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PARTIAL REGULARIZATION AND DESCENT METHOD FOR A EXTENDED PRIMAL-DUAL SYSTEM

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Abstract. In this works, we consider a system of variational inequality, which can be regarded as an extension of a primal-dual variational inequality system and involves multivalued mappings. The system does not possess monotonicity properties and the feasible set is unbounded in general. To solve the problem, we propose a completely implementable iterative scheme, whose convergence is proved under certain coercivity type conditions.

1. Introduction

Let X and Y be nonempty, closed and convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, moreover,

$$0 \in Y \subseteq \mathbb{R}^m_+ = \{y := (y_1, y_2, \cdots, y_m) \in \mathbb{R}^m \mid y_i \geq 0, i = 1, \cdots, m\},$$

$G : X \to \Pi(\mathbb{R}^n)$ be a convex smooth multi-valued mapping, $H : X \to \mathbb{R}^n$ be a continuous mapping with convex components $H_i : X \to \mathbb{R}, i = 1, \cdots, m$, $B : Y \to \Pi(\mathbb{R}^m)$ be a continuous multi-valued mapping, which is the gradient of some function $\varphi : Y \to \mathbb{R}$, *i.e.*, $\varphi'(y) = B(y)$. Here and below $\Pi(A)$ denotes the family of all nonempty subsets of a set A.

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Under these assumptions, in this work, we can define the extended primal-dual system: finding a pair $(x^*, y^*) \in X \times Y$ such that

$$\exists g^* \in G(x^*),\ \langle g^*, x - x^* \rangle + \langle y^*, H(x) - H(x^*) \rangle \geq 0,\ \forall x \in X, \tag{1.1}$$

$$\exists b^* \in B(y^*),\ \langle b^* - H(x^*), y - y^* \rangle \geq 0,\ \forall y \in Y. \tag{1.2}$$

Note that, (1.1) can be replaced by the equivalent optimization problem:

$$g - g^* + \langle y^*, H(x) - H(x^*) \rangle \geq 0,\ \forall x \in X, g \in G(x), g^* \in G(x^*). \tag{1.3}$$

Then, if $Y = \mathbb{R}^m_+$ and B is fixed, that is, $B(y) \equiv B$, and G is the subdifferential of a convex function, then the system (1.1)-(1.2) (or (1.2)-(1.3)) gives the necessary and sufficient Karush-Kuhn-Tucker optimality conditions for the constrained optimization problem:

$$\min_{x \in D} \to \varphi(x),$$

where $D = \{x \in X \mid H_i(x) \leq B_i, i = 1, \cdots, m\}$.

Various problems can be reduced to system (1.1)-(1.2) (see, [1, 5, 6, 11, 12, 13, 14]), moreover, under the condition that the mapping B is monotone, system (1.1)-(1.2) reduces to the saddle point problem of a convex concave function. The more difficult case when B is quite arbitrary was considered in [7, 8]. Then the main mapping of the variational inequality system (1.1)-(1.2) is neither monotone nor the gradient mapping of any function, which means that we have to develop special solution methods for such a problem. For instance, partial regularization and descent methods were constructed in [2, 3, 9, 10], but their convergence was established under the boundedness of X and/or Y, which may be quite a restrictive condition for applications. In the paper, this method is justified for the completely unbounded case.

2. Solutions of perturbed problem

Let $\varepsilon > 0$ be a fixed regularization parameter. Then we define the perturbed system: find $(x^{(\varepsilon)}, y^{(\varepsilon)}) \in X \times Y$ such that

$$\langle g' + \varepsilon x^{(\varepsilon)}, x - x^{(\varepsilon)} \rangle + \langle y^{(\varepsilon)}, H(x) - H(x^{(\varepsilon)}) \rangle \geq 0,\ \forall x \in X, g' \in G(x^{(\varepsilon)}), \tag{2.1}$$

$$\langle B(y^{(\varepsilon)}) - H(x^{(\varepsilon)}), y - y^{(\varepsilon)} \rangle \geq 0,\ \forall y \in Y. \tag{2.2}$$

Again we notice that (2.1) is equivalent to the following optimization problem with the strongly convex function:

$$g - g^{(\varepsilon)} + 0.5\varepsilon(\|x\|^2 - \|x^{(\varepsilon)}\|^2) + \langle y^{(\varepsilon)}, H(x) - H(x^{(\varepsilon)}) \rangle \geq 0,$$

for all $x \in X, g \in G(x), g^{(\varepsilon)} \in G(x^{(\varepsilon)})$. which always has a unique solution denoted by $x^{(\varepsilon)}(y^{(\varepsilon)})$, that is,

$$x^{(\varepsilon)}(y^{(\varepsilon)}) = arg\min\{g + 0.5\varepsilon\|x\|^2 + \langle y^{(\varepsilon)}, H(x)\rangle \mid x \in X, g \in G(x)\}.$$

For any $y \in Y$ we also set

$$S^{(\varepsilon)}(y) = -H(x^{(\varepsilon)}(y)),\ F^{(\varepsilon)}(y) = B(y) + S^{(\varepsilon)}(y),$$

$$\Psi_{(\varepsilon)}(y) = -[g^{(\varepsilon)} + 0.5\varepsilon\|x^{(\varepsilon)}(y)\|^2 + \langle y, H(x^{(\varepsilon)}(y))\rangle], g^{(\varepsilon)} \in G(x^{(\varepsilon)}(y)),$$

$$\Phi_{(\varepsilon)}(y) = \varphi(y) + \Psi_{(\varepsilon)}(y).$$

First of all we see that the assumptions made the mapping $y \to x^{(\varepsilon)}(y)$ is continuous on Y (for any fixed $\varepsilon > 0$), and that $S^{(\varepsilon)}$ is the gradient of the function $\Psi_{(\varepsilon)}$ (*see* [7], Lemmas 3.2 and 3.3). Further, the dual variational inequality for system (2.1)-(2.2) is to find a point $y^{(\varepsilon)} \in Y$ such that

$$\langle F^{(\varepsilon)}(y^{(\varepsilon)}), y - y^{(\varepsilon)}\rangle \geq 0,\ \forall y \in Y. \tag{2.3}$$

Obviously if $y^{(\varepsilon)}$ is a solution of problem (2.3), then the pair $(x^{(\varepsilon)}(y^{(\varepsilon)}), y^{(\varepsilon)})$ is a solution of system (2.1)-(2.2). Moreover, problem (2.3) is a necessary optimality condition for the optimization problem

$$\min_{y \in Y} \to \Phi_{(\varepsilon)}(y),$$

which is not sufficient in general, because the function φ is not convex and the same is true for the function $\Phi_{(\varepsilon)}$.

In order to solve problem (2.3) we apply a descent projection type method [4]. For fix $\lambda > 0$ and set

$$z^{(\varepsilon)}(y) = \pi_Y[y - \lambda F^{(\varepsilon)}(y)],$$

where $\pi_Y[\cdot]$ is the projection mapping onto the set Y. Note that the mapping $F^{(\varepsilon)}$ is continuous, hence so is the mapping $y \to z^{(\varepsilon)}(y)$. Further, y^* is a solution of problem (2.3) if and only if

$$y^* = \pi_Y[y^* - \lambda F^{(\varepsilon)}(y^*)], \tag{2.4}$$

besides

$$\langle \Phi'_{(\varepsilon)}(y), z^{(\varepsilon)}(y) - y\rangle \leq -\lambda^{-1}\|z^{(\varepsilon)}(y) - y\|^2. \tag{2.5}$$

To get (2.5) we rewrite the definition of $z^{(\varepsilon)}(y)$ as a solution of the optimization problem:

$$z^{(\varepsilon)}(y) = arg\min\{\|(y - \lambda F^{(\varepsilon)}(y)) - z\|^2 \mid z \in Y\}$$

or as the equivalent optimality condition:

$$z^{(\varepsilon)}(y) \in Y,\ \langle F^{(\varepsilon)}(y) + \lambda^{-1}(z^{(\varepsilon)}(y) - y), u - z^{(\varepsilon)}(y)\rangle \geq 0,\ \forall u \in Y. \tag{2.6}$$

Taking $u = y$ in (2.6) gives

$$\langle F^{(\varepsilon)}(y), z^{(\varepsilon)}(y) - y\rangle \leq -\lambda^{-1}\|z^{(\varepsilon)}(y) - y\|^2,$$

that is, (2.5) is satisfied.

Also set $d^{(\varepsilon)}(y) = z^{(\varepsilon)}(y) - y$ for brevity. Now we describe an algorithm to solve problem (2.3).

Algorithm 2.1. *Choose a point $u \in Y$ and numbers $\alpha, \beta \in (0,1)$. At the k^{th} iteration, $k = 0, 1, \cdots$, there is a point $u^k \in Y$. Compute $z^k = z^{(\varepsilon)}(u^{(\varepsilon)})$ and set $d^k = d^{(\varepsilon)}(u^k)$. If $d^k = 0$, then stop. Otherwise find p as the smallest nonnegative integer such that*

$$\Phi_{(\varepsilon)}(u^k + \beta^p d^k) \leq \Phi_{(\varepsilon)}(u^k) + \alpha\beta^p\langle \Phi'_{(\varepsilon)}(u^k), d^k\rangle,$$

set $\theta_k = \beta^p, u^{k+1} = u^k + \theta_k d^k$, the iteration has been completed.

Note that the termination of Algorithm 2.1 means that u^k is a solution of problem (2.3) due to (2.4), that is, $(x^{(\varepsilon)}(u^k), u^k)$ is then a solution of system (2.1)-(2.2). Hence it is sufficient to consider the case when Algorithm 2.1 generates an infinite iteration sequence.

First we give the convergence properties of Algorithm 2.1, which follow from [3, 4, 7]. We also note that its linesearch was also justified in these works, together with the convergence.

Proposition 2.2. *Assume that the level set $W_{(\varepsilon)}(u^0) = \{y \in Y \mid \Phi_{(\varepsilon)}(y) \leq \Phi_{(\varepsilon)}(u^0)\}$ is bounded, the sequence $\{u^k\}$ is defined by Algorithm 2.1 and $v^k = x^{(\varepsilon)}(u^k)$. Then the following statements hold:*

(a) *$d^k \to 0$ as $k \to \infty$,*
(b) *$\{u^k\}$ has limit points and each of them is a solution of problem (2.3),*
(c) *$\{(u^k, v^k)\}$ has limit points and each of them is a solution of system (2.1)- (2.2).*

3. Partial regularization and descent method

This method uses a sequence of the perturbed problems (2.1)-(2.2), which corresponds to a sequence of numbers $\varepsilon_\ell \to 0$. For any $\gamma \in Y$ set

$$x^\ell(y) = arg\min\{g + 0.5\varepsilon_\ell\|x\|^2 + \langle y, H(x)\rangle \mid x \in X, g \in G(x)\},$$
$$S^\ell(y) = -H(x^\ell(y)), F^\ell(y) = B(y) + S^\ell(y),$$
$$\Psi_\ell(y) = -[g^\ell + 0.5\varepsilon_\ell\|x^\ell(y)\|^2 + \langle y, H(x^\ell(y))\rangle], g^\ell \in G(x^\ell(y)),$$

$$\Phi_\ell(y) = \varphi(y) + \Psi_\ell(y), z^\ell(y) = z^{(\varepsilon_\ell)}(y).$$

Algorithm 3.1. *Step 0: Choose an arbitrary point* $y^0 \in Y$*, sequences* $\varepsilon_\ell \to 0, \delta_\ell \to 0$*, and numbers* $\alpha, \beta \in (0,1), \lambda > 0$*. Set* $\ell = 1$*.*
Step 1: Set $u^{\ell,0} = y^{\ell-1}, k = 0$*.*
Step 2: Compute $z^{\ell,k} = z^\ell(u^{\ell,k})$ *and set* $d^{\ell,k} = z^{\ell,k} - u^{\ell,k}$*.*
Step 3: If $\|d^{\ell,k}\| \leq \delta_\ell$*, then set* $y^\ell = u^{\ell,k}, \ell = \ell + 1$ *and go to Step 1.*
Step 4: Find p as the smallest nonnegative integer such that

$$\Phi_\ell(y^{\ell,k} + \beta^p d^{\ell,k}) \leq \Phi_\ell(y^{\ell,k}) + \alpha\beta^p \langle \Phi'_\ell(y^{\ell,k}), d^{\ell,k}\rangle,$$

set $\theta_k = \beta^p, u^{\ell,k+1} = u^{\ell,k} + \theta_k d^{\ell,k}, k = k + 1$ *and go to Step 2.*

So, we can see that for any fixed ℓ Steps 2-4 of Algorithm 3.1 contain Algorithm 2.1 applied to problem (2.3) with $\varepsilon = \varepsilon_\ell$. The index k change will be referred to as an inner step of Algorithm 3.1. We need the following additional assumptions.

(A1) There exists a point $\bar{x} \in X$ such that $H(\bar{x}) < 0$.
(A2) The mapping H is bounded on the set X.
(A3) It holds that $f(x) \to +\infty$ as $\|x\| \to \infty$, $x \in X$.
(A4) It holds that $\langle B(y), y\rangle \to +\infty$ as $\|y\| \to \infty$, $y \in Y$.
(A5) The mapping B is bounded on the set Y. Set

$$V_\varepsilon^\gamma = \{y \in Y \mid \Phi(\varepsilon)(y) \leq \gamma\}.$$

Theorem 3.2. *Assume that the set* $V_{\varepsilon_0}^\gamma$ *is bounded for any number* γ *and that conditions* (A1-A5) *are satisfied. If the sequence* $\{y^\ell\}$ *is defined by Algorithm* 3.1 *and the sequence* $\{x^\ell\}$ *is given by the rule* $x^\ell = x^\ell(y^\ell)$*, then*

(a) *for each fixed* ℓ *the number of inner steps is finite,*
(b) *the sequence* $\{(x^\ell, y^\ell)\}$ *has limit points and each of them is a solution of system* (1.1)-(1.2)*.*

Proof. First of all we show that the set $W_{\varepsilon_\ell}(y^{\ell-1})$ is always bounded. Note that $y^{\ell-1} \in Y$ by construction. Take an arbitrary point $z \in W_{\varepsilon_\ell}(y^{\ell-1})$ and set $\gamma = \Phi_\ell(y^{\ell-1})$, then

$$W_{\varepsilon_\ell}(y^{\ell-1}) = V_{\varepsilon_\ell}^\gamma, \text{ that is, } \Psi_\ell(z) + \varphi(z) \leq \gamma.$$

Next, for all $g^{(\varepsilon'')} \in G(x^{(\varepsilon'')})$ we have

$$\begin{aligned}
\Psi_{(\varepsilon')}(y) &= -[g^{(\varepsilon'')} + 0.5\varepsilon''\|x^{(\varepsilon'')}\|^2 + \langle y, H(x^{(\varepsilon'')})\rangle] \\
&\leq -[g^{(\varepsilon'')} + 0.5\varepsilon'\|x^{(\varepsilon'')}\|^2 + \langle y, H(x^{(\varepsilon'')})\rangle] \\
&\leq -[g^{(\varepsilon')} + 0.5\varepsilon'\|x^{(\varepsilon')}\|^2 + \langle y, H(x^{(\varepsilon')})\rangle], \quad \forall g^{(\varepsilon')}) \in G(x^{(\varepsilon')}) \\
&= \Psi_{(\varepsilon')}(y).
\end{aligned}$$

If $0 \le \varepsilon' \le \varepsilon''$. Therefore $\Psi_\ell(z) + \varphi(z) \le \gamma$ implies $\Psi_0(z) + \varphi(z) \le \gamma$ and $z \in V^\gamma_{\varepsilon_0}$. Thus, $W_{\varepsilon_\ell}(y^{\ell-1}) \subseteq V^\gamma_{\varepsilon_0}$. But the set $V^\gamma_{\varepsilon_0}$ is bounded and part (a) holds true due to Proposition 2.2(a). Set $z^\ell = z^\ell(y^\ell) = \pi_Y[y^\ell - \lambda F^\ell(y^\ell)]$, then for any pair $(x, y) \in X \times Y$ it holds that (cf. (2.1), (2.6))

$$\begin{gathered}\langle g' + \varepsilon_\ell x^\ell, x - x^\ell \rangle + \langle y^\ell, H(x) - H(x^\ell)\rangle \ge 0, \ \ \forall g' \in G(x^\ell),\\ \langle B(y^\ell) - H(x^\ell) + \lambda^{-1}(z^\ell - y^\ell), y - z^\ell \rangle \ge 0,\\ \|z^\ell - y^\ell\| \le \delta_\ell. \end{gathered} \tag{3.1}$$

Addition of the first and the second inequality in (3.1) with $x = \bar{x}, y = 0$ and $g' \in G(x^\ell), \bar{g} \in G(\bar{x}), g^\ell \in G(x^\ell)$ gives

$$\begin{aligned} 0 &\le \langle g', \bar{x} - x^\ell\rangle + \varepsilon_\ell \langle x^\ell, \bar{x} - x^\ell\rangle + \langle y^\ell, H(\bar{x})\rangle \\ &\quad - \langle y^\ell, H(x^\ell)\rangle - \langle B(y^\ell), z^\ell\rangle + \langle H(x^\ell), z^\ell\rangle - \lambda^{-1}\langle z^\ell - y^\ell, z^\ell\rangle \\ &\le \bar{g} - g^\ell + \varepsilon_\ell \|x^\ell\|(\|\bar{x}\| - \|x^\ell\|) + \langle y^\ell, H(\bar{x})\rangle - \langle y^\ell - z^\ell, H(x^\ell)\rangle \\ &\quad + \langle B(y^\ell), y^\ell - z^\ell\rangle - \langle B(y^\ell), y^\ell\rangle - \lambda^{-1}\|z^\ell - y^\ell\|^2 - \lambda^{-1}\langle z^\ell - y^\ell, y^\ell\rangle \\ &= \bar{g} - g^\ell + \varepsilon_\ell \|x^\ell\|(\|\bar{x}\| - \|x^\ell\|) + \langle y^\ell, H(\bar{x}) - \lambda^{-1}(z^\ell - y^\ell)\rangle \\ &\quad - \langle y^\ell - z^\ell, H(x^\ell)\rangle + \langle B(y^\ell), y^\ell - z^\ell\rangle - \langle B(y^\ell), y^\ell\rangle - \lambda^1 \|z^\ell - y^\ell\|^2 \\ &= \Delta_\ell. \end{aligned}$$

Suppose that $\|w^\ell\| \to \infty$ as $\ell \to \infty$, where $w^\ell = (x^\ell, y^\ell)$. Then the following three cases are possible.

Case 1:

$$\|x^\ell\| \to \infty, \ \ \|y^\ell\| \to \infty.$$

Then due to (A1)-(A5) we have $\Delta_\ell \to -\infty$, which is a contradiction.

Case 2:

$$\|x^\ell\| \to \infty, \ \ \|y^\ell\| \le C_1.$$

Then due to (A1)-(A5) it holds that

$$\Delta_\ell \le C - g^\ell + \varepsilon_\ell \|x^\ell\|(\|\bar{x}\| - \|x^\ell\|) \to -\infty,$$

which is a contradiction.

Case 3:

$$\|x^\ell\| \le C_2, \ \ \|y^\ell\| \to \infty.$$

Then due to (A1)-(A5), it holds that

$$\Delta_\ell \le C - \langle B(y^\ell), y^\ell\rangle \to -\infty,$$

which is again a contradiction. So, the sequence $\{(x^\ell, y^\ell)\}$ is bounded, hence it has limit points. Let (x', y') be an arbitrary pair of limit points of $\{(x^\ell, y^\ell)\}$. Then, taking the corresponding limit in (3.1), we obtain

$$\langle g', x - x' \rangle + \langle y', H(x) - H(x') \rangle \geq 0, \;\; \forall x \in X, g' \in G(x'),$$
$$\langle B(y') - H(x'), y - y' \rangle \geq 0, \;\; \forall y \in Y,$$

that is, (x', y') is a solution of system (1.1)-(1.2). The proof is complete. □

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