Nonlinear Functional Analysis and Applications
Vol. 25, No. 3 (2020), pp. 525-530
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2020.25.03.08
http://nfaa.kyungnam.ac.kr/journal-nfaa

FIXED POINTS OF HYBRID GENERALIZED WEAKLY CONTRACTIVE MAPPINGS IN METRIC SPACES

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Abstract. Let (E, d) be a complete metric space and $S, T : E \to E$ be two self-mappings such that

$$\varphi(F(d(Sx, Ty))) \leq \psi(F(M(x, y))),$$

for all $x, y \in E$, where

(i) $F : [0, +\infty) \to [0, +\infty)$ is a continuous function with $F(0) = 0$ and $F(t) > 0$ for all $t > 0$;
(ii) $\psi, \varphi : [0, +\infty) \to [0, +\infty)$ are two functions with $\psi(0) = \varphi(0) = 0$ and $\varphi(t) > \psi(t)$ and $\lim_{\tau \to t} \inf \varphi(\tau) > \lim_{\tau \to t} \sup \psi(\tau)$ for all $t > 0$.

Then S and T have a unique common fixed point.

1. Introduction

Throughout this paper, we assume that E is a complete metric space with the metric by d. We use $\mathcal{F}$ to denote the set of functions $F : [0, +\infty) \to [0, +\infty)$ satisfying the following hypotheses:

(h1) $F(0) = 0$ and $F(t) > 0$ for each $t > 0$;
(h2) F is continuous.

We denote by Ψ and Φ the sets of functions $\psi, \varphi : [0, +\infty) \to [0, +\infty)$ satisfying the following conditions, respectively

(c1) $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$;
(c2) $\varphi(t), \psi(t) > 0$ for all $t > 0$;
(c3) $\liminf_{\tau \to t} \varphi(\tau)$ and $\limsup_{\tau \to t} \psi(\tau)$ exist for all $t > 0$.

[0]Received January 16, 2020. Revised April 17, 2020. Accepted April 19, 2020.
[0]2010 Mathematics Subject Classification: 47H10, 47H17.
[0]Keywords: Complete metric space, generalized weakly contractive mapping, fixed point.

In 2007, Zhang [4] gave the common fixed point theorems for generalized contractive type mappings.

Theorem 1.1. ([4]) *Let (E, d) be a complete metric space and $S, T : E \to E$ be two self-mappings satisfying the inequality:*

$$F(d(Sx, Ty)) \leq \psi(F(M(x, y))), \text{ for all } x, y \in E, \tag{1.1}$$

where

(i) $M(x, y) = \max\{d(x, y), d(x, Sx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Sx)]\}$;
(ii) $F : [0, +\infty) \to [0, +\infty)$ *is a continuous nondecreasing function with* $F(0) = 0$ *and* $F(t) > 0$ *for each* $t > 0$;
(iii) $\psi : [0, +\infty) \to [0, +\infty)$ *is a nondecreasing and right upper semi-continuous function with* $\psi(0) = 0$ *and* $\psi(t) > 0$, $\lim_{n\to\infty} \psi^n(t) = 0$ *for each* $t > 0$.

Then there exists a unique common fixed point of S and T.

The main purpose of this paper is to improve and extend Zhang′s convergence theorems to more general form by virtue of new analysis techniques.

2. Main results

Theorem 2.1. *Let (E, d) be a complete metric space and $S, T : E \to E$ be two self-mappings satisfying*

$$\varphi(F(d(Sx, Ty))) \leq \psi(F(M(x, y))), \text{ for all } x, y \in E, \tag{2.1}$$

where

(i) $M(x, y) = \max\{d(x, y), d(x, Sx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Sx)]\}$;
(ii) $F \in \digamma, \psi \in \Psi, \varphi \in \Phi$ *with* $\varphi(t) > \psi(t)$ *for* $t > 0$;
(iii) $\liminf_{\tau\to t} \varphi(\tau) > \limsup_{\tau\to t} \psi(\tau)$ *for* $t > 0$.

Then there exists a unique common fixed point of S and T.

Proof. Let x_0 be an arbitrary point of E and define $\{x_n\}_{n=0}^{\infty}$ as follows

$$x_{2n+2} = Tx_{2n+1}, \;\; x_{2n+1} = Sx_{2n}, \;\; \forall n \geq 0.$$

If there exists N such that $x_{2N+1} = Sx_{2N} = x_{2N}$, then $x_{2N+2} = Tx_{2N+1} = x_{2N+1}$. We are done the proof. Without loss of generality, we assume that $x_{n+1} \neq x_n$ for all $n \geq 0$. Then

$$\begin{aligned} \varphi(F(d(x_{2n+2}, x_{2n+1}))) &= \varphi(F(d(Tx_{2n+1}, Sx_{2n}))) \\ &\leq \psi(F(M(x_{2n+1}, x_{2n}))), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} M(x_{2n+1}, x_{2n}) &= \max\{d(x_{2n+1}, x_{2n}), d(x_{2n+2}, x_{2n+1}), \tfrac{1}{2}d(x_{2n+2}, x_{2n})\} \\ &= \max\{d(x_{2n+1}, x_{2n}), d(x_{2n+2}, x_{2n+1})\}. \end{aligned} \tag{2.3}$$

Suppose there exists some n such that $d(x_{2n+1}, x_{2n}) < d(x_{2n+2}, x_{2n+1})$. Then it follows that

$$0 < \varphi(F(d(x_{2n+2}, x_{2n}))) \leq \psi(F(d(x_{2n+2}, x_{2n})), \tag{2.4}$$

which is a contradiction and so $d(x_{2n+2}, x_{2n+1}) \leq d(x_{2n+1}, x_{2n})$ for any $n \geq 0$.

Similarly, we also have

$$d(x_{2n+3}, x_{2n+2}) \leq d(x_{2n+2}, x_{2n+1}) \tag{2.5}$$

for any $n \geq 0$. Hence $\{d(x_{n+1}, x_n)\}$ is a monotone nonincreasing sequence, and so there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = r. \tag{2.6}$$

We claim that $r = 0$. Otherwise, $r > 0$. By (2.2), we have

$$\varphi(F(d(x_{n+1}, x_n))) \leq \psi(F(d(x_n, x_{n-1}))), \tag{2.7}$$

which implies that

$$\inf_{i\geq n} \varphi(F(d(x_{i+1}, x_i))) \leq \sup_{i\geq n} \psi(F(d(x_i, x_{i-1}))). \tag{2.8}$$

Then taking limit as $n \to \infty$ on (2.8), we get

$$0 < \liminf_{t\to r} \varphi(F(t)) \leq \limsup_{t\to r} \psi(F(t)). \tag{2.9}$$

This is a contradiction and so $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$.

Next we show that $\{x_n\}$ is a Cauchy sequence. Let $c_k = \sup\{d(x_i, x_j) : i, j \geq k\}$. Then $\{c_k\}$ is monotone decreasing and bounded. Denote $\lim_{k\to\infty} c_k = c$, then $c = 0$. Indeed, let $\{\epsilon_k\}$ be a sequence of positive numbers with $\epsilon_k \to 0$ as $k \to \infty$. Since $\lim_{k\to\infty} d(x_{k+1}, x_k) = 0$, by the definition of $\{c_k\}$, there exist two infinite subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $m(k)$ is odd and $n(k)$ is even for $k \geq 1$ such that

$$c_k - \epsilon_k \leq d(x_{m(k)}, x_{n(k)}) \leq c_k$$

for $k < m(k) < n(k)$. Hence

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = c. \tag{2.10}$$

By triangle inequality, we have

$$\begin{aligned} &d(x_{m(k)}, x_{n(k)}) - d(x_{m(k)}, x_{m(k)+1}) - d(x_{n(k)+1}, x_{n(k)}) \\ &\leq d(x_{m(k)+1}, x_{n(k)+1}) \\ &\leq d(x_{m(k)+1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)+1}). \end{aligned} \tag{2.11}$$

It implies that

$$\lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) = c. \tag{2.12}$$

Similarly, we get

$$\lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)}) = \lim_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) = c. \tag{2.13}$$

In view of (2.2), we have

$$\begin{aligned} \varphi(F(d(x_{m(k)+1}, x_{n(k)+1}))) &= \varphi(F(d(Tx_{m(k)}, Sx_{n(k)}))) \\ &\leq \psi(F(M(x_{m(k)}, x_{n(k)}))) \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} &d(x_{m(k)}, x_{n(k)}) \\ &\leq M(x_{m(k)}, x_{n(k)}) \\ &= \max\{d(x_{m(k)}, x_{n(k)}), d(x_{m(k)}, x_{m(k)+1}), d(x_{n(k)}, x_{n(k)+1}), \\ &\qquad \tfrac{1}{2}[d(x_{m(k)}, x_{n(k)+1}) + d(x_{n(k)}, x_{m(k)+1})]\}, \end{aligned} \tag{2.15}$$

which implies that $M(x_{m(k)}, x_{n(k)}) \to c$ as $k \to \infty$. So (2.15) follows that

$$\inf_{i\geq k} \varphi(F(d(x_{m(i)+1}, x_{n(i)+1}))) \leq \sup_{i\geq k} \psi(F(M(x_{m(i)}, x_{n(i)}))). \tag{2.16}$$

Taking limit as $k \to \infty$ on both side of the above inequality

$$0 < \liminf_{t\to c} \varphi(F(t)) \leq \limsup_{t\to c} \psi(F(t)), \tag{2.17}$$

which is a contradiction. This shows that $\{x_n\}$ is a Cauchy sequence and hence it is convergent by the completeness of X. Denote $\lim_{n\to\infty} x_n = q$.

Finally we show that q is a unique common fixed point of S and T. If $q \neq Tq$, then $d(q, Tq) > 0$. Consequently,

$$\begin{aligned} d(q, Tq) &\leq M(q, x_{2n}) \\ &= \max\{d(q, x_{2n}), d(q, Tq), d(x_{2n+1}, x_{2n}), \\ &\qquad \tfrac{1}{2}[d(q, x_{2n+1}) + d(x_{2n}, Tq)]\} \\ &\leq d(q, x_{2n}) + d(q, Tq) + d(x_{2n+1}, x_{2n}), \end{aligned} \tag{2.18}$$

so $M(q, x_{2n}) \to d(q, Tq)$ as $n \to \infty$. By taking $x = q, y = x_{2n}$ in (2.2), we get

$$\varphi(F(d(x_{2n+1}, Tq))) = \varphi(F(d(Sx_{2n}, Tq))) \leq \psi(F(M(q, x_{2n}))), \tag{2.19}$$

that is,

$$\inf_{i\geq n} \varphi(F(d(x_{2i+1}, Tq))) \leq \sup_{j\geq n} \psi(F(M(q, x_{2j}))). \tag{2.20}$$

Taking limit as $n \to \infty$ in (2.20), we have

$$0 < \liminf_{t\to d(Tq,q)} \varphi(F(t)) \leq \limsup_{t\to d(Tq,q)} \psi(F(t)), \tag{2.21}$$

which is a contradiction and so $q = Tq$. Suppose that $Sq \neq q$. Then we have

$$\begin{aligned} 0 < \varphi(F(d(Sq,q))) &= \varphi(F(d(Sq,Tq)))) \\ &\leq \psi(F(M(q,q))) \\ &= \psi(F(\max\{d(q,q), d(Sq,q), \frac{1}{2}[d(q,Tq) + d(Sq,q)]\})) \\ &= \psi(F(d(Sq,q))), \end{aligned}$$

which is a contradiction. Thus $q = Sq = Tq$.

For uniqueness, we assume that there exists another point $p \in E$ such that $Tp = Sp = p \neq q = Tq = Sq$. Observe that

$$\begin{aligned} 0 < \varphi(F(d(q,p))) &= \varphi(F(d(Sq,Tp))) \\ &\leq \psi(F(M(q,p))) \\ &= \psi(F(\max\{d(q,p), \frac{1}{2}[d(q,Tp) + d(Sq,p)]\})) \\ &= \psi(F(d(q,p))), \end{aligned}$$

we obtain a contradiction. Hence $p = q$. The proof is completed. □

Theorem 2.2. *Let (E, d) be a complete metric space and $S, T : E \to E$ be two self-mappings satisfying the inequality:*

$$\varphi(F(d(Sx,Ty))) \leq \psi(F(M(x,y))),\ \text{for all } x, y \in E, \tag{2.22}$$

where

(i) $M(x,y) = \max\{d(x,y), d(x,Sx), d(y,Ty), \frac{1}{2}[d(x,Ty) + d(y,Sx)]\}$;
(ii) $F \in \digamma, \psi \in \Psi, \varphi \in \Phi$ *with*
$\liminf_{\tau \to t} \varphi(\tau) \geq \varphi(t) > \psi(t) \geq \limsup_{\tau \to t} \psi(\tau)$ *for all* $t > 0$.

Then there exists a unique common fixed point of S and T.

If $\varphi(t) = t$ and ψ is an upper semi-continuous function, then we obtain from Theorem 2.2 the following result.

Theorem 2.3. *Let (E, d) be a complete metric space and $S, T : E \to E$ be two self-mappings satisfying the inequality:*

$$F(d(Sx,Ty)) \leq \psi(F(M(x,y))),\ \text{for all } x, y \in E, \tag{2.23}$$

where

(i) $M(x,y) = \max\{d(x,y), d(x,Sx), d(y,Ty), \frac{1}{2}[d(x,Ty) + d(y,Sx)]\}$;
(ii) $F \in \digamma, \psi \in \Psi$ *with* $t > \psi(t) \geq \limsup_{\tau \to t} \psi(\tau)$ *for all* $t > 0$.

Then there exists a unique common fixed point of S and T.

If $\psi(t) = t$ and φ is a lower semi-continuous function in Theorem 2.2, then we get the following conclusion.

Theorem 2.4. *Let (E, d) be a complete metric space and $S, T : E \to E$ be two self-mappings satisfying the inequality:*

$$\varphi(F(d(Sx, Ty))) \leq F(M(x, y)), \ \text{for all } x, y \in E, \tag{2.24}$$

where

(i) $M(x, y) = \max\{d(x, y), d(x, Sx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Sx)]\}$;

(ii) $F \in \digamma, \varphi \in \Phi$ *with* $\liminf_{\tau \to t} \varphi(\tau) \geq \varphi(t) > t$ *for all* $t > 0$.

Then there exists a unique common fixed point of S and T.

Remark 2.5. Our Theorem 2.3 extends Theorem 3.1 of Zhang [4] in the following aspect:

(i) The assumption that functions F and ψ are nondecreasing is not necessary.

(ii) The condition that $\lim_{n\to\infty} \psi^n(t) = 0$ for $t > 0$ is superfluous.

Remark 2.6. In Theorem 2.1, if $F(t) = t$ and $S = T$, then the corresponding result due to the author of this paper(see [6]). Therefore our Theorem 2.1 generalizes the result of [6]. On the other hand, our results contain the corresponding results in [1] -[5].

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