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A NEW MODIFIED PROXIMAL POINT ALGORITHM FOR A FINITE FAMILY OF MINIMIZATION PROBLEM AND FIXED POINT FOR A FINITE FAMILY OF DEMICONTRACTIVE MAPPINGS IN HADAMARD SPACES

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Abstract. In this paper, a new modified proximal point algorithm involving a finite family of minimization problem and fixed point for a finite family of demicontractive in Hadamard spaces is proposed. Some Δ-convergence and strong convergence theorems for the sequence generated by the algorithm are proved in Hadamard space withr suitable conditions. The results presented in the paper improve and generalize some recent results.

1. Introduction

Let X be a linear space and $g : X \to (-\infty, +\infty]$ be a proper and convex function. One of the major problems in optimization theory is to find $x \in X$

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such that

$$g(x) = \min_{y \in X} g(y).$$

In 1970, Martinet [14] introduced the proximal point algorithm for solving this type of problem. Later on, Rockafellar [18] studied the convergence results of solution of the convex minimization problem in the framework of Hilbert space using the proximal point algorithm which can be described as follows:

For a proper, lower semi-continuous and convex function $g : X \to (-\infty, +\infty]$, the sequence $\{x_n\}$ generated by $x_1 \in X$ and

$$x_{n+1} = J^g_{\lambda_n}(x_n), \quad \forall n \in \mathbb{N}, \tag{1.1}$$

where $\{\lambda_n\}$ is a positive real sequence and $J^g_{\lambda_n}$ is the resolvent of g defined by

$$J^g_{\lambda_n}(x) = \arg\min_{y \in X}[g(y) + \frac{1}{2\lambda_n}||y - x||^2], \quad \forall n \in \mathbb{N},$$

for all $x \in X$. He proved that the sequence $\{x_n\}$ defined by (1.1) converges weakly to its a minimizer of g.

Recently, many convergence results for solving minimization problems have been extended from the classical linear spaces such as Euclidean spaces, Hilbert spaces and Banach spaces to the setting of manifolds (see [4], [8], [10], [12], [15], [17]) and the references therein.

Very recently, Chang [5] gave the following modified proximal point algorithm for solving common solution of the minimization problem and common fixed point of k-strictly pseudononspreading mappings T_1, T_2 in Hadamard spaces:

$$\begin{cases} u_n = J^g_\lambda(x_n), \\ y_n = (1 - \beta_n)x_n \oplus \beta_n K_1 u_n, \\ x_{n+1} = (1 - \alpha_n)x_n \oplus \alpha_n K_2 y_n, \end{cases}$$

where $K_i(x) := \delta_i x \oplus (1-\delta_i)T_i x$, $x \in C$ with $k \leq \delta_i < 1$, $i = 1, 2$. They proved some convergence theorems under suitable mild conditions.

In 2020, Chang [6] gave an iterative algorithm to approximate a common solution of a finite family of minimization problem and fixed point for a pair of demicontractive mappings in Hadamard spaces.

Motived and inspired by the researches going on in this direction, the purpose of this paper is to study the following proximal point algorithm and fixed point problem: to find $x^* \in C$ such that

$$\begin{cases} J_\lambda^{g_j}(x^*) = \arg\min_{y\in X}(g_j(y) + \frac{1}{2\lambda}d^2(y,x^*)), j = 1,2,\cdots,l, \\ x^* = T_i(x^*), \quad i = 1,2,\cdots,m, \\ x^* = S(x^*), \end{cases} \tag{1.2}$$

where (X,d) is a Hadamard space, C is a nonempty and closed convex subset of X, $\lambda > 0$ is a given positive number, $g_j : C \to \mathbb{R}, j = 1,2,\cdots,l$ is a proper convex and lower semi-continuous function, $T_i : C \to C, i = 1,2,\cdots,m$ and S are k-demicontractive mappings. Problem (1.2) is equivalent to find a point $x^* \in C$ such that

$$x^* \in Fix(S) \bigcap \left(\bigcap_{i=1}^{m} Fix(T_i)\right) \bigcap \left(\bigcap_{j=1}^{l} Fix(J_\lambda^{g_j})\right), \tag{1.3}$$

where $Fix(T_i)$ is the set of fixed points of mapping T_i. Denote the solution set of problem (1.3) by Γ.

In this paper, an iterative algorithm to approximate a common solution of a finite family of minimization problem and fixed point for a finite family of demicontractive mappings in Hadamard spaces is proposed. Under suitable conditions, some Δ-convergence and strong convergence theorems of the sequence generated by the algorithm to an element in the intersection of the set of solutions of such kind of minimization problem and fixed point problem in Hadamard space are proved. Our results complement and extend some recent results in literature.

2. Preliminaries

Let (X,d) be a metric space and $x,y \in X$. A geodesic path joining x to y is an isometry $c : [0,d(x,y)] \to X$ such that $c(0) = x$ and $c(d(x,y)) = y$. The image of a geodesic path joining x to y is called a geodesic segment between x and y. The metric space (X,d) is said to be a geodesic space, if every two points of X are joined by a geodesic. X is said to be uniquely geodesic space, if there is exactly one geodesic joining x and y for each $x,y \in X$.

Lemma 2.1. ([9]) *A geodesic space X is a CAT(0) space, if and only if the following inequality*

$$d^2((1-t)x \oplus ty, z) \le (1-t)d^2(x,z) + td^2(y,z) - t(1-t)d^2(x,y) \tag{2.1}$$

is satisfied for all $x,y,z \in X$ and $t \in [0,1]$. In particular, if x,y,z are points in a CAT(0) space and $t \in [0,1]$, then

$$d((1-t)x \oplus ty, z) \le (1-t)d(x,z) + td(y,z). \tag{2.2}$$

Lemma 2.2. ([3]) *Let X be a CAT(0) space, $p, q, r, s \in X$ and $\lambda \in [0,1]$. Then*

$$d(\lambda p \oplus (1-\lambda)q, \lambda r \oplus (1-\lambda)s) \leq \lambda d(p,r) + (1-\lambda)d(q,s).$$

In the sequel, we denote by

$$\bigoplus_{m=1}^{n} \lambda_m x_m := \lambda_1 x_1 \oplus \lambda_2 x_2 \oplus \cdots \oplus \lambda_{n-1} x_{n-1} \oplus \lambda_n x_n. \tag{2.3}$$

Lemma 2.3. ([7]) *Let X be a CAT(0) space. Then for any sequence $\{\lambda_m\}_{m=1}^n$ in [0,1] satisfying $\Sigma_{m=1}^n \lambda_m = 1$ and for any $\{x_m\}_{m=1}^n \subset X$, the following conclusions hold:*

$$d\left(\bigoplus_{m=1}^{n} \lambda_m x_m, x\right) \leq \sum_{m=1}^{n} \lambda_m d(x_m, x),\ x \in X \tag{2.4}$$

and

$$d^2\left(\bigoplus_{m=1}^{n} \lambda_m x_m, x\right) \leq \sum_{m=1}^{n} \lambda_m d^2(x_m, x) - \sum_{i,j=1, i\neq j}^{n} \lambda_i \lambda_j d^2(x_i, x_j),\ x \in X. \tag{2.5}$$

It is well known that any complete and simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space.

A complete CAT(0) space is often called a *Hadamard space*. We write $(1-t)x \oplus ty$ for the unique point z in the geodesic segment joining from x to y such that $d(x,z) = td(x,y)$ and $d(y,z) = (1-t)d(x,y)$. We also denote by $[x,y]$ the geodesic segment joining x to y, that is, $[x,y] = \{(1-t)x \oplus ty : 0 \leq t \leq 1\}$. A subset C of a CAT(0) space is convex if $[x,y] \subset C$ for all $x, y \in C$.

Let $\{x_n\}$ be a bounded sequence in X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf_{x \in X} \{r(x, \{x_n\})\}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known that in a Hadamard space, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\} \subset X$ is said to Δ-converge to $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write $\Delta - \lim_{n\to\infty} x_n = x$.

The concept of Δ-convergence in CAT(0) space is very similar to the weak convergence in the setting of Banach space.

Lemma 2.4. ([13]) *Every bounded sequence in a complete CAT(0) space always has a Δ-convergent subsequence.*

Berg and Nikolaev [2] introduced the concept of quasilinearization as follows. Let us denote a pair $(a,b)\in X\times X$ by $\overrightarrow{ab}$ and call it a vector. Then quasilinearization is defined as a map $\langle\cdot,\cdot\rangle:(X\times X)\times(X\times X)\to\mathbb{R}$ defined by

$$\langle\overrightarrow{ab},\overrightarrow{cd}\rangle=\frac{1}{2}(d^2(a,d)+d^2(b,c)-d^2(a,c)-d^2(b,d)),\ \ \forall a,b,c,d\in X.$$

It is easily to see that $\langle\overrightarrow{ab},\overrightarrow{cd}\rangle=\langle\overrightarrow{cd},\overrightarrow{ab}\rangle$, $\langle\overrightarrow{ab},\overrightarrow{cd}\rangle=-\langle\overrightarrow{ba},\overrightarrow{cd}\rangle$ and $\langle\overrightarrow{ax},\overrightarrow{cd}\rangle+\langle\overrightarrow{xb},\overrightarrow{cd}\rangle=\langle\overrightarrow{ab},\overrightarrow{cd}\rangle$ for all $a,b,c,d\in X$. We say that X satisfies the Cauchy-Schwarz inequality if

$$\langle\overrightarrow{ab},\overrightarrow{cd}\rangle\le d(a,b)d(c,d),\ \ \forall a,b,c,d\in X. \tag{2.6}$$

It is known that a geodesically connected metric space is a CAT(0) space if and only if it is satisfies the Cauchy-Schwarz inequality.

In the sequel, we always assume that X is a Hadamard space, C is a nonempty and closed convex subset of X and $Fix(T)$ is the fixed point set of a mapping T.

Recall that a mapping $T:C\to C$ is said to be

(1) contractive if there exists a $k\in(0,1)$ such that

$$d(Tx,Ty)\le kd(x,y),\ \forall x,y\in C;$$

If $k=1$, then T is said to be nonexpansive;

(2) quasinonexpansive, if $F(T)\neq\emptyset$ and

$$d(Tx,p)\le d(x,p),\ \forall p\in F(T),x\in C;$$

(3) firmly nonexpansive if

$$d^2(Tx,Ty)\le\langle\overrightarrow{TxTy},\overrightarrow{xy}\rangle,\ \forall x,y\in C; \tag{2.7}$$

(4) nonspreading if

$$2d^2(Tx,Ty)\le d^2(Tx,y)+d^2(Ty,x),\ \forall x,y\in C;$$

(5) k-strict pseudononspreading if there exists a constant $k\in(0,1)$ such that for all $x,y\in C$

$$d^2(Tx,Ty)\le d^2(x,y)+kd^2(Tx,x)+kd^2(Ty,y)+2(1-k)\langle\overrightarrow{xT(x)},\overrightarrow{yT(y)}\rangle;$$

(6) k-demicontractive if $Fix(T) \neq \emptyset$ and there exists a constant $k \in [0, 1)$ such that

$$d^2(Tx, p) \leq d^2(x, p) + kd^2(x, Tx), \ \forall x \in C, \ p \in Fix(T). \tag{2.8}$$

Clearly, if T is a nonspreading mapping with $F(T) \neq \emptyset$, then T is a quasi-nonexpansive mapping, and it is also a 0-demicontractive mapping. If T is a k-strictly pseudononspreading mapping with $Fix(T) \neq \emptyset$, then it is a k-demicontractive mapping. But the converse is not true. This shows that the class of demicontractive mappings is more general than the class of nonspreading mappings and quasinonexpansive mapping.

Recall that a function $f : C \to (-\infty, +\infty]$ defined on a convex subset C of a CAT(0) space is convex if, for any x and y in C with geodesic segment $[x, y] := \{\gamma_{x,y}(\lambda) : 0 \leq \lambda \leq 1\} := \{\lambda x \oplus (1 - \lambda)y : 0 \leq \lambda \leq 1\}$, the function $f \circ \gamma$ is convex, that is,

$$f(\gamma_{x,y}(\lambda)) := f(\lambda x \oplus (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

Lemma 2.5. *Let $f : X \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. For any $\lambda > 0$, define the Moreau-Yosida resolvent of f in Hadamard space X as*

$$J_\lambda^f(x) = \operatorname{argmin}_{y \in X}[f(y) + \frac{1}{2\lambda} d^2(y, x)], \quad \forall x \in X. \tag{2.9}$$

Then

(i) *the set $Fix(J_\lambda^f)$ of fixed points of the resolvent of f coincides with the set* $\operatorname{argmin}_{y \in X} f(y)$ *of minimizers of f, and for any $\lambda > 0$, the resolvent J_λ^f of f is a firmly nonexpansive mapping. Hence it is nonexpansive* ([1])*;*

(ii) *Since J_λ^f is a firmly nonexpansive mapping, if $Fix(J_\lambda^f) \neq \emptyset$, then from* (2.7) *we have*

$$d^2(J_\lambda^f x, p) \leq d^2(x, p) - d^2(J_\lambda^f x, x), \ \forall x \in X, \ p \in Fix(J_\lambda^f). \tag{2.10}$$

Definition 2.6. Let X be a complete metric space and $Q \subset X$ be a nonempty set. A sequence $\{x_n\} \subset X$ is called Fejér monotone with respect to Q if for any $y \in Q$ and $n \geq 1$,

$$d(x_{n+1}, y) \leq d(x_n, y). \tag{2.11}$$

Lemma 2.7. ([11]) *Let X be a complete metric space, $Q \subset X$ be a nonempty set. If $\{x_n\} \subset X$ is Fejér monotone with respect to Q, then $\{x_n\}$ is bounded. Moreover, if a cluster point x of $\{x_n\}$ belongs to Q, then $\{x_n\}$ converges to x.*

Lemma 2.8. ([16]) *Let C be a nonempty closed convex subset of a Hadamard space X and $T : C \to C$ be a k-demicontractive mapping. Then $Fix(T)$ is closed and convex subset in C.*

Lemma 2.9. ([6]) *Let C be a nonempty closed convex subset of a Hadamard space X and $T : C \to C$ be a k-demicontractive mapping with $k \in [0, 1)$. If $Fix(T) \neq \emptyset$ and $k \leq \delta$, then the mapping $K : C \to C$ defined by*

$$K := \delta x \oplus (1-\delta)Tx, \ x \in C \tag{2.12}$$

is quasinonexpansive mapping and $Fix(K) = Fix(T)$.

3. Main results

Throughout this section, we assume that:

(1) (X, d) is a Hadamard space, C is a nonempty closed convex subset of X;

(2) $g_j : C \to \mathbb{R}, \ j = 1, 2, \cdots, l$ is a proper convex and lower semi-continuous function. For given $\lambda > 0$, define the Moreau-Yosida resolvent of g_j in C by

$$J_\lambda^{g_j}(x) = \text{argmin}_{y \in C} \left(g_j(y) + \frac{1}{2\lambda} d^2(y, x) \right), \ j = 1, 2, \cdots, l. \tag{3.1}$$

In the sequel, we denote by

$$R_\lambda^k := J_\lambda^{g_k} \circ J_\lambda^{g_{k-1}} \circ \cdots \circ J_\lambda^{g_2} \circ J_\lambda^{g_1}, \ k = 1, 2. \cdots, l. \tag{3.2}$$

(3) $T_i : C \to C, \ i = 1, 2, \cdots, m$ is a k_i-demicontractive mapping with $0 \leq k_i \leq \delta < 1, \ i = 1, 2, \cdots, m$, and T_i is demiclosed at zero (i.e., for any bounded sequence $\{x_n\}$ in C such that $\triangle - \lim x_n = p$ and $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$, then $T_i p = p$).

Denote by

$$G_i(x) := \delta x \oplus (1-\delta)T_i x, \ x \in C, \ i = 1, 2, \cdots, m.$$

(4) $S : C \to C$ is a k-demicontractive mapping with $0 \leq k \leq \delta < 1$, and S is demiclosed at zero (i.e., for any bounded sequence $\{x_n\}$ in C such that $\triangle - \lim x_n = p$ and $\lim_{n\to\infty} d(x_n, Sx_n) = 0$, then $Sp = p$).

Denote by

$$K(x) := \delta x \oplus (1-\delta)Sx, \ x \in C.$$

Theorem 3.1. *Let* X, C, S, $\{T_i\}_{i=1}^m$, $\{K_i\}_{i=1}^m$, $\{g_j\}_{j=1}^l$, $\{J_\lambda^{g_j}\}_{j=1}^l$, $\{R_\lambda^k\}_{k=1}^l$ *be the same as above. For any given* $x_0 \in C$, *let* $\{x_n\}$ *be the sequence generated by*

$$\begin{cases} u_n = R_\lambda^l(x_n), \\ y_n = \beta_{n,0}u_n \oplus (\oplus_{i=1}^m \beta_{n,i}G^i u_n),\ n \geq 0. \\ x_{n+1} = (1-\alpha_n)u_n \oplus \alpha_n K y_n, \end{cases} \tag{3.3}$$

where $\lambda > 0$, $\{\alpha_n\}, \{\beta_{n,i}\}_{i=0}^m$ *are sequences in [0,1] with* $\Sigma_{i=0}^m \beta_{n,i} = 1$ *and* $0 < a \leq \alpha_n, \beta_{n,i} < b < 1$, *for all* $n \geq 0, i = 0, 1, 2, \cdots, m$. *If the solution set* Γ *of problem* (1.2) *is nonempty, then the sequence* $\{x_n\}$ *defined by* (3.3) *is* Δ*-convergent to a point* $x^* \in \Gamma$ *which is a common minimization of* $\{g_j\}_{j=1}^l$*, as well as it is also a common fixed point of* $\{T_i\}_{i=1}^m$ *and* S *in* C.

Proof. (I) It follows from Lemma 2.5, Lemma 2.8 and Lemma 2.9 that:

(1) if $p \in \Gamma$, then $p \in \bigcap_{i=1}^m Fix(T_i)$ and $p \in Fix(S)$, p is a common minimizer of $\{g_j\}_{j=1}^l$ and $p \in \bigcap_{j=1}^l Fix(J_\lambda^{g_j})$;

(2) for each $i = 1, 2, \cdots, m$, $Fix(G_i) = Fix(T_i)$, $Fix(T_i)$ is a closed convex subset of C and G_i is a quasinonexpansive mapping;

(3) $Fix(K) = Fix(S)$, $Fix(S)$ is a closed convex subset of C and K is a quasinonexpansive mapping;

(4) Now we prove that for each $i = 1, 2, \cdots, m$, G_i is demiclosed at zero. In fact, for any bounded sequence $\{x_n\}$ in C such that $\triangle - \lim x_n = p$ and $\lim_{n\to\infty} d(x_n, G_i x_n) = 0$, then we have

$$d(x_n, G_i x_n) = d(x_n, \delta x_n \oplus (1-\delta)T_i x_n) = (1-\delta)d(x_n, T_i x_n) \to 0. \tag{3.4}$$

Since T_i is demiclosed at zero, hence we have $T_i p = p$. Since $Fix(T_i) = Fix(G_i)$, this implies that $G_i p = p$. Hence G_i is demiclosed at zero.

(5) Now we prove that K is demiclosed at zero.
In fact, for any bounded sequence $\{x_n\}$ in C such that $\Delta - \lim\limits_{n\to\infty} x_n = p$ and $\lim\limits_{n\to\infty} d(x_n, Kx_n) = 0$, then we have

$$d(x_n, Kx_n) = d(x_n, \delta x_n \oplus (1-\delta)Sx_n) = (1-\delta)d(x_n, Sx_n) \to 0.$$

Since S is demiclosed at zero, hence we have $Sp = p$. Since $Fix(K) = Fix(S)$, we have $Kp = p$. Hence K is demiclosed at zero.

(II) Next we prove that $\{x_n\}$ is Fejér monotone with respect to Γ.

In fact, by Lemma 2.5, for each $j = 1, 2, \cdots, l$, $J_\lambda^{g_j}$ is nonexpansive, therefore R_λ^l is also nonexpansive. Let $q \in \Gamma$. Then we have

$$d(u_n, q) = d(R_\lambda^l(x_n), R_\lambda^l(q)) \leq d(x_n, q). \tag{3.5}$$

By Lemma 2.9, for each $i = 1, 2, \cdots, m$, G_i is quasinonexpansive, hence from (3.3) and (3.5) we have

$$\begin{aligned}d(y_n, q) =& d\left(\beta_{n,0}u_n \oplus (\oplus_{i=1}^m \beta_{n,i}G_iu_n), q\right)\\ \leq& \beta_{n,0}d(u_n, q) + \sum_{i=1}^m \beta_{n,i}d(G_iu_n, q)\\ \leq& \beta_{n,0}d(u_n, q) + \sum_{i=1}^m \beta_{n,i}d(u_n, q)\\ =& d(u_n, q) \leq d(x_n, q).\end{aligned} \tag{3.6}$$

From (3.3), (3.5) and (3.6) we have

$$\begin{aligned}d(x_{n+1}, q) =& d((1-\alpha_n)u_n \oplus \alpha_n Ky_n, q)\\ \leq& (1-\alpha_n)d(u_n, q) + \alpha_n d(Ky_n, q)\\ \leq& (1-\alpha_n)d(u_n, q) + \alpha_n d(y_n, q)\\ \leq& d(u_n, q) \leq d(x_n, q),\ \forall n \geq 0.\end{aligned} \tag{3.7}$$

This shows that $\{d(x_n, q)\}$ is decreasing and bounded below. Hence the limit $\lim_{n\to\infty} d(x_n, q)$ exists for each $q \in \Gamma$. This implies that $\{x_n\}$ is Fejér monotone with respect to Γ. Without loss of generality, we can assume that

$$\lim_{n\to\infty} d(x_n, q) = c. \tag{3.8}$$

Hence the sequence $\{x_n\}$ is bounded. So are the sequences $\{y_n\}, \{u_n\}, \{Ky_n\}$ and $\{G_iu_n\}, i = 1, 2, \cdots, m$.

(III) Next we prove that

$$\lim_{n\to\infty} d(x_n, u_n) = 0. \tag{3.9}$$

In fact, it follows from Lemma 2.5 (ii) and (3.7) that for any given $p \in \Gamma$ we have

$$\begin{aligned}d^2(u_n, R_\lambda^{l-1}x_n) &= d^2(R_\lambda^l x_n, R_\lambda^{l-1}x_n)\\ &\leq d^2(R_\lambda^{l-1}x_n, p) - d^2(u_n, p)\\ &\leq d^2(x_n, p) - d^2(u_n, p)\\ &\leq d^2(x_n, p) - d^2(x_{n+1}, p).\end{aligned}$$

Therefore

$$\lim_{n\to\infty} d(u_n, R_\lambda^{l-1}x_n) = 0.$$

Similarly, by using the same method, we can prove that

$$\lim_{n\to\infty} d(R_\lambda^{l-j}x_n, R_\lambda^{l-(j+1)}x_n) = 0, \quad j = 0, 1, 2, \cdots l-1. \tag{3.10}$$

Therefore we have

$$\begin{aligned} d(u_n, x_n) &= d(R_\lambda^l x_n, x_n) \\ &\leq d(R_\lambda^l x_n, R_\lambda^{l-1} x_n) + d(R_\lambda^{l-1} x_n, R_\lambda^{l-2} x_n) \\ &\quad + \cdots + d(R_\lambda^2 x_n, R_\lambda^1 x_n) + d(R_\lambda^1 x_n, x_n) \to 0 \ (n \to \infty). \end{aligned}$$

The conclusion (3.9) is proved.

(IV) Next we prove that

$$\lim_{n\to\infty} (u_n, G_i u_n) = 0, \ \ \forall i = 1, 2, \cdots, \ m$$

and

$$\lim_{n\to\infty} d(x_n, y_n) = 0, \ \lim_{n\to\infty} d(y_n, Ky_n) = 0.$$

Indeed, it follows from (3.3) and (2.5) that

$$\begin{aligned} d^2(y_n, q) =& d^2(\beta_{n,0} u_n \oplus (\oplus_{i=1}^m \beta_{n,i} G_i u_n), q) \\ \leq& \beta_{n,0} d^2(u_n, q) + \sum_{i=1}^m \beta_{n,i} d^2(G_i u_n, q) - \beta_{n,0}\beta_{n,s} d^2(u_n, G_s u_n) \\ \leq& \beta_{n,0} d^2(u_n, q) + \sum_{i=1}^m \beta_{n,i} d^2(u_n, q) - \beta_{n,0}\beta_{n,s} d^2(u_n, G_s u_n) \\ =& d^2(u_n, q) - \beta_{n,0}\beta_{n,s} d^2(u_n, G_s u_n), \ \forall s = 1, 2, \cdots, l. \end{aligned} \tag{3.11}$$

Also it follows from (3.3), (2.5) and (3.11) that

$$\begin{aligned} d^2(x_{n+1}, q) =& d^2((1-\alpha_n)u_n \oplus \alpha_n K y_n, q) \\ \leq& (1-\alpha_n) d^2(u_n, q) + \alpha_n d^2(Ky_n, q) - \alpha_n(1-\alpha_n) d^2(u_n, Ky_n) \\ \leq& (1-\alpha_n) d^2(u_n, q) + \alpha_n d^2(y_n, q) - \alpha_n(1-\alpha_n) d^2(u_n, Ky_n) \\ \leq& (1-\alpha_n) d^2(u_n, q) + \alpha_n (d^2(u_n, q) - \beta_{n,0}\beta_{n,s} d^2(u_n, G_s u_n)) \\ & - \alpha_n(1-\alpha_n) d^2(u_n, Ky_n) \\ =& d^2(u_n, q) - \alpha_n\beta_{n,0}\beta_{n,s} d^2(u_n, G_s u_n) - \alpha_n(1-\alpha_n) d^2(u_n, Ky_n) \\ \leq& d^2(x_n, q) - \alpha_n\beta_{n,0}\beta_{n,s} d^2(u_n, G_s u_n) - \alpha_n(1-\alpha_n) d^2(u_n, Ky_n). \end{aligned}$$

After simplifying and by using the condition

$$0 < a \leq \alpha_n, \beta_{n,i} < b < 1 \ \ (i = 1, 2, \cdots, m),$$

we have that

$$\begin{aligned} & a^3 d^2(u_n, G_s u_n) + a(1-b) d^2(u_n, Ky_n) \\ & \leq \alpha_n \beta_{n,0}\beta_{n,i} d^2(u_n, G_s u_n) + \alpha_n(1-\alpha_n) d^2(u_n, Ky_n) \\ & \leq d^2(x_n, q) - d^2(x_{n+1}, q) \to 0 \ \ (n \to \infty). \end{aligned}$$

This implies that

$$\lim_{n\to\infty} d(u_n, Ky_n) = 0 \;\; \textit{and} \;\; \lim_{n\to\infty} d(u_n, G_i u_n) = 0 \;\; (\forall i = 1, 2, \cdots, m). \tag{3.12}$$

Hence from (3.3) and (3.12) we have

$$\begin{aligned} d(y_n, u_n) =& d(\beta_{n,0} u_n \oplus (\oplus_{i=1}^m \beta_{n,i} G_i u_n), u_n) \\ \le& \sum_{i=1}^m \beta_{n,i} d(G_i u_n, u_n) \to 0 \;\; (n \to \infty). \end{aligned} \tag{3.13}$$

It follows from (3.9), (3.12) and (3.13) that

$$\begin{cases} d(u_n, G_i u_n) \to 0 \;\; (n \to \infty) \;\; (\forall i = 1, 2, \cdots, m), \\ d(y_n, Ky_n) \le d(y_n, u_n) + d(u_n, Ky_n) \to 0 \;\; (n \to \infty), \\ d(R_\lambda^l(x_n), x_n) = d(u_n, x_n) \to 0 \;\; (n \to \infty), \\ d(x_n, y_n) \le d(x_n, u_n) + d(u_n, y_n) \to 0 \;\; (n \to \infty). \end{cases} \tag{3.14}$$

(V) Finally we prove that $\{x_n\}$ is Δ-convergent to some point in Γ.

In fact, in (II) we have proved that $\{x_n\}$ is a bounded sequence in C, and it is also Fejér monotone with respect to Γ. By Lemma 2.7, in order to prove $\{x_n\}$ is Δ-convergent to some point in Γ, it suffices to prove that each Δ-sequential cluster point of $\{x_n\}$ belongs to Γ.

Indeed, let x^* be a Δ-sequential cluster point of $\{x_n\}$. Then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\Delta - \lim_{n\to\infty} x_{n_j} = x^*$. This together with (3.9) and (3.13) shows that $\Delta - \lim_{n\to\infty} u_{n_j} = x^*$ and $\Delta - \lim_{n\to\infty} y_{n_j} = x^*$.

On the other hand, by (3.14) we have

$$\lim_{n\to\infty} d(u_{n_j}, Ky_{n_j}) = 0, \quad \lim_{n\to\infty} d(R_\lambda^l(x_{n_j}), x_{n_j}) = 0$$

and

$$\lim_{n\to\infty} d(u_{n_j}, G_i u_{n_j}) = 0 \;\; (\forall i = 1, 2, \cdots, m).$$

Since R_λ^l is nonexpansive, it is demiclosed at zero. Also in (I) we have proved that G_i and K are demiclosed at zero, this implies that

$$x^* \in Fix(K) \bigcap \left(\bigcap_{i=1}^m Fix(G_i) \right) \bigcap Fix(R_\lambda^l).$$

In order to prove that $x^* \in \Gamma$, it should be proved that

$$Fix(R_\lambda^l) = \bigcap_{j=1}^l Fix(J_\lambda^{g_j}). \tag{3.15}$$

It is obvious that $\bigcap_{j=1}^{m} Fix(J_\lambda^{g_j}) \subseteqq Fix(R_\lambda^l)$. Next we prove that

$$Fix(R_\lambda^l) \subseteqq \bigcap_{j=1}^{l} Fix(J_\lambda^{g_j}). \tag{3.16}$$

Let $q \in Fix(R_\lambda^l)$ and $p \in \bigcap_{j=1}^{l} Fix(J_\lambda^{g_j})$, we have

$$\begin{aligned} d(q,p) &= d(R_\lambda^l q,\ p) = d(J_\lambda^{g_l} R_\lambda^{l-1} q, J_\lambda^{g_l} p) \le d(R_\lambda^{l-1} q, p) \\ &\le d(R_\lambda^{l-2} q, p) \le \cdots \le d(R_\lambda^1 q, p) = d(J_\lambda^{g_1} q, p) \\ &\le d(q,p). \end{aligned}$$

This implies that

$$\begin{aligned} d(q,p) &= d(R_\lambda^l q,\ p) = d(R_\lambda^{l-1} q, p) = d(R_\lambda^{l-2} q, p) \\ &= \cdots = d(R_\lambda^1 q, p) = d(J_\lambda^{g_1} q, p). \end{aligned} \tag{3.17}$$

It follows from (3.17) and (2.10) that for each $j = 1, 2, \cdots, l$, we have

$$d(R_\lambda^j q, p) + d(R_\lambda^j q, R_\lambda^{j-1} q) \le d(R_\lambda^{j-1} q, p) = d(q,p).$$

Since $d(R_\lambda^j q, p) = d(q,p)$, this implies that for each $j = 1, 2, \cdots, l$

$$d(R_\lambda^j q, R_\lambda^{j-1} q) = 0, \quad i.e., \ R_\lambda^{j-1} q \in Fix(J_\lambda^{g_j}) q. \tag{3.18}$$

Taking $j = 1$ in (3.18), we have $q = J_\lambda^{g_1}(q)$. Taking $j = 2$ in (3.18), we have that

$$q = J_\lambda^{g_1}(q) = J_\lambda^{g_2} q.$$

Taking $j = 1, 2, \cdots, l$ in (3.18) we can prove that

$$q = J_\lambda^{g_1}(q) = J_\lambda^{g_2} q = \cdots = J_\lambda^{g_{l-1}} q = J_\lambda^{g_l} q, \ i.e., \quad q \in \bigcap_{j=1}^{l} Fix(J_\lambda^{g_j}).$$

This completes the proof of Theorem 3.1. □

4. Some strong convergence theorems in Hadamard spaces

Let (X, d) be a Hadamard space and C be a nonempty, closed and convex subset of X.

Recall that a mapping $T : C \to C$ is said to be demi-compact, if for any bounded sequence $\{x_n\}$ in C such that $d(x_n, Tx_n) \to 0$ (as $n \to \infty$), then there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to some point $p \in C$.

Theorem 4.1. *In addition to satisfying the conditions in Theorem* 3.1*, and if one of the mappings* $\{T_i\}_{i=1}^m$ *or* S *is demi-compact, then the sequence* $\{x_n\}$ *defined by* (3.3) *converges strongly to a point* $x^* \in \Gamma$.

Proof. In fact, it follows from (3.14) that

$$\lim_{n\to\infty} d(u_n, G_i u_n) = 0, \quad \lim_{n\to\infty} d(y_n, K y_n) = 0, \quad \lim_{n\to\infty} d(R_\lambda^l(x_n), x_n) = 0. \tag{4.1}$$

Since $K = \delta I \oplus (1-\delta)S$ and $G_i = \delta I \oplus (1-\delta)T_i (i = 1, 2, \cdots, m)$, we have

$$d(y_n, K y_n) = d(y_n, \delta y_n \oplus (1-\delta)S y_n) = (1-\delta) d(y_n, S y_n),$$
$$d(u_n, G_i u_n) = d(u_n, \delta u_n \oplus (1-\delta)T_i u_n) = (1-\delta) d(u_n, T_i u_n).$$

This together with (4.1) implies that

$$\begin{aligned} d(u_n, T_i u_n) &= \frac{1}{1-\delta} d(u_n, G_i u_n) \to 0 (n \to \infty), \\ d(y_n, S y_n) &= \frac{1}{1-\delta} d(y_n, K y_n) \to 0 (n \to \infty). \end{aligned} \tag{4.2}$$

By the assumption that one of $\{T_i\}_{i=1}^m$ and S is demi-compact, without loss of generality, we can assume S is demi-compact. Therefore there exists a subsequence $\{y_{n_j}\} \subset \{y_n\}$ such that $\{y_{n_j}\}$ converges strongly to some point $x^* \in C$. Since S is demiclosed at zero, $x^* \in Fix(S)$.

Furthermore, by (3.8) and (3.13), $d(x_n, u_n) \to 0$ and $d(u_n, y_n) \to 0$. These show that $u_{n_j} \to x^*$ and $x_{n_j} \to x^*$. Since T_i and R_λ^l both are demiclosed at zero, we have

$$x^* \in \left(\bigcap_{i=1}^m Fix(T_i) \right) \bigcap Fix(R_\lambda^l).$$

From (3.15), we get

$$x^* \in \left(\bigcap_{i=1}^m Fix(T_i) \right) \bigcap \left(\bigcap_{j=1}^l Fix(J_\lambda^{g_j}) \right).$$

This together with $x^* \in Fix(S)$ implies that $x^* \in \Gamma$. Again by (3.8) the limit $\lim_{n\to\infty} d(x_n, x^*)$ exists. Hence we have $\lim_{n\to\infty} d(x_n, x^*) = 0$. This completes the proof of Theorem 4.1. □

Theorem 4.2. *In addition to satisfying the conditions in Theorem* 3.1*, and if there exists a nondecreasing function* $f : [0, \infty) \to [0, \infty)$ *with* $f(0) = 0, f(r) > 0$ *for all* $r > 0$*, such that*

$$f(d(x, \Gamma)) \le d(x, R_\lambda^l x), \ \forall x \in C.$$

Then the sequence $\{x_n\}$ *defined by* (3.3) *converges strongly to a point* $x^* \in \Gamma$.

Proof. In fact, it follows from (3.14) that

$$\lim_{n\to\infty} d(x_n, R_\lambda^l x_n) = 0. \tag{4.3}$$

Therefore we have $\lim_{n\to\infty} f(d(x_n,\Gamma)) = 0$. Since f is nondecreasing with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$, we have

$$\lim_{n\to\infty} d(x_n, \Gamma) = 0. \tag{4.4}$$

This implies that

$$\lim_{n,m\to\infty} d(x_n, x_m) \leq \lim_{n\to\infty} d(x_n, \Gamma) + \lim_{m\to\infty} d(x_m, \Gamma) = 0. \tag{4.5}$$

This implies that $\{x_n\}$ is a Cauchy sequence in C. Since C is a closed subset in a Hadamard space X, it is complete. Without loss of generality, we can assume that $\{x_n\}$ converges strongly to some point $x^* \in C$. It is easy to see that $Fix(S)$, $Fix(R_\lambda^l)$ and $Fix(T_i), i = 1, 2, \cdots, m$ all are closed subsets in C, so is Γ. Since $d(x^*, \Gamma) = \lim_{n\to\infty} d(x_n, \Gamma) = 0$, we have $x^* \in \Gamma$. This completes the proof of Theorem 4.2. □

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