Nonlinear Functional Analysis and Applications
Vol. 25, No. 3 (2020), pp. 531-543
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2020.25.03.09
http://nfaa.kyungnam.ac.kr/journal-nfaa

DEGREE THEORY FOR SET-VALUED OPERATORS OF MONOTONE TYPE IN REFLEXIVE BANACH SPACES

In-Sook Kim

Department of Mathematics, Sungkyunkwan University
Suwon 16419, Republic of Korea
e-mail: iskim@skku.edu

Abstract. We are concerned with degree theory for some classes of upper demicontinuous set-valued operators of monotone type with weakly compact convex values in reflexive separable Banach spaces. As extensions of the celebrated Leray-Schauder degree, the basic idea is to use an elliptic super-regularization method by means of suitable compact embeddings due to Browder and Ton.

1. Introduction

Degree theory may be one of the most effective tools in the study of nonlinear equations, with application to nonlinear problems in partial differential equations. Leray and Schauder [10] introduced a degree theory for compact perturbations of the identity in Banach spaces, based on the classical Brouwer degree [3] for continuous functions in the Euclidean space.

Browder [4] constructed a topological degree for demicontinuous operators of class (S_+) in reflexive Banach spaces in the technique of Galerkin approximation; see also [13, 14]. Berkovits and Tienari [2] developed a degree theory for set-valued operators of class (S_+) in reflexive separable Banach spaces with a method of elliptic super-regularization, with application to elliptic problems with discontinuous nonlinearity. In [2], a compact embedding theorem of Browder and Ton [5] is used to apply a set-valued form of the Leray-Schauder

[0]Received January 23, 2020. Revised July 1, 2020. Accepted July 3, 2020.
[0]2010 Mathematics Subject Classification: 47H04, 47H05, 47H11.
[0]Keywords: Set-valued operators, operators of monotone type, degree theory.

degree due to Ma [11], still speaking of the Leray-Schauder degree; see also [6, 12]. This approach is more elegant than the Galerkin method.

Moreover, Berkovits [1] considered an extension of the Leray-Schauder degree by replacing the compact perturbation by a composition of operators of monotone type, called an abstract Hammerstein operator. Actually, a given boundary value problem can be transformed into an abstract Hammerstein equation which will be solved.

In this direction, we focus on degree theory for set-valued operators of monotone type in reflexive Banach spaces in two kind of ways, as extensions of the Leray-Schauder degree.

In the present paper, the first goal is to investigate the degree theory of Berkovits and Tienari for upper demicontinuous set-valued operators of class (S_+) in a more precise manner. The study is based on the Leray-Schauder degree by means of compact embeddings. To do this, it is supposed that the operators considered have at least weakly compact values. It is emphasized that the closed-valued condition in [2] is not sufficient for the use of the Leray-Schauder degree.

In a similar situation, the second goal is to develop a topological degree theory for bounded upper demicontinuous set-valued operators of class $(S_+)_T$ with weakly compact convex values, where T is a bounded continuous operator of class (S_+). As a set-valued version of [1], the method of approach is to use the degree theory for the class (S_+); see [7]. It is remarkable that weak compactness is only required in place of compactness. This is due to the compact embedding theorem.

Applying the (S_+)-degree theory, some elliptic problems with discontinuous nonlinearity were dealt with in [2, 8]. Based on the $(S_+)_T$-degree, the Dirichlet boundary value problem related to the p-Laplacian with discontinuous nonlinearity was discussed in [7], via an abstract Hammerstein equation; see [1, 9] for the continuous case.

In this note, we first introduce the Leray-Schauder degree for compact set-valued perturbations of the identity in normed spaces in Section 2. This is applied to construct a degree theory for upper demicontinuous operators of class (S_+) with weakly compact convex values in reflexive Banach spaces in Section 3. Based on the (S_+)-degree, we demonstrate a degree theory for upper demicontinuous operators of class $(S_+)_T$ in Section 4.

2. The Leray-Schauder degree

As a set-valued version of the celebrated Leray-Schauder degree, we introduce a degree theory of Ma [11] for compact set-valued perturbations of the identity in normed spaces; see also [6, 12].

Definition 2.1. Let X and Y be two normed spaces. A set-valued operator $F : \Omega \subset X \to 2^Y$ is said to be

(1) *upper semicontinuous (u.s.c.)* if the set $F^{-1}(A) = \{u \in \Omega \mid Fu \cap A \neq \emptyset\}$ is closed for each closed set A in Y;
(2) *upper demicontinuous (u.d.c.)* if $F^{-1}(A)$ is closed for each weakly closed set A in Y;
(3) *bounded* if it takes bounded sets into bounded sets;
(4) *compact* if it is upper semicontinuous and the image of any bounded set is relatively compact;
(5) *of Leray-Schauder type* if it is of the form $I + C$, where I denotes the identity operator and C is compact.

Given a nonempty set Ω in a normed space X, let $\overline{\Omega}$ and $\partial\Omega$ denote the closure and the boundary of Ω in X, respectively. Let $B_r(u)$ denote the open ball in X of positive radius r centered at u.

For our aim, we need the topological degree for set-valued operators of Leray-Schauder type in infinite dimensional normed spaces given in [11], still speaking of the Leray-Schauder degree. The basic idea is to use the Brouwer degree [3] by reduction to continuous single-valued operators in finite dimensional normed spaces.

Theorem 2.2. *Let G be any bounded open set in a normed space X and suppose that $F : \overline{G} \to 2^X$ is a compact set-valued operator with nonempty compact convex values. If $h \notin (I + F)(\partial G)$, then the (LS)-degree of $I + F$ on G over h is defined as an integer, denoted by $d_{LS}(I + F, G, h)$, and it has the following properties:*

(a) (Existence) *If $d_{LS}(I + F, G, h) \neq 0$, then $h \in (I + F)(G)$.*
(b) (Additivity) *If G_1 and G_2 are two disjoint open subsets of G such that $h \notin (I + F)(\overline{G}\backslash(G_1 \cup G_2))$, then we have*

$$d_{LS}(I + F, G, h) = d_{LS}(I + F, G_1, h) + d_{LS}(I + F, G_2, h).$$

(c) (Homotopy Invariance) *Suppose that $H : [0, 1] \times \overline{G} \to 2^X$ is a compact set-valued homotopy with nonempty compact convex values. If $h : [0, 1] \to X$ is a continuous map such that $h(t) \notin (I + H(t, \cdot))(\partial G)$ for all $t \in [0, 1]$, then the value of $d_{LS}(I + H(t, \cdot), G, h(t))$ is constant for all $t \in [0, 1]$.*
(d) (Normalization) *For any $h \in G$, we have $d_{LS}(I, G, h) = 1$.*

The Leray-Schauder degree stated in Theorem 2.2 will be a main ingredient for the introduction to degree function for set-valued operators of monotone type in the next section.

3. The (S_+)-degree

In this section, we introduce a degree theory for the class of upper demicontinuous operators of class (S_+) in reflexive separable Banach spaces, due to Berkovits and Tienari [2]. The study is mainly based on the Leray-Schauder degree with the aid of compact embeddings.

Let X be a Banach space with dual space X^*. The symbol $\langle \cdot, \cdot \rangle_X$ denotes the dual pairing between X^* and X in this order. The symbol $\to$ ($\rightharpoonup$) stands for strong (weak) convergence.

Definition 3.1. (1) A set-valued operator $F : \Omega \subset X \to 2^{X^*}$ is said to be of class (S_+) if for any sequence (u_n) in Ω and any sequence (v_n) in X^* with $v_n \in Fu_n$ such that

$$u_n \rightharpoonup u \text{ in } X \text{ and } \limsup_{n\to\infty} \langle v_n, u_n - u \rangle \leq 0,$$

we have $u_n \to u$ in X.
(2) A homotopy $H \colon [0,1] \times \Omega \to 2^{X^*}$ is said to be of class (S_+) if for any sequence (t_n, u_n) in $[0,1]\times\Omega$ and any sequence (w_n) in X^* with $w_n \in H(t_n, u_n)$ such that

$$t_n \to t \text{ in } [0,1], \ u_n \rightharpoonup u \text{ in } X, \text{ and } \limsup_{n\to\infty} \langle w_n, u_n - u \rangle \leq 0,$$

we have $u_n \to u$ in X.

For the discussion later, we now consider the duality operator which is of class (S_+). In fact, the existence of the operator lies in the Hahn-Banach theorem; see [4, Proposition 8].

Proposition 3.2. *Let $(X, \|\cdot\|)$ be a reflexive Banach space which is renormed so that both X and X^* are locally uniformly convex. Then there exists a unique bicontinuous operator J of X onto X^*, called the duality operator, such that $\langle Ju, u \rangle = \|u\|^2$ and $\|Ju\| = \|u\|$ for all $u \in X$. Moreover, the duality operator $J : X \to X^*$ is of class (S_+).*

For the construction of a new degree, we need the following compact embedding theorem of Browder and Ton [5, Theorem 1]. This enables us to apply the Leray-Schauder degree or the (S_+)-degree.

Proposition 3.3. *Let Y be a reflexive separable Banach space. Then there exists a separable Hilbert space W and a compact linear injection $\phi : W \to Y$ such that $\phi(W)$ is dense in Y.*

In what follows, let X be a real reflexive separable Banach space, renormed if necessary, such that X and X^* are locally uniformly convex.

In the sense of Proposition 3.3, let $\phi : W \to X$ be a compact linear injection defined on a separable Hilbert space W such that $\phi(W)$ is dense in X. Define another operator $\hat{\phi} : X^* \to W$ by setting

$$(\hat{\phi}(v), w)_W = \langle v, \phi(w) \rangle_X \quad \text{for all } w \in W \text{ and all } v \in X^*, \tag{3.1}$$

where $(\cdot,\cdot)_W$ denotes the inner product of the space W. Obviously, $\hat{\phi}$ is also a compact linear injection.

Suppose that $F : \overline{G} \subset X \to 2^{X^*}$ is a bounded upper demicontinuous operator with nonempty weakly compact convex values, where G is an open set in X. To this F, we associate a family of operators defined by

$$F_\lambda := I + \lambda \phi \hat{\phi} F \quad \text{for any positive number } \lambda.$$

Then each $F_\lambda : \overline{G} \to 2^X$ is an operator of Leray-Schauder type with nonempty compact convex values.

Remark 3.4. In fact, the condition "F has closed-values" in [2] is not sufficient for applying the Leray-Schauder degree given in Theorem 2.2. For this reason, it should be required that F has weakly compact values. This implies, by the strong continuity of $\hat{\phi}$, that $\phi\hat{\phi}F$ has compact values.

Let $k(X^*)$ denote the collection of nonempty weakly compact convex subsets of X^*. For any bounded open set G in X, we consider the following class of operators:

$$\mathcal{F}_{S_+}(\overline{G}) := \{F : \overline{G} \to k(X^*) \mid F \text{ is bounded, u.d.c., and of class } (S_+)\}.$$

We begin with a fundamental result needed for the construction of the (S_+)-degree and its properties.

Lemma 3.5. *Let G be any bounded open set in X and A be any closed subset of $\overline{G}$. Suppose that $H : [0,1] \times \overline{G} \to k(X^*)$ is a bounded upper demicontinuous homotopy of class (S_+). If $h : [0,1] \to X^*$ is a continuous map such that $h(t) \notin H(t, A)$ for all $t \in [0,1]$, then there is a positive number λ_0 such that*

$$h_\lambda(t) \notin H(t,\cdot)_\lambda(A) \quad \textit{for all } t \in [0,1] \textit{ and all } \lambda \in [\lambda_0, \infty),$$

where $H(t,\cdot)_\lambda = I + \lambda\phi\hat{\phi}H(t,\cdot)$ and $h_\lambda(t) = \lambda\phi\hat{\phi}h(t)$.

Proof. Let A be any closed subset of $\overline{G}$ such that $h(t) \notin H(t, A)$ for all $t \in [0,1]$. Assume to the contrary that there are sequences (λ_n) in $(0,\infty)$ with $\lambda_n \to \infty$, (t_n) in $[0,1]$, and (u_n) in A such that

$$u_n + \lambda_n \phi\hat{\phi}(w_n - h(t_n)) = 0 \quad \text{for each } n \in \mathbb{N}, \tag{3.2}$$

where $w_n \in H(t_n, u_n)$. Passing to subsequences if necessary, we may suppose that

$$t_n \to t \text{ in } [0,1], \quad u_n \rightharpoonup u \text{ in } X, \quad \text{and} \quad w_n \rightharpoonup w \text{ in } X^*. \tag{3.3}$$

Then we obtain from (3.2) and (3.3) that

$$\phi\hat{\phi}(w_n) \to \phi\hat{\phi}h(t) = \phi\hat{\phi}(w) \text{ in } X,$$

which implies by the injectivity of $\phi\hat{\phi}$ that $w = h(t)$. Hence it follows from (3.2) and (3.3) that

$$\begin{aligned} \limsup_{n\to\infty} \langle w_n, u_n - u\rangle &= \limsup_{n\to\infty} \langle w_n - h(t_n), u_n\rangle \\ &= \limsup_{n\to\infty} \langle w_n - h(t_n), -\lambda_n\phi\hat{\phi}(w_n - h(t_n))\rangle \\ &= \limsup_{n\to\infty} \left[-\lambda_n\|\hat{\phi}(w_n - h(t_n))\|_W^2\right] \\ &\le 0, \end{aligned}$$

where $\|\cdot\|_W$ denotes the norm of the Hilbert space W in the sense of (3.1). Since H is of class (S_+) and is upper demicontinuous with weakly closed values, we have $u_n \to u \in A$ and $h(t) \in H(t,u)$, which contradicts the hypothesis that $h(t) \notin H(t,A)$. This completes the proof. □

Corollary 3.6. *Suppose that G is a bounded open set in X and $F \in \mathcal{F}_{S_+}(\overline{G})$. If $h \notin F(\partial G)$, then there exists a positive number λ_0 such that $h_\lambda \notin F_\lambda(\partial G)$ for all $\lambda \in [\lambda_0, \infty)$ and the value of $d_{LS}(F_\lambda, G, h_\lambda)$ is constant for all $\lambda \in [\lambda_0, \infty)$.*

Proof. Applying Lemma 3.5 with $H(t,\cdot) = F$ and $h(t) = h$ for all $t \in [0,1]$ and $A = \partial G$, we can choose a positive number λ_0 such that $h_\lambda \notin F_\lambda(\partial G)$ for all $\lambda \in [\lambda_0, \infty)$. Next, let λ_1 and λ_2 be arbitrary elements of $[\lambda_0, \infty)$ such that $\lambda_1 < \lambda_2$. Let $H : [0,1] \times \overline{G} \to 2^X$ be defined by

$$H(t,u) := F_{\lambda(t)}(u) \quad \text{for } (t,u) \in [0,1] \times \overline{G},$$

where $\lambda(t) = (1-t)\lambda_1 + t\lambda_2$ for $t \in [0,1]$. Then H is a homotopy of Leray-Schauder type with nonempty compact convex values such that

$$h_{\lambda(t)} \notin H(t, \partial G) \quad \text{for all } t \in [0,1].$$

Hence it follows from the homotopy invariance of the Leray-Schauder degree in Theorem 2.2 that

$$\begin{aligned} d_{LS}(F_{\lambda_1}, G, h_{\lambda_1}) &= d_{LS}(H(0,\cdot), G, h_{\lambda(0)}) \\ &= d_{LS}(H(1,\cdot), G, h_{\lambda(1)}) \\ &= d_{LS}(F_{\lambda_2}, G, h_{\lambda_2}). \end{aligned}$$

Since λ_1 and λ_2 were arbitrarily chosen in $[\lambda_0, \infty)$, We conclude the value of $d_{LS}(F_\lambda, G, h_\lambda)$ is constant for all $\lambda \in [\lambda_0, \infty)$. This completes the proof. □

In view of Corollary 3.6, we are now in a position to define a topological degree for the class $\mathcal{F}_{S_+}$.

Definition 3.7. Suppose that $F \in \mathcal{F}_{S_+}(\overline{G})$, where G is a bounded open set in X. If $h \notin F(\partial G)$, then we define a degree function as follows:

$$d_{S_+}(F, G, h) := \lim_{\lambda \to \infty} d_{LS}(F_\lambda, G, h_\lambda),$$

where $F_\lambda = I + \lambda \phi \hat{\phi} F$ and $h_\lambda = \lambda \phi \hat{\phi} h$.

In order to justify our degree in a more precise manner, we replaced the closed-valued condition on F in [2] by weakly compact-valued one, as mentioned in Remark 3.4.

Using the Leray-Schauder theory, we can deduce some of the basic properties of the (S_+)-degree.

Theorem 3.8. *Let G be any bounded open subset of X and suppose that $F \in \mathcal{F}_{S_+}(\overline{G})$. Then the following properties are satisfied:*

(a) (Existence) *If $d_{S_+}(F, G, h) \neq 0$, then the inclusion $h \in Fu$ has a solution in G.*

(b) (Additivity) *If G_1 and G_2 are two disjoint open subsets of G such that $h \notin F(\overline{G} \backslash (G_1 \cup G_2))$, then we have*

$$d_{S_+}(F, G, h) = d_{S_+}(F, G_1, h) + d_{S_+}(F, G_2, h).$$

(c) (Homotopy Invariance) *Suppose that $H : [0,1] \times \overline{G} \to k(X^*)$ is a bounded upper demicontinuous homotopy of class (S_+). If $h : [0,1] \to X^*$ is a continuous map such that $h(t) \notin H(t, \partial G)$ for all $t \in [0,1]$, then the value of $d_{S_+}(H(t,\cdot), G, h(t))$ is constant for all $t \in [0,1]$.*

(d) (Normalization) *If $h \in J(G)$, then we have $d_{S_+}(J, G, h) = 1$.*

Proof. (a) If $h \notin Fu$ for all $u \in \overline{G}$, then a special case of constant homotopy of Lemma 3.5 implies that there is a positive number λ_0 such that $h_\lambda \notin F_\lambda(\overline{G})$ for all $\lambda \in [\lambda_0, \infty)$. It follows from part (a) of Theorem 2.2 that $d_{LS}(F_\lambda, G, h_\lambda) = 0$ for all $\lambda \in [\lambda_0, \infty)$. By Definition 3.7, we have $d_{S_+}(F, G, h) = 0$.
(b) Applying Lemma 3.5 with $A = \overline{G} \backslash (G_1 \cup G_2)$, we take a positive number λ_0 such that

$$h_\lambda \notin F_\lambda(\overline{G} \backslash (G_1 \cup G_2)) \quad \text{for all } \lambda \in [\lambda_0, \infty).$$

By the additivity of the Leray-Schauder degree in Theorem 2.2, we have

$$d_{LS}(F_\lambda, G, h_\lambda) = d_{LS}(F_\lambda, G_1, h_\lambda) + d_{LS}(F_\lambda, G_2, h_\lambda) \quad \text{for all } \lambda \in [\lambda_0, \infty),$$

which implies by Definition 3.7 that

$$d_{S_+}(F, G, h) = d_{S_+}(F, G_1, h) + d_{S_+}(F, G_2, h).$$

(c) By Lemma 3.5, we can choose a positive number λ_0 such that

$$h_\lambda(t) \notin H(t,\cdot)_\lambda(\partial G) \quad \text{for all } t \in [0,1] \text{ and all } \lambda \in [\lambda_0, \infty).$$

Let $\lambda \in (\lambda_0, \infty)$ be arbitrary but fixed. Consider $\tilde{H} : [0,1] \times \overline{G} \to 2^X$ given by

$$\tilde{H}(t,u) := H(t,\cdot)_\lambda(u) \quad \text{for } (t,u) \in [0,1] \times \overline{G}.$$

Then $\tilde{H}$ is a homotopy of Leray-Schauder type with nonempty compact convex values such that

$$h_\lambda(t) \notin \tilde{H}(t,u) \quad \text{for all } (t,u) \in [0,1] \times \partial G.$$

Hence it follows from the homotopy invariance of the degree in Theorem 2.2 that the value of $d_{LS}(\tilde{H}(t,\cdot), G, h_\lambda(t))$ is constant for all $t \in [0,1]$. For any $t_1, t_2 \in [0,1]$, we have by Definition 3.7

$$\begin{aligned} d_{S_+}(H(t_1,\cdot), G, h(t_1)) &= \lim_{\lambda\to\infty} d_{LS}(H(t_1,\cdot)_\lambda, G, h_\lambda(t_1)) \\ &= \lim_{\lambda\to\infty} d_{LS}(H(t_2,\cdot)_\lambda, G, h_\lambda(t_2)) \\ &= d_{S_+}(H(t_2,\cdot), G, h(t_2)). \end{aligned}$$

(d) Let h be any element of $J(G)$. Then there is an element $u_0 \in G$ with $Ju_0 = h$. We may choose a positive number R with $\|u_0\| < R$ such that

$$d_{S_+}(J, G, h) = d_{S_+}(J, B_R(0), h).$$

Since the duality operator J is positively homogeneous, it is clear that $Ju \neq th$ for all $(t,u) \in [0,1] \times \partial B_R(0)$. Taking $h(t) = th$ for $t \in [0,1]$, we obtain from part (c) that

$$d_{S_+}(J, B_R(0), h) = d_{S_+}(J, B_R(0), 0).$$

Moreover, we have by Definition 3.7

$$d_{S_+}(J, B_R(0), 0) = \lim_{\lambda\to\infty} d_{LS}(J_\lambda, B_R(0), 0),$$

where $J_\lambda = I + \lambda\phi\hat{\phi}J$. Note by Proposition 3.2 and (3.1) that

$$\langle J(tu), u + \lambda\phi\hat{\phi}J(tu)\rangle = t\|u\|^2 + \lambda\|\hat{\phi}J(tu)\|_W^2$$

for $(t,u) \in [0,\infty) \times X$ and $\lambda \in (0,\infty)$. For any positive number λ, we have

$$Iu + t\lambda\phi\hat{\phi}Ju \neq 0 \quad \text{for all } (t,u) \in [0,1] \times \partial B_R(0),$$

which implies by the homotopy invariance and normalization of the degree in Theorem 2.2 that

$$d_{LS}(J_\lambda, B_R(0), 0) = d_{LS}(I, B_R(0), 0) = 1.$$

Therefore, $d_{S_+}(J, G, h) = 1$. This completes the proof. □

Applying the degree theory for the class $\mathcal{F}_{S_+}$, we can consider elliptic problems with discontinuous nonlinearity; see [2, 8].

4. The $(S_+)_T$-degree

In this section, we demonstrate a degree theory for another class of upper demicontinuous operators of class $(S_+)_T$ with elliptic super-regularization method, as in the previous section.

Let X be a reflexive Banach space with dual space X^*. Identifying the bidual space X^{**} with X, we sometimes write $\langle y, x\rangle$ for $\langle x, y\rangle_{X^*}$ for $x \in X$ and $y \in X^*$.

Definition 4.1. Let $T : \Omega_1 \subset X \to X^*$ be a bounded operator such that $\Omega \subset \Omega_1$. A set-valued operator $F : \Omega \subset X \to 2^X$ is said to be:

(1) of class $(S_+)_T$ if for any sequence (u_n) in Ω and any sequence (v_n) in X with $v_n \in Fu_n$ such that $u_n \rightharpoonup u$ in X, $Tu_n \rightharpoonup y$ in X^*, and

$$\limsup_{n\to\infty} \langle v_n, Tu_n - y\rangle \leq 0,$$

we have $u_n \to u$ in X;

(2) T-*quasimonotone*, written $F \in (QM)_T$, if for any sequence (u_n) in Ω and any sequence (v_n) in X with $v_n \in Fu_n$ such that $u_n \rightharpoonup u$ in X and $Tu_n \rightharpoonup y$ in X^*, we have

$$\liminf_{n\to\infty} \langle v_n, Tu_n - y\rangle \geq 0.$$

Notice that if $F : \Omega \subset X \to 2^X$ is a bounded operator of class $(S_+)_T$ and $T : \Omega \to X^*$ is a bounded continuous operator, where Ω is closed in X, then F is T-quasimonotone. Moreover, the operators of class $(S_+)_T$ are invariant under $(QM)_T$-perturbations. See [9] for the single-valued case.

In the following, let X be a real reflexive separable Banach space which has been renormed so that both X and X^* are locally uniformly convex.

According to Proposition 3.3 with $Y = X^*$, let $\phi : W \to X^*$ be a compact linear injection on a separable Hilbert space W such that $\phi(W)$ is dense in X^*. Let $\hat{\phi} : X \to W$ be defined by

$$(\hat{\phi}(v), w)_W = \langle v, \phi(w)\rangle_{X^*} \quad \text{for all } w \in W \text{ and all } v \in X, \tag{4.1}$$

where $(\cdot,\cdot)_W$ denotes the inner product of the space W.

Suppose that $F : \overline{G} \subset X \to 2^X$ is a bounded upper demicontinuous operator with nonempty weakly compact convex values and $T : \overline{G} \to X^*$ is a bounded continuous operator of class (S_+), where G is a bounded open set in X. To this F, we associate a family of operators given by

$$F_\lambda := T + \lambda\phi\hat{\phi}F \quad \text{for any positive number } \lambda.$$

Then it is obvious that each $F_\lambda : \overline{G} \to 2^{X^*}$ is a bounded upper semicontinuous operator of class (S_+) with nonempty compact convex values.

Let $k(X)$ denote the collection of nonempty weakly compact convex subsets of X. For any bounded open set G in X, we consider the following classes of operators:

$$\mathcal{F}_1(\overline{G}) := \{T\colon \overline{G} \to X^* \mid T \text{ is bounded, continuous, and of class } (S_+)\},$$
$$\mathcal{F}_T(\overline{G}) := \{F\colon \overline{G} \to k(X) \mid F \text{ is bounded, u.d.c., and of class } (S_+)_T\},$$

with $T \in \mathcal{F}_1(\overline{G})$, called an *essential inner map* to F.

We need an elementary result for the construction of the $(S_+)_T$-degree. For completeness, we give the proof; see also [7, Lemma 2.3].

Lemma 4.2. *Let G be any bounded open set in X and A be any closed subset of $\overline{G}$. Suppose that $F \in \mathcal{F}_T(\overline{G})$, where $T \in \mathcal{F}_1(\overline{G})$. If $h \notin F(A)$, then there exists a positive number λ_0 such that $h_\lambda \notin F_\lambda(A)$ for all $\lambda \in [\lambda_0, \infty)$, where $h_\lambda = \lambda\phi\hat{\phi}h$.*

Proof. Let A be any closed subset of $\overline{G}$ such that $h \notin F(A)$. Assume that there exist sequences (λ_n) in $(0,\infty)$ with $\lambda_n \to \infty$ and (u_n) in A such that $h_{\lambda_n} \in F_{\lambda_n}(u_n)$ for all $n \in \mathbb{N}$, that is,

$$Tu_n + \lambda_n\phi\hat{\phi}(v_n - h) = 0, \tag{4.2}$$

where $v_n \in Fu_n$. Without loss of generality, we may suppose that

$$u_n \rightharpoonup u \ \text{ in } X,\ v_n \rightharpoonup v \ \text{ in } X,\ \text{and } y_n := Tu_n \rightharpoonup y \ \text{ in } X^*. \tag{4.3}$$

As before, we get $\phi\hat{\phi}(v_n) \to \phi\hat{\phi}(h) = \phi\hat{\phi}(v)$ in X^*, which implies $v = h$, that is, $v_n \rightharpoonup h$ in X. Hence it follows from (4.2) and (4.3) that

$$\begin{aligned}\limsup_{n\to\infty} \langle v_n, y_n - y\rangle &= \limsup_{n\to\infty} \langle v_n - h, -\lambda_n\phi\hat{\phi}(v_n - h)\rangle \\ &= \limsup_{n\to\infty} \left[-\lambda_n\|\hat{\phi}(v_n - h)\|_W^2\right] \\ &\le 0,\end{aligned}$$

where $\|\cdot\|_W$ denotes the norm of the Hilbert space W in the sense of (4.1). Since F is of class $(S_+)_T$, we have $u_n \to u \in A$ and $h \in Fu$, in contradiction to the hypothesis that $h \notin F(A)$. This completes the proof. □

Corollary 4.3. *Suppose that G is a bounded open set in X and $F \in \mathcal{F}_T(\overline{G})$, where $T \in \mathcal{F}_1(\overline{G})$. If $h \notin F(\partial G)$, then there is a positive number λ_0 such that $h_\lambda \notin F_\lambda(\partial G)$ for all $\lambda \in [\lambda_0, \infty)$ and the value of $d_{S_+}(F_\lambda, G, h_\lambda)$ is constant for all $\lambda \in [\lambda_0, \infty)$.*

Proof. According to Lemma 4.2 with $A = \partial G$, we find a positive number λ_0 such that $h_\lambda \notin F_\lambda(\partial G)$ for all $\lambda \in [\lambda_0, \infty)$. Let $\lambda_1, \lambda_2 \in [\lambda_0, \infty)$ with $\lambda_1 < \lambda_2$. Then $F_\lambda, \lambda \in [\lambda_1, \lambda_2]$, defines a bounded upper semicontinuous homotopy of class (S_+) with nonempty compact convex values such that $h_\lambda \notin F_\lambda(\partial G)$ for all $\lambda \in [\lambda_1, \lambda_2]$. Hence it follows from the homotopy invariance of the (S_+)-degree in Theorem 3.8 that

$$d_{S_+}(F_{\lambda_1}, G, h_{\lambda_1}) = d_{S_+}(F_{\lambda_2}, G, h_{\lambda_2}).$$

Consequently, the value of $d_{S_+}(F_\lambda, G, h_\lambda)$ is constant for all $\lambda \in [\lambda_0, \infty)$. This completes the proof. □

Now we can define a topological degree for the class $\mathcal{F}_T$. As a set-valued version of [1], it is emphasized that the closed-valued condition on F in [7] is replaced by weakly compact-valued one.

Definition 4.4. Suppose that $F \in \mathcal{F}_T(\overline{G})$ with $T \in \mathcal{F}_1(\overline{G})$, where G is a bounded open set in X. If $h \notin F(\partial G)$, then we define a degree function by

$$d(F, G, h) := \lim_{\lambda \to \infty} d_{S_+}(F_\lambda, G, h_\lambda),$$

where $F_\lambda = T + \lambda \phi \hat{\phi} F$ and $h_\lambda = \lambda \phi \hat{\phi} h$.

Actually, the value of $d(F, G, h)$ is independent of the choice of essential inner map T. In the single-valued case, it was proved in [1, Corollary 6.2].

Definition 4.5. For a bounded operator $T : \overline{G} \subset X \to X^*$, a homotopy $H\colon [0,1] \times \overline{G} \to 2^X$ is said to be of class $(S_+)_T$ if for any sequence (t_n, u_n) in $[0,1] \times \overline{G}$ and any sequence (w_n) in X with $w_n \in H(t_n, u_n)$ such that

$$t_n \to t \text{ in } [0,1],\ u_n \rightharpoonup u \text{ in } X,\ Tu_n \rightharpoonup y \text{ in } X^*, \text{ and } \limsup_{n\to\infty} \langle w_n, Tu_n - y\rangle \le 0,$$

we have $u_n \to u$ in X.

The following result shows that every affine homotopy with a common essential inner map T is of class $(S_+)_T$.

Lemma 4.6. *Suppose that $F, S \in \mathcal{F}_T(\overline{G})$ with $T \in \mathcal{F}_1(\overline{G})$, where G is a bounded open set in X. Then affine homotopy $H\colon [0,1] \times \overline{G} \to 2^X$ defined by*

$$H(t, u) := (1 - t)Fu + tSu \quad \text{for } (t, u) \in [0,1] \times \overline{G}$$

is bounded, upper demicontinuous, and of class $(S_+)_T$ and it has nonempty weakly compact convex values. It is called an admissible affine homotopy with the common essential inner map T.

Proof. For the proof of the fact that H is of class $(S_+)_T$, we refer to [7, Lemma 1.6]. It is easy to verify that $H(t, u)$ is weakly compact and convex for each $(t, u) \in [0,1] \times \overline{G}$. □

The degree function d defined above has the usual properties whose proof is mainly based on the (S_+)-degree in the previous section.

Theorem 4.7. *Let G be any bounded open set in X and suppose that $F \in \mathcal{F}_T(\overline{G})$, where $T \in \mathcal{F}_1(\overline{G})$. Then the following properties are satisfied:*

(a) (Existence) If $d(F, G, h) \neq 0$, then the inclusion $h \in Fu$ has a solution in G.

(b) (Additivity) If G_1 and G_2 are two disjoint open subsets of G such that $h \notin F(\overline{G} \setminus (G_1 \cup G_2))$, then we have

$$d(F, G, h) = d(F, G_1, h) + d(F, G_2, h).$$

(c) (Homotopy invariance) Suppose that $H\colon [0,1] \times \overline{G} \to k(X)$ is an admissible affine homotopy of class $(S_+)_T$ with a common essential inner map $T \in \mathcal{F}_1(\overline{G})$. If $h : [0,1] \to X$ is a continuous map such that $h(t) \notin H(t, \partial G)$ for all $t \in [0,1]$, then the value of $d(H(t,\cdot), G, h(t))$ is constant for all $t \in [0,1]$.

(d) (Normalization) For any $h \in G$, we have $d(I, G, h) = 1$.

Proof. Assertions (a)-(c) follow from the corresponding properties of the (S_+)-degree stated in Theorem 3.8, together with Lemma 4.2 and Definition 4.4.
(d) Note by Proposition 3.2 that the duality operators $J : X \to X^*$ and $J^{-1} : X^* \to X$ are bounded, continuous, and of class (S_+). It is known in [1, 9] that the identity operator $I = J^{-1} \circ J$ belongs to $\mathcal{F}_J(\overline{G})$. Let h be any element of G. Let R be a positive number with $\|h\| < R$ such that

$$d(I, G, h) = d(I, B_R(0), h).$$

Since $Iu \neq th$ for all $(t, u) \in [0,1] \times \partial B_R(0)$, this implies that

$$d(I, B_R(0), h) = d(I, B_R(0), 0) = \lim_{\lambda \to \infty} d_{S_+}(I_\lambda, B_R(0), 0),$$

where $I_\lambda = J + \lambda \phi \hat{\phi} I$. Note by (4.1) that

$$\langle u, Ju + t\lambda\phi\hat{\phi}(u)\rangle_{X^*} = \|u\|^2 + t\lambda\|\hat{\phi}(u)\|_W^2$$

for $(t, u) \in [0, \infty) \times X$ and $\lambda \in (0, \infty)$. For any positive number λ, we have

$$Ju + t\lambda\phi\hat{\phi}(u) \neq 0$$

for all $(t, u) \in [0,1] \times \partial B_R(0)$, which implies by parts (c) and (d) of Theorem 3.8 that

$$d_{S_+}(I_\lambda, B_R(0), 0) = d_{S_+}(J, B_R(0), 0) = 1.$$

Therefore, $d(I, G, h) = 1$, what we wanted to prove. □

Based on the degree theory for the class $\mathcal{F}_T$, the Dirichlet boundary value problem related to the p-Laplacian with discontinuous nonlinearity was considered in [7]; see also [1, 9] for the continuous case.

Remark 4.8. So far we have observed two degree functions for upper demicontinuous set-valued operators of monotone type in reflexive separable Banach spaces, as extensions of the Leray-Schauder degree. The main point in this note was that (weak) compactness of values should be needed instead of closedness when handling certain compositions with set-valued operators in the construction of our degree, as we saw in Remark 3.4.

References

[1] J. Berkovits, *Extension of the Leray-Schauder degree for abstract Hammerstein type mappings*, J. Differ. Equ., **234** (2007), 289–310.

[2] J. Berkovits and M. Tienari, *Topological degree theory for some classes of multis with applications to hyperbolic and elliptic problems involving discontinuous nonlinearities*, Dyn. Syst. Appl., **5** (1996), 1–18.

[3] L.E.J. Brouwer, *Über Abbildungen von Mannigfaltigkeiten*, Math. Ann., **71** (1912), 97–115.

[4] F.E. Browder, *Fixed point theory and nonlinear problems*, Bull. Amer. Math. Soc., **9** (1983), 1–39.

[5] F.E. Browder and B.A. Ton, *Nonlinear functional equations in Banach spaces and elliptic super-regularization*, Math. Z., **105** (1968), 177–195.

[6] A. Granas, *Sur la notion du degré topologique pour une certaine classe de transformations multivalentes dans les espaces de Banach*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., **7** (1959b), 191–194.

[7] I.-S. Kim, *Topological degree and applications to elliptic problems with discontinuous nonlinearity*, J. Nonlinear Sci. Appl., **10** (2017), no. 2 , 612–624.

[8] I.-S. Kim and J.-H. Bae, *Elliptic boundary value problems with discontinuous nonlinearities*, J. Nonlinear Convex Anal., **17** (2016), no. 1, 27–38.

[9] I.-S. Kim and S.-J. Hong, *A topological degree for operators of generalized* (S_+) *type*, Fixed Point Theory Appl., **2015** (2015), 16 pages.

[10] J. Leray and J. Schauder, *Topologie et equationes functionelles*, Ann. Sci. Éc. Norm. Sup. **51** (1934), 45–78.

[11] T.-W. Ma, *Topological degrees of set-valued compact fields in locally convex spaces*, Dissertationes Math. Rozprawy Mat., **92** (1972), 43 pages.

[12] D. O'Regan, Y.J. Cho, and Y.Q. Chen, *Topological Degree Theory and Applications*, Taylor & Francis, Chapman & Hall/CRC, Florida, 2006.

[13] I.V. Skrypnik, *Nonlinear Higher Order Elliptic Equations*, Naukova Dumka, Kiev (1973) (in Russian).

[14] I.V. Skrypnik, *Methods for Analysis of Nonlinear Elliptic Boundary Value Problems*, Amer. Math. Soc., Transl., Ser. II., vol. 139, AMS, Providence, 1994.