problems.

[0]Received December 9, 2019. Revised December 29, 2019. Accepted December 31, 2019.
[0]2010 Mathematics Subject Classification: 47H09, 47H10, 37C25.
[0]Keywords: Beride type contraction, simulation function, fixed point, coincidence point theorems.
[0]Corresponding author: J. K. Kim(jongkyuk@kyungnam.ac.kr).

Later, Berinde [3] extended the Zamfirescu fixed point theorem [2] to almost contractions, a class of contractive type mappings.

Khojasteh et al. [6] originated the notion of $\mathcal{Z}$-contractions using a specific family of functions called simulation functions. Subsequently, many researchers generalized this idea in many ways (see [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18] and proved many interesting results in the arena of fixed point theory.

In this paper, we define a Berinde type contraction mappings via the simulation functions in metric spaces.

2. Preliminaries

Theorem 2.1. ([1]) *Let (X, d) be a complete metric space and S be a self-mapping on X such that there exist $k \in [0, 1)$,*

$$d(S\theta, S\vartheta) \leq kd(\theta, \vartheta), \quad \forall\, \theta, \vartheta \in X. \tag{2.1}$$

Then, S has a unique fixed point in X.

Theorem 2.2. ([3]) *Let (X, d) be a complete metric space and a self-mapping S on X be an almost contraction, that is, there exist $\delta \in [0, 1)$ and $L \geq 0$ such that*

$$d(S\theta, S\vartheta) \leq \delta d(\theta, \vartheta) + Ld(\vartheta, S\theta), \quad \forall\, \theta, \vartheta \in X. \tag{2.2}$$

Then, we have the followings:

(i) $\mathrm{Fix}(S) \neq \emptyset$, *where* $\mathrm{Fix}(S) = \{\theta \in X : S\theta = \theta\}$;
(ii) *for any $\theta_0 \in X$, the Picard iteration $\{\theta_{n+1}\}$ given by $\theta_{n+1} = S\theta_n$ for each $n \geq 0$ converges to some $\theta^* \in \mathrm{Fix}(S)$;*
(iii) *the following estimate holds*

$$d(\theta_{n+i-1}, \theta^*) \leq \frac{\delta^i}{1-\delta} d(\theta_{n+1}, \theta_n), \quad \forall\, n \geq 0,\ i \geq 1.$$

Theorem 2.3. ([4]) *Let (X, d) be a complete metric space and a self-mapping S ont X be a Ćirić almost contraction, that is, there exist $\delta \in [0, 1)$ and $L \geq 0$ such that for all $\theta, \vartheta \in X$,*

$$\begin{aligned} d(S\theta, S\vartheta) \leq\ & \delta \max\{d(\theta, \vartheta), d(\theta, S\theta), d(\vartheta, S\vartheta), d(\theta, S\vartheta), d(\vartheta, S\theta)\} \\ & + Ld(\vartheta, S\theta). \end{aligned} \tag{2.3}$$

Then, we have the followings:

(i) $\mathrm{Fix}(S) \neq \emptyset$, *where* $\mathrm{Fix}(S) = \{\theta \in X : S\theta = \theta\}$;
(ii) *for any $\theta_0 = \theta \in X$, the Picard iteration $\{\theta_n\}$ given by $\theta_{n+1} = S\theta_n$ for each $n \geq 0$ converges to some $\theta^* \in \mathrm{Fix}(S)$;*

(iii) *the following estimate holds*

$$d(\theta_n, \theta^*) \leq \frac{\delta^n}{1-\delta} d(\theta, S\theta), \quad \forall\, n \geq 1.$$

Subsequently, Babu et al. [5] defined the class of mappings satisfying axiom (B) as follows:

Definition 2.4. ([5]) Let (X, d) be a metric space and a self-mapping S on X is said to satisfy axiom (B) if there exist a constant $\delta \in (0, 1)$ and $L \geq 0$ such that

$$\begin{aligned} &d(S\theta, S\vartheta) \\ &\leq \delta d(\theta, \vartheta) + L \min\{d(\theta, S\theta), d(\vartheta, S\vartheta), d(\theta, S\vartheta), d(\vartheta, S\theta)\},\ \forall\, \theta, \vartheta \in X. \end{aligned} \tag{2.4}$$

They proved a fixed point theorem for such mappings in complete metric spaces. They also discussed quasi-contraction, almost contraction and the class of mappings that satisfy axiom (B) in detail.

Definition 2.5. A mapping $\zeta : [0, \infty)^2 \to \mathbb{R}$ is called a simulation function if it satisfies the following axioms:

(ζ_1) $\zeta(0, 0) = 0$;

(ζ_2) $\zeta(t, s) < s - t$ for all $t, s > 0$;

(ζ_3) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0; \tag{2.5}$$

(ζ_4) if $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$ for all $n \in \mathbb{N}$, then equation (2.5) is satisfied.

If the function ζ satisfies the axioms (ζ_1)-(ζ_3), we say that ζ is a simulation function according to the sense of Khojasteh et al. [6] and if it satisfies (ζ_1), (ζ_2), and (ζ_4), then it is a simulation function according to the sense of Roldán-López-de-Hierro et al. [10]. Denoted by $\mathcal{Z}$ is the set of all simulation functions.

Example 2.6. ([6]) We give some examples of simulation functions.

(i) Let $\zeta : [0, \infty)^2 \to \mathbb{R}$ be defined by $\zeta(t, s) = f(s) - g(t)$ for all $t, s \in [0, \infty)$, where $f, g : [0, \infty) \to [0, \infty)$ are two continuous functions such that $f(t) = g(t) = 0$ if and only if $t = 0$, and $f(t) < t < g(t)$ for all $t > 0$. Then ζ is a simulation function.

(ii) Let $\zeta : [0, \infty)^2 \to \mathbb{R}$ be defined by $\zeta(t, s) = s - \frac{f(t,s)}{g(t,s)} t$ for all $t, s \in [0, \infty)$, where $f, g : [0, \infty)^2 \to [0, \infty)$ are two continuous functions with respect to each variable such that $f(t,s) > g(t,s)$ for all $t,s > 0$. Then ζ is a simulation function.

(iii) Let $\zeta : [0,\infty)^2 \to \mathbb{R}$ be defined by $\zeta(t,s) = s - f(s) - t$ for all $t,s \in [0,\infty)$, where $f : [0,\infty) \to [0,\infty)$ is a continuous function such that $f(t) = 0$ if and only if $t = 0$. Then ζ is a simulation function.

Definition 2.7. ([6]) Let (X,d) be a metric space and $\zeta \in \mathcal{Z}$. A mapping $S : X \to X$ is called a $\mathcal{Z}$-contraction with respect to ζ if

$$\zeta(d(S\theta, S\vartheta), d(\theta,\vartheta)) \geq 0$$

holds for all $\theta, \vartheta \in X$.

Let S and Υ be two self-maps defined on a non-empty set X. If $\theta^* = S\theta = \Upsilon\theta$ for some $\theta \in X$, then θ is called a coincidence point of S and Υ and θ^* is called a point of coincidence of S and Υ. Moreover θ^* is called a common fixed point of S and Υ if $\theta = \theta^*$. A pair (S,Υ) of self-maps is called weakly compatible if they commute at their coincidence points.

Theorem 2.8. ([18]) *Let S and Υ be weakly compatible self-maps defined on a nonempty set X. If S and Υ have a unique point of coincidence $\eta = S\theta = \Upsilon\theta$, then η is the unique common fixed point of S and Υ.*

Motivated and inspired by Definition 2.4, Definition 2.7 and Theorem 2.2, we define a Berinde type contraction mappings via the simulation functions in metric spaces as follows:

Definition 2.9. Let (X,d) be a metric space and let $S,\Upsilon : X \to X$ be self-mappings. We say that S is a Berinde type $(\mathcal{Z},\Upsilon)$-contraction if there exists $\zeta \in \mathcal{Z}$ and a constant $L \geq 0$ such that

$$\zeta(d(S\theta,S\vartheta), M(\theta,\vartheta) + LN(\theta,\vartheta)) \geq 0, \quad \forall\, \theta,\vartheta \in X, \tag{2.6}$$

holds with $\Upsilon\theta \neq \Upsilon\vartheta$, where

$$M(\theta,\vartheta) = \max\left\{ d(\Upsilon\theta,\Upsilon\vartheta), d(\Upsilon\theta,S\theta), d(\Upsilon\vartheta,S\vartheta), \frac{d(\Upsilon\theta,S\vartheta) + d(\Upsilon\vartheta,S\theta)}{2} \right\}$$

and

$$N(\theta,\vartheta) = \min\left\{ d(\Upsilon\theta,S\theta), d(\Upsilon\vartheta,S\vartheta), d(\Upsilon\theta,S\vartheta), d(\Upsilon\vartheta,S\theta) \right\}.$$

Remark 2.10. If S is a Berinde type $(\mathcal{Z},\Upsilon)$-contraction with respect to $\zeta \in \mathcal{Z}$, then

$$d(S\theta,S\vartheta) < M(\theta,\vartheta) + LN(\theta,\vartheta), \quad \forall\, \theta,\vartheta \in X. \tag{2.7}$$

3. Main results

Firstly, the following lemma shows that a point of coincidence of a Berinde type $(\mathcal{Z}, \Upsilon)$-contraction is unique.

Lemma 3.1. *Let (X, d) be a metric space. If S be a Berinde type $(\mathcal{Z}, \Upsilon)$-contraction with respect to $\zeta \in \mathcal{Z}$ with a point of coincidence in X, then it is unique.*

Proof. We prove that if a point of coincidence of S and Υ exists then it is unique. If η_1 and η_2 are two distinct points of coincidence of S and Υ, then there exist two points $\varrho_1, \varrho_2 \in X$ such that $S\varrho_1 = \Upsilon\varrho_1 = \eta_1 \neq \eta_2 = \Upsilon\varrho_2 = S\varrho_2$. Thus, it follows from equation (2.6) and (ζ_2) that

$$0 \leq \zeta(d(S\varrho_1, S\varrho_2), M(\varrho_1, \varrho_2) + LN(\varrho_1, \varrho_2)), \tag{3.1}$$

where

$$\begin{aligned} &M(\varrho_1, \varrho_2) \\ &= \max\left\{d(\Upsilon\varrho_1, \Upsilon\varrho_2), d(\Upsilon\varrho_1, S\varrho_2), d(\Upsilon\varrho_2, S\varrho_2), \frac{d(\Upsilon\varrho_1, S\varrho_2) + d(\Upsilon\varrho_2, S\varrho_1)}{2}\right\} \\ &= d(\eta_1, \eta_2) \end{aligned}$$

and

$$N(\varrho_1, \varrho_2) = \min\{d(\Upsilon\varrho_1, S\varrho_1), d(\Upsilon\varrho_2, S\varrho_2), d(\Upsilon\varrho_1, S\varrho_2), d(\Upsilon\varrho_2, S\varrho_1)\} = 0.$$

This together with (3.1) shows that

$$\begin{aligned} 0 &\leq \zeta(d(S\varrho_1, S\varrho_2), M(\varrho_1, \varrho_2) + LN(\varrho_1, \varrho_2) \\ &= \zeta(d(\eta_1, \eta_2), d(\eta_1, \eta_2)) \\ &< d(\eta_1, \eta_2) - d(\eta_1, \eta_2) \\ &= 0 \end{aligned} \tag{3.2}$$

which is a contradiction. Hence, the point of coincidence of S and Υ in X is unique. □

Theorem 3.2. *Let (X, d) be a complete metric space, S be a Berinde type $(\mathcal{Z}, \Upsilon)$-contraction with respect to $\zeta \in \mathcal{Z}$ and suppose that there exists a Picard-Jungck sequence $\{\theta_n\}$ of (S, Υ). Then*

$$\lim_{n\to\infty} d(\Upsilon\theta_n, \Upsilon\theta_{n+1}) = 0. \tag{3.3}$$

Proof. We consider the Picard-Jungck sequence such that $\Upsilon\theta_{n+1} = S\theta_n$, where $n \in \mathbb{N}$. If $\Upsilon\theta_n = \Upsilon\theta_{n+1}$, for some $n \in \mathbb{N}$, then θ_n is a coincidence point. Thus, we assume that $\Upsilon\theta_n \neq \Upsilon\theta_{n+1}$ which implies that $d(\Upsilon\theta_n, \Upsilon\theta_{n+1}) > 0$ for all $n \in \mathbb{N}$. Letting $\theta = \theta_n$ and $\vartheta = \theta_{n+1}$ in equation (2.6), we obtain

$$0 \leq \zeta(d(S\theta_n, S\theta_{n+1}), M(\theta_n, \theta_{n+1}) + LN(\theta_n, \theta_{n+1})), \tag{3.4}$$

where

$$\begin{aligned}
&M(\theta_n, \theta_{n+1}) \\
&= \max\left\{d(\Upsilon\theta_n, \Upsilon\theta_{n+1}), d(\Upsilon\theta_n, S\theta_n), d(\Upsilon\theta_{n+1}, S\theta_{n+1}),\right. \\
&\qquad \left.\frac{d(\Upsilon\theta_n, S\theta_{n+1}) + d(\Upsilon\theta_{n+1}, S\theta_n)}{2}\right\} \\
&= \max\left\{d(\Upsilon\theta_n, \Upsilon\theta_{n+1}), d(\Upsilon\theta_n, \Upsilon\theta_{n+1}), d(\Upsilon\theta_{n+1}, \Upsilon\theta_{n+2}),\right. \\
&\qquad \left.\frac{d(\Upsilon\theta_n, \Upsilon\theta_{n+2}) + d(\Upsilon\theta_{n+1}, \Upsilon\theta_{n+1})}{2}\right\} \\
&= \max\left\{d(\Upsilon\theta_n, \Upsilon\theta_{n+1}), d(\Upsilon\theta_{n+1}, \Upsilon\theta_{n+2}), \frac{d(\Upsilon\theta_n, \Upsilon\theta_{n+2})}{2}\right\}.
\end{aligned} \tag{3.5}$$

The triangle inequality yields

$$\frac{d(\Upsilon\theta_n, \Upsilon\theta_{n+2})}{2} \leq \max\{d(\Upsilon\theta_n, \Upsilon\theta_{n+1}), d(\Upsilon\theta_{n+1}, \Upsilon\theta_{n+2})\}. \tag{3.6}$$

Since

$$\begin{aligned}
&N(\theta_n, \theta_{n+1}) \\
&= \min\{d(\Upsilon\theta_n, S\theta_n), d(\Upsilon\theta_{n+1}, S\theta_{n+1}), d(\Upsilon\theta_n, S\theta_{n+1}), d(\Upsilon\theta_{n+1}, S\theta_n)\} \\
&= \min\{d(\Upsilon\theta_n, \Upsilon\theta_{n+1}), d(\Upsilon\theta_{n+1}, \Upsilon\theta_{n+2}), d(\Upsilon\theta_n, \Upsilon\theta_{n+1}), d(\Upsilon\theta_{n+1}, \Upsilon\theta_{n+1})\} \\
&= 0,
\end{aligned} \tag{3.7}$$

this together with (3.4) shows that

$$\begin{aligned}
0 &\leq \zeta(d(S\theta_n, S\theta_{n+1}), M(\theta_n, \theta_{n+1}) + LN(\theta_n, \theta_{n+1})) \\
&= \zeta(d(S\theta_n, S\theta_{n+1}), \max\{d(\Upsilon\theta_n, \Upsilon\theta_{n+1}), d(\Upsilon\theta_{n+1}, \Upsilon\theta_{n+2})\}) \\
&< \max\{d(\Upsilon\theta_n, \Upsilon\theta_{n+1}), d(\Upsilon\theta_{n+1}, \Upsilon\theta_{n+2})\} - d(\Upsilon\theta_{n+1}, \Upsilon\theta_{n+2}).
\end{aligned} \tag{3.8}$$

The inequality (3.8) shows that

$$M(\theta_n, \theta_{n+1}) = d(\Upsilon\theta_n, \Upsilon\theta_{n+1}), \quad \forall n \in \mathbb{N} \tag{3.9}$$

which implies that

$$d(\Upsilon\theta_{n+1}, \Upsilon\theta_{n+2}) < d(\Upsilon\theta_n, \Upsilon\theta_{n+1}), \quad \forall n \in \mathbb{N}. \tag{3.10}$$

Therefore, the sequence $\{d(\Upsilon\theta_n, \Upsilon\theta_{n+1})\}$ is decreasing, so there is some $\varphi \geq 0$ such that

$$\lim_{n\to\infty} d(\Upsilon\theta_n, \Upsilon\theta_{n+1}) = \varphi.$$

Suppose $\varphi > 0$. Let the sequences $\{t_n\}$ and $\{s_n\}$ as $t_n = d(\Upsilon\theta_{n+1}, \Upsilon\theta_{n+2})$ and $s_n = d(\Upsilon\theta_n, \Upsilon\theta_{n+1})$. Since $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \varphi$ and $t_n < s_n$ for all n, by the axiom (ζ_4) and equation (2.6) we get

$$0 \leq \limsup_{n\to\infty} \zeta(d(\Upsilon\theta_{n+1}, \Upsilon\theta_{n+2}), d(\Upsilon\theta_n, \Upsilon\theta_{n+1})) < 0,$$

which is a contradiction. Hence, $\varphi = 0$, that is, equation (3.3) holds. □

Theorem 3.3. *Let (X, d) be a metric space and S be a Berinde type $(\mathcal{Z}, \Upsilon)$-contraction with respect to $\zeta \in \mathcal{Z}$ and suppose that there exists a Picard-Jungck sequence $\{\theta_n\}$ of (S, Υ). Then the sequence $\{\Upsilon\theta_n\}$ is a Cauchy sequence.*

Proof. We know that $\{\Upsilon\theta_n\}$ is a sequence in (X, d) such that (3.3) holds. We now show that $\{\Upsilon\theta_n\}$ is a Cauchy sequence.

Suppose that $\{\Upsilon\theta_n\}$ is not a Cauchy sequence. Then there exist $\xi > 0$ and two sequences $\{n_k\}$ and $\{m_k\}$ of natural numbers with $m_k > n_k > k > 0$,

$$d(\Upsilon\theta_{m_k}, \Upsilon\theta_{n_k}) \geq \xi \quad \text{and} \quad d(\Upsilon\theta_{m_k-1}, \Upsilon\theta_{n_k}) < \xi.$$

So, we obtain

$$\begin{aligned} \xi &\leq d(\Upsilon\theta_{m_k}, \Upsilon\theta_{n_k}) \\ &\leq d(\Upsilon\theta_{m_k}, \Upsilon\theta_{m_k-1}) + d(\Upsilon\theta_{m_k-1}, \Upsilon\theta_{n_k}) \\ &< d(\Upsilon\theta_{m_k}, \Upsilon\theta_{m_k-1}) + \xi. \end{aligned}$$

Letting $k \to \infty$ in above inequality, we have

$$\lim_{k\to\infty} d(\Upsilon\theta_{m_k}, \Upsilon\theta_{n_k}) = \xi \tag{3.11}$$

Using (3.3) and (3.11), we obtain

$$\lim_{k\to\infty} d(\Upsilon\theta_{m_k+1}, \Upsilon\theta_{n_k+1}) = \xi. \tag{3.12}$$

Hence,

$$\begin{aligned} M(\theta_{m_k}, \theta_{n_k}) = \max\Bigg\{ & d(\Upsilon\theta_{m_k}, \Upsilon\theta_{n_k}), d(\Upsilon\theta_{m_k}, S\theta_{m_k}), d(\Upsilon\theta_{n_k}, S\theta_{n_k}), \\ & \frac{d(\Upsilon\theta_{m_k}, S\theta_{n_k}) + d(\Upsilon\theta_{n_k}, S\theta_{m_k})}{2} \Bigg\}. \end{aligned} \tag{3.13}$$

Taking $k \to \infty$ in equation (3.13) and using equation (3.11) and (3.12), we get

$$\lim_{k\to\infty} M(\theta_{m_k}, \theta_{n_k}) = \xi. \tag{3.14}$$

Additionally, with the aid of equation (2.6), we have

$$\lim_{k\to\infty} N(\theta_{m_k},\theta_{n_k}) = 0. \tag{3.15}$$

Indeed, we take two sequences $\{t_k\}$ and $\{s_k\}$ with we get

$$t_k = d(S\theta_{m_k}, S\theta_{n_k} n) = d(\Upsilon\theta_{m_k+1}, \Upsilon\theta_{n_k+1}) > 0$$

and

$$s_k = M(\theta_{m_k},\theta_{n_k}) + LN(\theta_{m_k},\theta_{n_k}) > 0, \forall\, k \in \mathbb{N}.$$

Also, we have

$$M(\theta_{m_k},\theta_{n_k}) + LN(\theta_{m_k} \geq d(\Upsilon\theta_{m_k}, \Upsilon\theta_{n_k}) > \xi.$$

Thus, we can apply the axiom (ζ_4) to these sequences, that is,

$$0 \leq \limsup_{k\to\infty} \zeta(d(\Upsilon\theta_{m_k+1}, \Upsilon\theta_{n_k+1}), M(\theta_{m_k},\theta_{n_k}) + LN(\theta_{m_k},\theta_{n_k})) < 0.$$

which is a contradiction. That is, $\{\Upsilon\theta_n\}$ is a Cauchy sequence. □

Theorem 3.4. *Let (X,d) be a metric space, S be a Berinde type $(\mathcal{Z},\Upsilon)$-contraction with respect to $\zeta \in \mathcal{Z}$ and suppose that there exists a Picard-Jungck sequence $\{\theta_n\}$ of (S,Υ). Also assume that, at least, one of the following axioms holds.*

(i) *$(S(X),d)$ or $(\Upsilon(X),d)$ is complete;*
(ii) *(X,d) is complete, Υ is continuous and S and Υ are compatible.*

Then S and Υ have a unique coincidence point.

Proof. Suppose that $(\Upsilon(X),d)$ is complete. Then there exists $\omega \in X$ such that $\Upsilon\theta_{n+1} \to \Upsilon\omega$ as $n \to \infty$ which implies

$$\lim_{n\to\infty} d(\Upsilon\theta_{n+1}, \Upsilon\omega) = 0. \tag{3.16}$$

Next, we prove that $S\omega = \Upsilon\omega$. Assume $S\omega \neq \Upsilon\omega$ and so, $d(S\omega,\Upsilon\omega) = \varepsilon > 0$. From equation (3.16), there exists $n_0 \in \mathbb{N}$ such that

$$d(S\theta_n, \Upsilon\eta) < \varepsilon = d(S\eta, \Upsilon\eta)$$

for all $n \geq n_0$. This leads us to

$$S\theta_n \neq S\eta \Rightarrow d(S\theta_n, S\eta) > 0. \tag{3.17}$$

for all $n \geq n_0$. Now, there does not exist some $n_3 \in \mathbb{N}$ such that for all $n \geq n_3$

$$\Upsilon\theta_{n+1} = \Upsilon\eta.$$

Hence, there exists a partial subsequence $\{\Upsilon\theta_{p_k}\}$ of $\{\Upsilon\theta_{n+1}\}$ such that

$$\Upsilon\theta_{p_k} \neq \eta. \tag{3.18}$$

for all $k \in \mathbb{N}$. Let $n_2 \in \mathbb{N}$ be such that $p_{n_2} \geq n_0$. Thus, by using equation (3.17) and (3.18). By the previous facts and axiom (ζ_2), we get

$$\begin{aligned}0 &\leq \zeta(d(S\eta, S\theta_{p_n}), M(\eta, \theta_{n+1}) + LN(\eta, \theta_{n+1}))\\ &< M(\eta, \theta_{n+1}) + LN(\eta, \theta_{n+1}) - d(S\eta, S\theta_{p_n}).\end{aligned}$$

Taking $n \to \infty$, we obtain

$$\begin{aligned}0 &< M(\eta, \theta_{n+1}) + LN(\eta, \theta_{n+1}) - d(S\omega, \Upsilon\omega)\\ &= 0 - d(S\omega, \Upsilon\omega).\end{aligned}$$

This implies that $\eta = \Upsilon\omega = S\omega$ and η is the (unique) point of coincidence of S and Υ.

In the same way, we can prove that $\varrho = S\omega = \Upsilon\omega$ is a (unique) point of coincidence of S and Υ, when $(S(X), d)$ is complete.

Suppose that (X, d) is complete, Υ is continuous and S and Υ are compatible. Since (X, d) is complete, there exists $\omega \in X$ such that $S\theta_n = \Upsilon\theta_{n+1} \to \omega$ when $n \to \infty$. As Υ is continuous, we obtain

$$\lim_{n\to\infty} \Upsilon(S\theta_n) = \Upsilon\omega \Rightarrow d(\Upsilon(S\theta_n), \Upsilon\omega) = 0 \tag{3.19}$$

and

$$\lim_{n\to\infty} \Upsilon(\Upsilon\theta_{n+1}) = \Upsilon\omega \Rightarrow d(\Upsilon(\Upsilon\theta_{n+1}), \Upsilon\omega) = 0. \tag{3.20}$$

We claim that $\lim_{n\to\infty} S(\Upsilon\theta_n) = S\omega$. If not, then there exists a subsequence $\{S(\Upsilon\theta_{p_k})\}$ of $\{S(\Upsilon\theta_n)\}$ such that

$$S(\Upsilon\theta_{p_\gamma}) \neq S\omega \tag{3.21}$$

for all $k \in \mathbb{N}$. There does not exist some $k_1 \in \mathbb{N}$ such that for all $n \geq k_1$, we get $\Upsilon(\Upsilon\theta_{n+1}) = \Upsilon\omega$. Thus, there exists a partial subsequence $\{\Upsilon(\Upsilon\theta_{p_\tau})\}$ of $\{\Upsilon(\Upsilon\theta_{n+1})\}$ such that

$$\Upsilon(\Upsilon\theta_{p_\tau}) \neq \Upsilon\omega. \tag{3.22}$$

for all $\tau \in \mathbb{N}$. Hence, by (3.21) and (3.22), we have $d(S(\Upsilon\theta_{p_\gamma}), S\omega) > 0$ and $d(\Upsilon(\Upsilon\theta_{p_\tau}), \Upsilon\omega) > 0$ for all $\gamma, \tau \in \mathbb{N}$. By using axiom (ζ_2), we obtain

$$\begin{aligned}0 &\leq \zeta(d(S(\Upsilon\theta_{p_\gamma}), S\omega), M(\Upsilon(\Upsilon\theta_{p_\tau}), \Upsilon\omega)) + LN(\Upsilon(\Upsilon\theta_{p_\tau}), \Upsilon\omega)\\ &< M(\Upsilon(\Upsilon\theta_{p_\tau}), \Upsilon\omega)) + LN(\Upsilon(\Upsilon\theta_{p_\tau}), \Upsilon\omega) - d(S(\Upsilon\theta_{p_\gamma}), S\omega)\\ &= d(\Upsilon(\Upsilon\theta_{p_\tau}), \Upsilon\omega) - d(S(\Upsilon\theta_{p_\gamma}), S\omega).\end{aligned}$$

Thus, we have

$$d(S(\Upsilon\theta_{p_\gamma}), S\omega) < d(\Upsilon(\Upsilon\theta_{p_\tau}), \Upsilon\omega) \to 0 \quad \text{as} \quad \gamma \to \infty$$

which is a contradiction. This implies that

$$\lim_{n\to\infty} d(S(\Upsilon\theta_n), S\omega) = 0. \tag{3.23}$$

Moreover, as S and Υ are compatible, we have

$$\lim_{n\to\infty} d(S(\Upsilon\theta_n), \Upsilon(S\theta_n) = 0. \tag{3.24}$$

Using equation (3.19), (3.23) and (3.24), we get

$$\begin{aligned} d(S\omega, \Upsilon\omega) &\leq d(S\omega, S(\Upsilon\theta_n)) + d(S(\Upsilon\theta_n), \Upsilon(S\theta_n)) \\ &\quad + d(\Upsilon(S\theta_n), \Upsilon\omega) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore,

$$d(S\omega, \Upsilon\omega) = 0.$$

This implies that $\varrho := \Upsilon\omega = S\omega$ and ϱ is the (unique) point of coincidence of S and Υ. □

Theorem 3.5. *Let (X, d) be a complete metric space, S be a Berinde type $(\mathcal{Z}, \Upsilon)$-contraction with respect to $\zeta \in \mathcal{Z}$ and suppose that there exists a Picard-Jungck sequence $\{\theta_n\}$ of (S, Υ). Also assume that, $(S(X), d)$ or $(\Upsilon(X), d)$ is complete and S and Υ are weakly compatible. Then S and Υ have a unique common fixed point in X.*

Proof. It follows Theorem 3.4, S and Υ have a unique point of coincidence. Moreover, as S and T are weakly compatible, then according to Theorem 2.8, they have a unique common fixed point in X. □

Example 3.6. Let $X = \{0, 4, 5\}$ and $d : X \times X \to [0, \infty)$ be defined by $d(\theta, \vartheta) = |\theta - \vartheta|$. Define $S, \Upsilon : X \to X$ as

$$S = \begin{pmatrix} 0 & 4 & 5 \\ 4 & 4 & 4 \end{pmatrix} \quad \text{and} \quad \Upsilon = \begin{pmatrix} 0 & 4 & 5 \\ 5 & 4 & 0 \end{pmatrix}.$$

Suppose $\zeta(t, s) = \dfrac{s}{s+1} - t$.

Case (i). For $\theta = 0, \vartheta = 4$. From (2.6), we have

$$\zeta(d(S0, S4), M(0, 4) + LN(0, 4)),$$

where

$$\begin{aligned} M(0,4) &= \max\left\{d(\Upsilon 0, \Upsilon 4), d(\Upsilon 0, S0), d(\Upsilon 4, S4), \frac{d(\Upsilon 0, S4) + d(\Upsilon 4, S0)}{2}\right\} \\ &= \max\left\{d(5,4), d(5,4), d(4,4), \frac{d(5,4) + d(4,4)}{2}\right\} \\ &= \max\left\{1, 1, 0, \frac{1}{2}\right\} \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} N(0,4) &= \min\{d(\Upsilon 0, S0), d(\Upsilon 4, S4), d(\Upsilon 0, S4), d(\Upsilon 4, S0)\} \\ &= \min\{d(5,4), d(4,4), d(5,4), d(4,4)\} \\ &= \min\{1,0,1,0\} \\ &= 0. \end{aligned} \tag{3.25}$$

Thus,

$$\zeta(d(S0,S4), M(0,4)+LN(0,4)) = \zeta(0,1) = \frac{1}{1+1} - 0 = \frac{1}{2} \geq 0.$$

Case (ii). For $\theta = 0, \vartheta = 5$. From (2.6), we have

$$\zeta(d(S0,S5), M(0,5)+LN(0,5)),$$

where

$$\begin{aligned} M(0,5) &= \max\left\{d(\Upsilon 0, \Upsilon 5), d(\Upsilon 0, S0), d(\Upsilon 5, S5), \frac{d(\Upsilon 0, S5)+d(\Upsilon 5, S0)}{2}\right\} \\ &= \max\left\{d(5,0), d(5,4), d(0,4), \frac{d(5,4)+d(0,4)}{2}\right\} \\ &= \max\left\{5,1,4,\frac{5}{2}\right\} \\ &= 5 \end{aligned}$$

and

$$\begin{aligned} N(0,5) &= \min\{d(\Upsilon 0, S0), d(\Upsilon 5, S5), d(\Upsilon 0, S5), d(\Upsilon 5, S0)\} \\ &= \min\{d(5,4), d(0,4), d(5,4), d(0,4)\} \\ &= \min\{1,4,1,4\} \\ &= 1. \end{aligned} \tag{3.26}$$

Thus,

$$\zeta(d(S0,S5), M(0,5)+LN(0,5)) = \zeta(0,6) = \frac{1}{6+1} - 0 = \frac{1}{7} \geq 0.$$

Case (iii). For $\theta = 4, \vartheta = 5$. From (2.6), we have

$$\zeta(d(S4,S5), M(4,5)+LN(4,5)),$$

where

$$\begin{aligned} M(4,5) &= \max\left\{ d(\Upsilon 4, \Upsilon 5), d(\Upsilon 4, S4), d(\Upsilon 5, S5), \frac{d(\Upsilon 4, S5) + d(\Upsilon 5, S4)}{2} \right\} \\ &= \max\left\{ d(4,0), d(4,4), d(0,4), \frac{d(4,4) + d(0,4)}{2} \right\} \\ &= \max\{4,0,4,2\} \\ &= 4 \end{aligned}$$

and

$$\begin{aligned} N(4,5) &= \min\{d(\Upsilon 4, S4), d(\Upsilon 5, S5), d(\Upsilon 4, S5), d(\Upsilon 5, S4)\} \\ &= \min\{d(4,4), d(0,4), d(4,4), d(0,4)\} \\ &= \min\{0,4,0,4\} \\ &= 0. \end{aligned} \tag{3.27}$$

Thus,

$$\zeta(d(S4, S5), M(4,5) + LN(4,5)) = \zeta(0,4) = \frac{1}{4+1} - 0 = \frac{1}{5} \geq 0.$$

Therefore, all the assumptions of Theorem 3.5 are satisfied and by the conclusion of it, S and Υ have a unique point of coincidence $\theta = 4$ and also it is their unique common fixed point.

Acknowledgments: The first author was supported by Rambhai Barni Rajabhat University and the second author was supported by the Basic Science Research Program through the National Research Foundation(NRF) Grant funded by Ministry of Education of Korea (2018R1D1A1B07045427).

References

[1] S. Banach, *Sur les opérations dans lesensembles abstraits et leurs applications aux équationsintégrales*, Fund Math., **3** (1922), 133–181.

[2] T. Zamfirescu, *Fix point theorems in metric spaces*, Arch. Math., (Basel) **23** (1972), 292–298.

[3] V. Berinde, *Approximating fixed points of weak contractions using the Picard iteration*, Nonlinear. Anal. Forum., **9**(1) (2004), 43–53.

[4] V. Berinde, *General constructive fixed point theorems for Ćirić-type almost contractions in metric spaces*, Carpathian J. Math., **24**(2) (2008), 10–19.

[5] G.V.R. Babu, M.L. Sandhya and M.V.R. Kameswari, *A note on a fixed point theorem of Berinde on weak contractions*, Carpathian J. Math., **24** (2008), 8–12.

[6] F. Khojasteh, S. Shukla and S. Radenović, *A new approach to the study of fixed point theory for simulation functions*, Filomat, **29**(6) (2015), 1189–1194.

[7] S. Radenović and S. Chandok, *Simulation type functions and coincidence points*, Filomat, **32**(1) (2018), 141–147.

[8] S. Radenović, F. Vetro and J. Vujaković, *An alternative and easy approach to fixed point results via simulation functions*, Demonstr. Math., **50**(1) (2017), 223–230.

[9] A.F. Roldán-López-de-Hierro, E. Karapnar, C. Roldán-López-de-Hierro and J. Mart´nez Moreno, *Coincidence point theorems on metric spaces via simulation functions*, J. Comput. Appl. Math., **275** (2015), 345–355.

[10] A.F. Roldán-López-de-Hierro and B. Samet, *φ-admissibility results via extended simulation functions*, J. Fixed Point Theory Appl., **19**(3) (2017), 1997–2015.

[11] A. Chanda, B. Damjanović and L.K. Dey, *Fixed point results on θ-metric spaces via simulation functions*, Filomat, **31**(11) (2017), 3365–3375.

[12] S. Komal, P. Kumam and D. Gopal, *Best proximity point for $\mathcal{Z}$-contraction and Suzuki type $\mathcal{Z}$-contraction mappings with an application to fractional calculus*, Appl. Gen. Topol., **17**(2) (2016), 185–198.

[13] P. Kumam, D. Gopal and L. Budhiya, *A new fixed point theorem under Suzuki type $\mathcal{Z}$-contraction mappings*, J. Math. Anal., **8**(1) (2017), 113–119.

[14] C. Mongkolkeha, Y.J. Cho and P. Kumam, *Fixed point theorems for simulation functions in b-metric spaces via the wt-distance*, Appl. Gen. Topol., **18**(1) (2017), 91–105.

[15] A. Nastasi and P. Vetro, *Fixed point results on metric and partial metric spaces via simulation functions*, J. Nonlinear Sci. Appl., **8**(6) (2015), 1059–1069.

[16] A. Padcharoen, P. Kumam, P. Saipar and P. Chaipunya, *Generalized Suzuki type $\mathcal{Z}$-contraction in complete metric spaces*, Kragujevac J. Math., **42**(3) (2018), 419–430.

[17] S. Aleksić, S. Chandok and S. Radenović, *Simulation functions and Boyd-Wong type results*, Tbilisi Math. J., **12**(1) (2019), 105–115.

[18] M. Abbas and G. Jungck, *Common fixed point results for noncommuting mappings without continuity in cone metric spaces*, J. Math. Anal. Appl., **341**(1) (2008), 416–420.