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CONDITIONS TO GUARANTEE THE EXISTENCE AND UNIQUENESS OF THE SOLUTION TO STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. The main aim of this paper show the conditions to guarantee the existence and uniqueness of the solution to stochastic differential equations. To make this stochastic analysis theory more understandable, we impose a weakened Hölder condition and a weakened linear growth condition. Furthermore, we give some properties of the solutions to the stochastic differential equations.

1. Introduction

Stochastic system has come to play an important role in many branches of natural and applied science where more and more researcher have encountered stochastic differential equations(short for SDEs). The problems of the existence and uniqueness of the solution to the SDEs has become an important field of study because the solution of the SDEs does not have an explicit

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expression except for linear cases as well as the question of the existence of stochastic integral part in the equations.

Mao [10] had investigated the SDEs in his book. The main one of this paper is the proof of the existence and uniqueness of the SDEs. In his book [10], He had introduced the stochastic differential equations studied by previous researchers;

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t), \tag{1.1}$$

on the closed interval $[t_0, T], t_0 \leq T$. And he obtained that if Lipschiz condition and linear growth condition hold, then the SDEs (1.1) had a unique solution $x(t)$, moreover, $x(t) \in \mathcal{M}^2([t_0, T]; R^{d\times m})$ which means that we denoted by $\mathcal{M}^2$ the family of processes $\{f(t)\}$ in $\mathcal{L}^p$ such that $E\int_{t_0}^T |f(t)|^2 dt < \infty$.

After that the study of the existence and uniqueness theorem for the SDEs has been developed into some new uniqueness theorem for SDEs under special conditions. See the references to this [2], [4], [5], and [12], [13], [15], [16]. Moreover, as for the studies related to this research, see [3], [6]-[11], [14], and references therein for details.

Especially, Wei et al. [16] obtained that if two condition (1.2) and (1.3) hold: For all $y, z \in R^d$ and $t \in [t_0, T]$, it follows that

$$|f(y,t) - f(z,t)|^2 \vee |g(y,t) - g(z,t)|^2 \leq \kappa\ (|y-z|^2), \tag{1.2}$$

where $\kappa(\cdot)$ is a concave non-decreasing function. For all $t \in [t_0, T]$, it follows that $f(0,t), g(0,t) \in R^d \times [t_0, T]$ such that

$$|f(0,t)|^2 \vee |g(0,t)|^2 \leq K, \tag{1.3}$$

then there exists a unique solution $x(t)$ to equation (1.1) and the solution belongs to $\mathcal{M}^2([t_0, T]; R^d)$.

And Bae et al. [2] obtained that if two condition (1.4) and (1.5) hold: For any $y, z \in R^d$ and $t \in [t_0, T]$, we assume that

$$|f(y,t) - f(z,t)|^2 \vee |g(y,t) - g(z,t)|^2 \leq \overline{K}|y-z|^{2\alpha}, \tag{1.4}$$

where $\overline{K}$ is a positive constant and $0 < \alpha \leq 1$ is a constant. For any $t \in [t_0, T]$ it follows that $f(0,t), g(0,t) \in \mathcal{L}^2([t_0, T])$ it follows that

$$|f(0,t)|^2 \vee |g(0,t)|^2 \leq K, \tag{1.5}$$

where K is a positive constant, then there exists a unique solution $x(t)$ to equation(1.1) and the solution belongs to $\mathcal{M}^2([t_0, T]; R^d)$.

In the paper [2], by employing non-Lipschitz condition and non-linear growth condition, authors established the results for d-dimensional stochastic functional differential equation.

Motivated by [2], [10], and [16], we will investigate the existence and uniqueness theorem of the solution for SDEs at a phase space $\mathcal{M}^2\left([t_0, T]; R^d\right)$ in this paper. We still take $t_0 \in R$ as our initial time throughout this paper. And we want to prove our main results as follows; first, under the weakened Hölder condition and the weakened linear growth condition, we estimate bounded of the solution for SDEs. Next, we prove the existence and uniqueness theorem of the solution for SDEs. Finally, we derived the estimate for the error between Picard iterations $\{x_n(t)\}$ and the unique solution $x(t)$ of SDEs.

2. Definitions and basic properties

Let $(\Omega, \mathcal{F}, P)$, throughout this paper unless otherwise specified, be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq t_0}$ satisfying the usual conditions (that is, it is right continuous and $\mathcal{F}_{t_0}$ contains all P-null sets). Let $|\cdot|$ denote Euclidean norm in R^n. If A is a vector or a matrix, its transpose is denoted by A^T; if A is a matrix, its trace norm is represented by $|A| = \sqrt{\text{trace}(A^T A)}$. Assume that $B(t)$ is an m-dimensional Brownian motion defined on complete probability space, that is, $B(t) = (B_1(t), B_2(t), ..., B_m(t))^T$.

Consider the d-dimensional stochastic differential equation of Itô type

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t) \quad on \quad t_0 \leq t \leq T \tag{2.1}$$

with initial value $x(t_0) = x_0$. By the definition of stochastic differential, this equation is equivalent to the following stochastic integral equation:

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s), s)ds + \int_{t_0}^{t} g(x(s), s)dB(s) \quad on \quad t_0 \leq t \leq T. \tag{2.2}$$

First, let us define the solution of the stochastic differential equations.

Definition 2.1. ([10]) An R^d-valued stochastic process $\{x(t)\}_{t_0\leq t\leq T}$ is called a solution of equation (2.1) if it has the following properties:

(i) $\{x(t)\}$ is continuous and $\mathcal{F}_t$-adapted;
(ii) $\{f(x(t), t)\} \in \mathcal{L}^1([t_0, T]; R^d)$ and $\{g(x(t), t)\} \in \mathcal{L}^2([t_0, T]; R^{d\times m})$;
(iii) equation (2.1) holds for every $t \in [t_0, T]$ with probability 1.

A solution $x(t)$ is said to be unique if any other solution $\bar{x}(t)$ is indistinguishable from $x(t)$, that is

$$P\{x(t) = \bar{x} \ \ for \ \ all \ \ t_0 \leq t \leq T\} = 1.$$

For the convenience of the reader, we state following lemmas.

Lemma 2.2. ([1, 10]) (Bihari's inequality). *Let $x(t)$ and $y(t)$ be non-negative continuous functions defined on R_+. Let $z(u)$ be a non-decreasing continuous* *function* R_+ *and* $z(u) > 0$ *on* $(0, \infty)$. *If*

$$x(t) \leq a + \int_0^t y(s)z(x(s))ds,$$

for $t \in R_+$, *where* $a \geq 0$ *is a constant, then for* $0 \leq t \leq t_1$,

$$x(t) \leq L^{-1}\left(L(a) + \int_0^t y(s)ds\right),$$

where $L(r) = \int_{r_0}^r \frac{ds}{z(s)}, \quad r > 0, \quad r_0 > 0$, *and* L^{-1} *is the inverse function of* L *and* $t_1 \in R_+$ *is chosen so that* $L(a) + \int_0^t y(s)ds \in Dom(L^{-1})$ *for all* $t \in R^+$ *lying in the interval* $0 \leq t \leq t_1$.

Lemma 2.3. ([10]) *Let* $p \geq 2$. *Let* $f \in \mathcal{M}^2([0,T]; R^{d\times m})$ *such that*

$$E\int_0^T |f(s)|^p ds < \infty.$$

Then

$$E\left|\int_0^T f(s)dB(s)\right|^p \leq \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} T^{\frac{p-2}{2}} E\int_0^T |f(s)|^p ds.$$

Lemma 2.4. ([10]) *If* $p \geq 2$, $f \in \mathcal{M}^2\left([0,T]; R^{d\times m}\right)$ *such that*

$$E\int_0^T |f(s)|^p ds < \infty,$$

then

$$E\left(\sup_{0\leq t\leq T}\left|\int_0^t f(s)dB(s)\right|^p\right) \leq \left(\frac{p^3}{2(p-1)}\right)^{\frac{p}{2}} T^{\frac{p-2}{p}} E\int_0^T |f(s)|^p ds.$$

Lemma 2.5. ([1, 10]) (Hölder's inequality) *If* $\frac{1}{p} + \frac{1}{q} = 1$ *for any* $p, q > 1$, $f \in \mathcal{L}^p$, *and* $g \in \mathcal{L}^q$, *then* $fg \in \mathcal{L}^1$ *and* $\int_a^b fg dx \leq \left(\int_a^b |f|^p dx\right)^{\frac{1}{p}} \left(\int_a^b |g|^q dx\right)^{\frac{1}{q}}$.

Lemma 2.6. ([1]) *Let* $a(t)$ *and* $u(t)$ *be continuous functions on* $[0,T]$. *Let* $K \geq 1$ *and* $0 < p \leq 1$ *be constants. If* $u(t) \leq K + \int_{t_0}^t a(s)u^p(s)ds$ *for* $t \in [t_0, T]$ *then*

$$u(t) \leq K \exp\left(\int_{t_0}^t a(s)ds\right).$$

Lemma 2.7. ([1]) (Stachurska's inequality) *Let $x(t)$ and $y(t)$ be non-negative continuous functions for $t \geq \alpha$, and let*

$$x(t) \leq a(t) + b(t) \int_\alpha^t y(s) x^p(s) ds,$$

$t \in J = [\alpha, \beta)$, where $\frac{a}{b}$ is non-decreasing function and $0 < p < 1$. Then

$$x(t) \leq a(t) \left(1 - (p-1) \left[\frac{a(t)}{b(t)} \right]^{p-1} \int_\alpha^t y(s) b^p(s) ds \right)^{\frac{-1}{p-1}}.$$

3. Existence of solution

In order to attain the solution of (2.1) we impose following assumptions:

(H1) For all $y, z \in R^d$ and $t \in [t_0, T]$, it follows that

$$|f(y,t) - f(z,t)|^2 \vee |g(y,t) - g(z,t)|^2 \leq \kappa\ (|y-z|^{2\alpha}), \tag{3.1}$$

where $0 < \alpha \leq 1$ and $\kappa(\cdot)$ is a concave non-decreasing function from R_+ to R_+ such that $\kappa(0) = 0$, $\kappa(u) > 0$, for $u > 0$ and $\int_{0+} \frac{1}{\kappa(u)} du = \infty$.

(H2) For all $t \in [t_0, T]$, it follows that $f(0,t), g(0,t) \in R^d \times [t_0, T]$ such that

$$|f(0,t)|^2 \vee |g(0,t)|^2 \leq K. \tag{3.2}$$

To demonstrate the generality of our results, let us illustrate it using a concave function $\kappa(\cdot)$. Let $K > 0$ and let $\delta \in (0,1)$ be sufficiently small. Define

$$\kappa_1(u) = Ku,\ u > 0;$$

$$\kappa_2(u) = \begin{cases} u \log(u^{-1}), & 0 \leq u < \delta \\ \delta \log(\delta^{-1}) + \dot{\kappa_2}(\delta-)(u-\delta), & u > \delta; \end{cases}$$

$$\kappa_3(u) = \begin{cases} u \log(u^{-1}) \log\log(u^{-1}), & 0 \leq u < \delta \\ \delta \log(\delta^{-1}) \log\log(\delta^{-1}) + \dot{\kappa_3}(\delta-)(u-\delta), & u > \delta. \end{cases}$$

They are all concave non-decreasing functions satisfying $\kappa_i(u) > 0$, for $u > 0$, and $\int_{0+} \frac{1}{\kappa_i(u)} du = \infty$. In particular, we see that condition (1.4) is a special case of our proposed condition (3.1).

Since our goal is to find the conditions that guarantee the existence and uniqueness of the solution to stochastic differential equation (2.1). We start with following an exponential estimate.

Lemma 3.1. *Assume that conditions* (3.1) *and* (3.2) *hold. If* $x(t)$ *is the solution of* (2.1), *then*

$$E\left(\sup_{t_0\le t\le T}|x(t)|^2\right)\le C_2\exp[6b(T-t_0)(T-t_0+4)], \tag{3.3}$$

where $C_1=3E|x_0|^2+6K(T-t_0+4)(T-t_0)$ *and* $C_2=C_1+6a(T-t_0+4)(T-t_0)\ge 1$.

Proof. For each number $n\ge 1$, define the stopping time

$$\tau_n=T\wedge\inf\{t\in[t_0,T]:|x(t)|\ge n\}.$$

Obviously, as $n\to\infty,\tau_n\uparrow T$ a.s. Let $x^n(t)=x(t\wedge\tau_n),t\in[t_0,T]$. Then $x^n(t)$ satisfy the following equation

$$x^n(t)=x_0+\int_{t_0}^t f(x^n(s),s)I_{[t_0,\tau_n]}(s)ds+\int_{t_0}^t g(x^n(s),s)I_{[t_0,\tau_n]}(s)dB(s).$$

Using the elementary inequality $(y+z+w)^2\le 3(y^2+z^2+w^2)$, we have the following

$$|x^n(t)|^2\le 3|x_0|^2+3\left|\int_{t_0}^t f(x^n,s)I_{[t_0,\tau_n]}(s)ds\right|^2+3\left|\int_{t_0}^t g(x^n,s)I_{[t_0,\tau_n]}(s)dBs\right|^2.$$

Taking the expectation on both sides, using the condition (3.1) and (3.2), Hölder's inequality, and the elementary inequality $(y+z)^2\le 2y^2+2z^2$, we have the following

$$\begin{aligned}
&E\left(\sup_{t_0\le s\le t}|x^n(s)|^2\right)\\
&\le 3E|x_0|^2+3E\sup_{t_0\le s\le t}\left|\int_{t_0}^s f(x^n(s),s)I_{[t_0,\tau_n]}(s)ds\right|^2\\
&\quad+3E\sup_{t_0\le s\le t}\left|\int_{t_0}^s g(x^n(s),s)I_{[t_0,\tau_n]}(s)dBs\right|^2\\
&\le 3E|x_0|^2+6(T-t_0)E\int_{t_0}^t\left(|f(x^n(s),s)-f(0,s)|^2+|f(0,s)|^2\right)ds\\
&\quad+24E\int_{t_0}^t\left((|g(x^n(s),s)-g(0,s)|^2+|g(0,s)|^2\right)ds\\
&\le C_1+6(T-t_0+4)E\int_{t_0}^t\kappa(|x^n(s)|^{2\alpha})ds,
\end{aligned}$$

where $C_1=3E|x_0|^2+6K(T-t_0+4)(T-t_0)$. If $\kappa(\cdot)$ is concave and $\kappa(0)=0$, then we can find the positive constants a and b such that $\kappa(u)\le a+bu$ for all

$u \geq 0$. Therefore, we find that

$$\begin{aligned} E\left(\sup_{t_0 \leq s \leq t} |x^n(s)|^2\right) &\leq C_1 + 6(T - t_0 + 4)E\int_{t_0}^{t}[a + b|x^n(s)|^{2\alpha}]ds, \\ &\leq C_2 + 6b(T - t_0 + 4)\int_{t_0}^{t} E|x^n(s)|^{2\alpha}ds, \end{aligned}$$

where $C_2 = C_1 + 6a(T - t_0 + 4)(T - t_0)$.

One further obtains that

$$E\left(\sup_{t_0 \leq s \leq t} |x^n(s)|^2\right) \leq C_2 + 6b(T - t_0 + 4)\int_{t_0}^{t} E\left(\sup_{t_0 \leq r \leq s} |x^n(r)|^{2\alpha}\right)ds.$$

By Lemma 2.6, we derive that

$$E\left(\sup_{t_0 \leq s \leq t} |x^n(s)|^2\right) \leq C_2 \exp\left[6b(T - t_0 + 4)(T - t_0)\right].$$

Letting t=T, it then follows that

$$E\left(\sup_{t_0 \leq s \leq T} |x^n(s)|^2\right) \leq C_2 \exp\left[6b(T - t_0 + 4)(T - t_0)\right].$$

This is as follows

$$E\left(\sup_{t_0 \leq s \leq T} |x(s \wedge \tau_n)|^2\right) \leq C_2 \exp\left[6b(T - t_0 + 4)(T - t_0)\right].$$

Consequently, we see that

$$E\left(\sup_{t_0 \leq t \leq \tau_n} |x(t)|^2\right) \leq C_2 \exp\left[6b(T - t_0 + 4)(T - t_0)\right].$$

Letting $n \to \infty$, it then implies the following inequality

$$E\left(\sup_{t_0 \leq t \leq T} |x(t)|^2\right) \leq C_2 \exp\left[6b(T - t_0 + 4)(T - t_0)\right],$$

which is the required inequality. The proof is complete. □

The following theorem gives an existence and uniqueness theorem of the solution to the stochastic differential equations under new conditions.

Theorem 3.2. *Assume that* (3.1) *and* (3.2) *hold. Then there exists a unique solution to the SDEs* (2.1)*. Moreover, the solution belongs to* $\mathcal{M}^2\left([t_0, T]; R^d\right)$.

Proof. **(uniqueness):** Let $x(t), \bar{x}(t)$ be any two solutions of the equation. By Lemma 3.1, we see $x(t), \bar{x}(t) \in \mathcal{M}^2([t_0, T]; R^d)$. Note that

$$x(t) - \bar{x}(t) = \int_{t_0}^{t}[f(x(s), s) - f(\bar{x}(s), s)]ds + \int_{t_0}^{t}[g(x(s), s) - g(\bar{x}(s), s)]dB(s).$$

By the elementary inequality $(y+z)^2 \leq 2y^2+2z^2$,we see that

$$\begin{aligned} |x(t)-\bar{x}(t)|^2 &\leq 2\left|\int_{t_0}^t [f(x(s),s)-f(\bar{x}(s),s)]ds\right|^2 \\ &\quad +2\left|\int_{t_0}^t [g(x(s),s)-g(\bar{x}(s),s)]dB(s)\right|^2. \end{aligned}$$

Taking the expectation on both sides, we find that

$$\begin{aligned} E\bigg(\sup_{t_0\leq s\leq t}|x(s)-\bar{x}(s)|^2\bigg) &\leq 2(t-t_0)E\int_{t_0}^t |f(x,s)-f(\bar{x},s)|^2 ds \\ &\quad +8E\int_{t_0}^t |g(x,s)-g(\bar{x},s)|^2 ds. \end{aligned}$$

By the condition (3.1), we can show that

$$E\bigg(\sup_{t_0\leq s\leq t}|x(s)-\bar{x}(s)|^2\bigg) \leq 2(T-t_0+4)E\int_{t_0}^t \bigg(\kappa(|x(s)-\bar{x}(s)|^{2\alpha})\bigg)ds.$$

Since $\kappa(\cdot)$ is concave, by the Jensen inequality, we have

$$E\kappa(|x(s)-\bar{x}(s)|^{2\alpha}) \leq \kappa(E(|x(s)-\bar{x}(s)|^{2\alpha})).$$

Therefore, this is induced as follows

$$E\bigg(\sup_{t_0\leq s\leq t}|x(s)-\bar{x}(s)|^2\bigg) \leq 2(T-t_0+4)\int_{t_0}^t \kappa\bigg(E(|x(s)-\bar{x}(s)|^{2\alpha})\bigg)ds.$$

Consequently, for any $\varepsilon>0$, we find that

$$E\bigg(\sup_{t_0\leq s\leq t}|x(s)-\bar{x}(s)|^2\bigg) \leq \varepsilon + 2(T-t_0+4)\int_{t_0}^t \kappa\bigg(E\sup_{t_0\leq r\leq s}|x-\bar{x}|^{2\alpha}\bigg)ds.$$

By the Bihari inequality, we deduces that, for all sufficiently small $\varepsilon>0$

$$E\sup_{t_0\leq s\leq t}|x(s)-\bar{x}(s)|^2 \leq G^{-1}[G(\varepsilon)+2(T-t_0+4)(T-t_0)], \tag{3.4}$$

where $G(r)=\int_1^r \frac{1}{\kappa_1(u)}du$ on $r>0$, $\kappa_1(u)=\kappa(u^\alpha)$, $z(t)=E\sup_{t_0\leq s\leq t}|x(s)-\overline{x}(s)|^2$ and $G^{-1}(\cdot)$ be the inverse function of $G(\cdot)$. By assumption $\int_{0+}\frac{1}{\kappa(u)}du=\infty$ and the definition of $\kappa(\cdot)$, we see that $\lim_{\varepsilon\downarrow 0}G(\varepsilon)=-\infty$ and then

$$\lim_{\varepsilon\downarrow 0}G^{-1}[G(\varepsilon)+2(T-t_0+4)(T-t_0)]=0.$$

Therefore, by letting $\varepsilon\to 0$ in (3.4), we have that

$$E\sup_{t_0\leq s\leq t}|x(s)-\overline{x}(s)|^2=0.$$

This implies that $x(t)=\overline{x}(t)$ for $t_0\leq t\leq T$. The uniqueness has been proved.

(existence): Next to check the existence, define $x_0(t) = x_0$, and for $n = 1, 2, \cdots$, define the Picard iterations

$$x_n(t) = x_0 + \int_{t_0}^{t} f(x_{n-1}(s), s)ds + \int_{t_0}^{t} g(x_{n-1}(s), s)dB(s),\ t_0 \le t \le T. \quad (3.5)$$

Obviously, $x_0(\cdot) \in \mathcal{M}^2([t_0, T]; R^d)$. Moreover, it is easy to show that $x_n(\cdot) \in \mathcal{M}^2([t_0, T]; R^d)$. Taking the expectation on both sides and using the inequality $|y + z + w|^p \le 3^{p-1}\left[|y|^p + |z|^p + |w|^p\right]$ we see that

$$\begin{aligned} E \sup_{t_0 \le s \le t} |x_n(s)|^2 &\le 3E|x_0|^2 + 3E \sup_{t_0 \le s \le t} \left| \int_{t_0}^{s} f(x_{n-1}, r)dr \right|^2 \\ &\quad + 3E \sup_{t_0 \le s \le t} \left| \int_{t_0}^{s} g(x_{n-1}, r)dB(r) \right|^2. \end{aligned}$$

By the Hölder inequality and the moment inequality, we have

$$\begin{aligned} E \sup_{t_0 \le s \le t} |x_n(s)|^2 &\le 3E|x_0|^2 + 3(T - t_0)E \int_{t_0}^{t} |f(x_{n-1}(s), s)|^2 ds \\ &\quad + 12E \int_{t_0}^{t} |g(x_{n-1}(s), s)|^2 ds. \end{aligned}$$

By the condition (3.1) and (3.2), we can show that

$$\begin{aligned} &E \sup_{t_0 \le s \le t} |x_n(s)|^2 \\ &\le 3E|x_0|^2 + 3(T - t_0)E \int_{t_0}^{t} |f(x_{n-1}(s), s) - f(0, s) + f(0, s)|^2 ds \\ &\quad + 12E \int_{t_0}^{t} |g(x_{n-1}(s), s) - g(0, s) + g(0, s)|^2 ds \\ &\le 3E|x_0|^2 + 6(T - t_0 + 4)E \int_{t_0}^{t} [\kappa(|x_{n-1}(s)|^{2\alpha}) + K]ds. \end{aligned}$$

If $\kappa(\cdot)$ is concave and $\kappa(0) = 0$, we can find the positive constants a and b such that $\kappa(u) \le a + bu$ for all $u \ge 0$. Therefore

$$\begin{aligned} &E \sup_{t_0 \le s \le t} |x_n(s)|^2 \\ &\le 3E|x_0|^2 + 6(T - t_0 + 4)E \int_{t_0}^{t} (K + a + b|x_{n-1}(s)|^{2\alpha})ds \\ &\le C_3 + 6b(T - t_0 + 4) \int_{t_0}^{t} E \sup_{t_0 \le r \le s} |x_{n-1}(r)|^{2\alpha} ds, \end{aligned}$$

where $C_3 = 3E|x_0|^2 + 6(K+a)(T-t_0+4)(T-t_0)$. Hence for any $k \geq 1$, we can derive that

$$\begin{aligned}
&\max_{1\leq n\leq k} E \sup_{t_0\leq s\leq t} |x_n(t)|^2 \\
&\leq C_3 + 6b(T-t_0+4)\int_{t_0}^{t}\left(E|x_0|^{2\alpha} + \max_{1\leq n\leq k} E \sup_{t_0\leq r\leq s} |x_n(r)|^{2\alpha}\right)ds \\
&\leq C_4 + 6b(T-t_0+4)\int_{t_0}^{t} \max_{1\leq n\leq k} E \sup_{t_0\leq r\leq s} |x_n(r)|^{2\alpha} ds,
\end{aligned}$$

where $C_4 = C_3 + 6b(T-t_0+4)(T-t_0)E|x_0|^{2\alpha}$. From the Lemma 2.6, we have

$$\max_{1\leq n\leq k} E|x_n(t)|^2 \leq C_4 \exp[6b(T-t_0+4)(T-t_0)].$$

Since k is arbitrary, for all $n = 0, 1, 2, ...$, we deduce that

$$E|x_n(t)|^2 \leq C_4 \exp[6b(T-t_0+4)(T-t_0)],$$

which shows the boundedness of the sequence $\{x_n(t)\}$.

Next we check that the sequence $\{x_n(t)\}$ is Cauchy. For all $n \geq 0$ and $t_0 \leq t \leq T$, we have

$$\begin{aligned}
x_{n+1}(t) - x_n(t) &= \int_{t_0}^{t}[f(x_n(s), s) - f(x_{n-1}(s), s)]ds \\
&\quad + \int_{t_0}^{t}[g(x_n(s), s) - g(x_{n-1}(s), s)]dB(s).
\end{aligned}$$

Using the elementary inequality $(y+z)^2 \leq 2y^2 + 2z^2$ and taking the expectation on both sides, we derive that

$$\begin{aligned}
&E \sup_{t_0\leq s\leq t} |x_{n+1}(s) - x_n(s)|^2 \\
&\leq 2E \sup_{t_0\leq s\leq t}\left|\int_{t_0}^{s}[f(x_n(r), r) - f(x_{n-1}(r), r)]dr\right|^2 \\
&\quad + 2E \sup_{t_0\leq s\leq t}\left|\int_{t_0}^{s}[g(x_n(r), r) - g(x_{n-1}(r), r)]dB(r)\right|^2.
\end{aligned}$$

By Hölder's inequality, Jensen's inequality, Lemma 2.4 and condition (3.1), we can show that

$$
\begin{aligned}
&E \sup_{t_0 \le s \le t} |x_{n+1}(s) - x_n(s)|^2 \\
&\le 2(T - t_0) E \int_{t_0}^{t} |f(x_n(s), s) - f(x_{n-1}(s), s)|^2 ds \\
&\quad + 8E \int_{t_0}^{t} |g(x_n(s), s) - g(x_{n-1}(s), s)|^2 ds \\
&\le 2(T - t_0 + 4) \int_{t_0}^{t} \kappa[E(|x_n(s) - x_{n-1}(s)|^{2\alpha})] ds.
\end{aligned}
$$

This yields that

$$
\begin{aligned}
&\limsup_{n\to\infty} E \sup_{t_0 \le s \le t} |x_{n+1}(s) - x_n(s)|^2 \\
&\le 2(T - t_0 + 4) \int_{t_0}^{t} \kappa[\limsup_{n\to\infty} E(|x_n(s) - x_{n-1}(s)|^{2\alpha})] ds.
\end{aligned}
$$

Let $z(t) = \limsup_{n\to\infty} E \sup_{t_0 \le s \le t} |x_{n+1}(s) - x_n(s)|^2$. Then we get

$$
z(t) \le \varepsilon + 2(T - t_0 + 4) \int_{t_0}^{t} \kappa[z(s)^{\alpha}] ds.
$$

By the Bihari inequality, we deduce that, for all sufficiently small $\varepsilon > 0$

$$
z(t) \le G^{-1}[G(\varepsilon) + 2(T - t_0 + 4)(T - t_0)], \tag{3.6}
$$

where $G(r) = \int_1^r \frac{1}{\kappa_1(u)} du$ on $r > 0$, $\kappa_1(u) = \kappa(u^{\alpha})$,

$$
z(t) = \limsup_{n\to\infty} E \sup_{t_0 \le s \le t} |x_{n+1}(s) - x_n(s)|^2,
$$

and $G^{-1}(\cdot)$ is the inverse function of $G(\cdot)$. By assumption $\int_{0+} \frac{1}{\kappa(u)} du = \infty$ and the definition of $\kappa(\cdot)$, we see that $\lim_{\varepsilon \downarrow 0} G(\varepsilon) = -\infty$ and then

$$
\lim_{\varepsilon \downarrow 0} G^{-1}[G(\varepsilon) + 2(T - t_0 + 4)(T - t_0)] = 0.
$$

Therefore, by letting $\varepsilon \to 0$ in (3.6), we derive that

$$
\limsup_{n\to\infty} E \sup_{t_0 \le s \le t} |x_{n+1}(s) - x_n(s)|^2 = 0.
$$

This show the sequence $\{x_n(t)\}$ is Cauchy sequence in L^2. Hence, as $n \to \infty$, $x_n(t) \to x(t)$ that is $E|x_n(t) - x(t)|^2 \to 0$. Therefore, we obtain that $x(t) \in \mathcal{M}^2([t_0, T]; R^d)$. It remains to show that $x(t)$ satisfies equation (2.1). Note that

$$
\begin{aligned}
&E\left|\int_{t_0}^{t}[f(x_n(s),s)-f(x(s),s)]ds\right|^2+E\left|\int_{t_0}^{t}[g(x_n(s),s)-g(x(s),s)]dB(s)\right|^2\\
&\le (T-t_0)E\int_{t_0}^{t}|f(x_n,s)-f(x,s)|^2ds+E\int_{t_0}^{t}|g(x_n,s)-g(x,s)|^2ds\\
&\le (T-t_0+1)\int_{t_0}^{t}\kappa(E|x_n(s)-x(s)|^{2\alpha})ds.
\end{aligned}
$$

The right-hand term of the inequality converge to zero, as $n\to\infty$. Noting that sequence $\{x_n(t)\}$ is uniformly converge on $[t_0,T]$, it means that $E\sup_{t_0\le s\le t}|x_n(s)-x(s)|^{2\alpha}\to 0$. Hence, taking limits on both sides in the Picard sequence, we obtain that

$$
x(t)=x_0+\int_{t_0}^{t}f(x(s),s)ds+\int_{t_0}^{t}g(x(s),s)dB(s)\quad on\quad t_0\le t\le T.
$$

This above expression demonstrates that $x(t)$ is the solution of (2.1). So far, the existence of theorem is complete. □

4. Continuous of the solutions

The following theorem shows that the Picard sequence of the equation (2.1) has a continuity under new conditions.

Theorem 4.1. *Assume that conditions* (3.1) *and* (3.2) *hold. Let* $x_n(t)$ *be the Picard iteration defined by* (3.5). *Then for all* $n\ge 1$, *it follows that*

$$
E\left(\sup_{t_0\le s\le T}|x_n(s)-x_{n-1}(s)|^2\right)\le C_8,
$$

where $C_5=4(T-t_0+4)(T-t_0)[a+b|x_0|^{2\alpha}+K]$, $C_6=2(T-t_0+4)(T-t_0)(a+bC_5^{\alpha})$, $C_7=C_6\left(1-2b(\alpha-1)C_6^{\alpha-1}(T-t_0)(T-t_0+4)\right)^{\frac{1}{1-\alpha}}$ *and* $C_8=C_5+C_7$.

Proof. Taking the expectation on both sides, and using the Hölder inequality, conditions (3.1) and (3.2), and the elementary inequality $(y+z)^2\le 2y^2+2z^2$, we derive that

$$
E\sup_{t_0\le s\le t}|x_1(s)-x_0|^2\le 4(T-t_0+4)E\int_{t_0}^{t}[\kappa(|x_0|^{2\alpha})+K]ds.
$$

If $\kappa(\cdot)$ is concave and $\kappa(0) = 0$, then we can find the positive constants a and b such that $\kappa(u) \leq a + bu$ for all $u \geq 0$. Therefore, this in induced as follows

$$E \sup_{t_0 \leq s \leq t} |x_1(s) - x_0|^2 \leq 4(T - t_0 + 4)E \int_{t_0}^{t} [a + b|x_0|^{2\alpha} + K]ds$$
$$\leq C_5,$$

where $C_5 = 4(T - t_0 + 4)(T - t_0)[a + b|x_0|^{2\alpha} + K]$. On the other hand, the properties maximum, we take

$$\max_{1 \leq n \leq k} E \sup_{t_0 \leq s \leq t} |x_{n+1}(s) - x_n(s)|^2$$
$$\leq C_6 + 2b(T - t_0 + 4) \int_{t_0}^{t} \max_{1 \leq n \leq k} E \sup_{t_0 \leq r \leq s} (|x_{n+1}(r) - x_n(r)|^{2\alpha})ds,$$

where $C_6 = 2(T - t_0 + 4)(T - t_0)(a + bC_5^{\alpha})$. Therefore, by the Stachurska inequality, we see that

$$\max_{1 \leq n \leq k} E \sup_{t_0 \leq s \leq t} |x_{n+1}(s) - x_n(s)|^2$$
$$\leq C_6 \left(1 - 2b(\alpha - 1)C_6^{\alpha - 1}(T - t_0)(T - t_0 + 4)\right)^{\frac{1}{1-\alpha}} := C_7.$$

That is, we have that

$$\begin{aligned} &\max_{1 \leq n \leq k} E \sup_{t_0 \leq s \leq T} |x_n(s) - x_{n-1}(s)|^2 \\ &\leq E \sup_{t_0 \leq s \leq T} |x_1(s) - x_0|^2 + \max_{1 \leq n \leq k} E \sup_{t_0 \leq s \leq T} |x_{n+1}(s) - x_n(s)|^2 \\ &\leq C_5 + C_7 \qquad (4.1) \\ &:= C_8, \end{aligned}$$

which is the required inequality. The proof is complete. □

The following theorem shows that an estimate for a difference between the approximate solution $x_n(t)$ and the accurate solution $x(t)$.

Theorem 4.2. *Assume that conditions* (3.1) *and* (3.2) *hold. Let $x(t)$ be the unique solution of* (2.1)*, and $x_n(t)$ be the Picard iteration defined by* (3.5)*. Then for all $n \geq 1$, we have the following characteristics*

$$E\left(\sup_{t_0 \leq t \leq T} |x_n(t) - x(t)|^2\right) \leq C_9 \left(1 - 4b(\alpha - 1)C_9^{\alpha - 1}(T - t_0 + 4)(T - t_0)\right)^{\frac{1}{1-\alpha}},$$

where $C_9 = 4(T - t_0 + 4)(T - t_0)(2a + bC_5^{\alpha})$.

Proof. From the Picard sequence and the accurate solution $x(t)$, we have

$$x_n(t) - x(t) = \int_{t_0}^{t} [f(x_{n-1}(s), s) - f(x(s), s)]ds \\ + \int_{t_0}^{t} [g(x_{n-1}(s), s) - g(x(s), s)]dB(s).$$

Taking expectation and by Hölder's inequality and condition (3.1), thus we have

$$E \sup_{t_0 \le s \le t} |x_n(s) - x(s)|^2 \\ \le 4(T - t_0 + 4) \int_{t_0}^{t} [\kappa(E|x_n(s) - x_{n-1}(s)|^{2\alpha}) + \kappa(E|x_n(s) - x(s)|^{2\alpha})]ds.$$

By Theorem 4.1, we get

$$E \sup_{t_0 \le s \le t} |x_n(s) - x(s)|^2 \le 4(T - t_0 + 4) \int_{t_0}^{t} [\kappa(C_8^{\alpha}) + \kappa(E|x_n(s) - x(s)|^{2\alpha})]ds.$$

If $\kappa(\cdot)$ is concave and $\kappa(0) = 0$, then we can find the positive constants a and b such that $\kappa(u) \le a + bu$ for all $u \ge 0$. Therefore, we obtain that

$$E \sup_{t_0 \le s \le t} |x_n(s) - x(s)|^2 \\ \le C_9 + 4b(T - t_0 + 4) \int_{t_0}^{t} E \sup_{t_0 \le r \le s} |x_n(r) - x(r)|^{2\alpha} dr,$$

where $C_9 = 4(T - t_0 + 4)(T - t_0)(2a + bC_8^{\alpha})$. By the Stachurska inequality, we find that

$$E \sup_{t_0 \le s \le t} |x_n(s) - x(s)|^2 \le C_9 \left(1 - 4b(\alpha - 1)C_9^{\alpha-1}(T - t_0 + 4)(T - t_0)\right)^{\frac{1}{1-\alpha}}.$$

Thus, we derive that

$$E\left(\sup_{t_0 \le t \le T} |x_n(t) - x(t)|^2\right) \le C_9 \left(1 - 4b(\alpha - 1)C_9^{\alpha-1}(T - t_0 + 4)(T - t_0)\right)^{\frac{1}{1-\alpha}},$$

where $C_9 = 4(T - t_0 + 4)(T - t_0)(2a + bC_5^{\alpha})$, which is the required inequality. The proof is complete. □

The following theorem shows that an exponential estimate for accurate solution of SDEs.

Theorem 4.3. *Let $p \geq 2$ and $x_0 \in \mathcal{L}^p\left(\Omega; R^d\right)$. Assume that the conditions* (3.1) *and* (3.2) *hold. Then, we have the following characteristics*

$$E \sup_{t_0 \leq t \leq T} |x(t)|^p \leq C_{11}\left(1 - 3^{\frac{p-2}{2}} b^{\frac{p}{2}} C_{10} C_{11}^{\alpha-1}(\alpha-1)(T-t_0)\right)^{\frac{1}{1-\alpha}} := C_{12},$$

where $C_{10} = 2^{\frac{p}{2}} 3^{p-1}\left[(T-t_0)^{p-1} + \left(\frac{p^3}{2(p-1)}\right)^{\frac{p}{2}}(T-t_0)^{\frac{p-2}{p}}\right]$ *and* $C_{11} = 3^{p-1}E|x_0|^p + 3^{\frac{p-2}{2}} C_{10}(T-t_0)(a^{\frac{p}{2}} + K^{\frac{p}{2}})$.

Proof. By the elementary inequality $|y+z+w|^p \leq 3^{p-1}(|y|^p + |z|^p + |w|^p)$, Hölder's inequality and Lemma 2.4, we have that

$$E \sup_{t_0 \leq s \leq t} |x(s)|^p \leq 3^{p-1}E|x_0|^p + C_{10}E\int_{t_0}^{t} [\kappa(|x(s)|^{2\alpha}) + K]^{\frac{p}{2}} ds,$$

where $C_{10} = 2^{\frac{p}{2}} 3^{p-1}\left[(T-t_0)^{p-1} + \left(\frac{p^3}{2(p-1)}\right)^{\frac{p}{2}}(T-t_0)^{\frac{p-2}{p}}\right]$. If $\kappa(\cdot)$ is concave and $\kappa(0) = 0$, then we can find the positive constants a and b such that $\kappa(u) \leq a + bu$ for all $u \geq 0$. Therefore, we find that

$$E \sup_{t_0 \leq s \leq t} |x(s)|^p \leq C_{11} + 3^{\frac{p-2}{2}} b^{\frac{p}{2}} C_{10} \int_{t_0}^{t} E \sup_{t_0 \leq r \leq s} |x(r)|^{p\alpha} ds,$$

where $C_{11} = 3^{p-1}E|x_0|^p + 3^{\frac{p-2}{2}} C_{10}(T-t_0)(a^{\frac{p}{2}} + K^{\frac{p}{2}})$. By Stachurska's inequality, we derive that

$$E \sup_{t_0 \leq s \leq T} |x(s)|^p \leq C_{11}\left(1 - 3^{\frac{p-2}{2}} b^{\frac{p}{2}} C_{10} C_{11}^{\alpha-1}(\alpha-1)(T-t_0)\right)^{\frac{1}{1-\alpha}}.$$

which is the required inequality. The proof is complete. □

The following theorem shows that L^p-continuity of the accurate solution of SPDs.

Theorem 4.4. *Let $p \geq 2$ and $x_0 \in \mathcal{L}^p\left(\Omega; R^d\right)$. Assume that conditions* (3.1) *and* (3.2) *hold. Then, we have the following characteristics*

$$E|x(t) - x(s)|^p \leq 3^{\frac{p-2}{2}} C_{13}[a^{\frac{p}{2}} + K^{\frac{p}{2}} + b^{\frac{p}{2}} C_{12}^{\alpha}](t-s)^{\frac{p}{2}},$$

for all $t_0 \leq s < t \leq T$, where $C_{13} = 2^{\frac{p}{2}}[2^{p-1}(T-t_0)^{\frac{p}{2}} + \frac{1}{2}(2p(p-1))^{\frac{p}{2}}]$.

Proof. By the elemantary inequality $|y+z|^p \leq 2^{p-1}(|y|^p + |z|^p)$, it is easy to see that

$$E|x(t) - x(s)|^p \leq 2^{p-1}E\left|\int_s^t f(x(r), r)dr\right|^p + 2^{p-1}E\left|\int_s^t g(x(r), r)dB(r)\right|^p.$$

Using Hölder's inequality, Lemma 2.3, condition (3.1) and (3.2), we have that

$$
\begin{aligned}
&E|x(t)-x(s)|^p \\
&\le [2(t-s)]^{p-1} E\int_s^t [2|f(x(r),r)-f(0,r)|^2+2|f(0,r)|^2]^{\frac{p}{2}}dr \\
&\quad +\frac{1}{2}\left(2p(p-1)\right)^{\frac{p}{2}}(t-s)^{\frac{p-2}{2}} E\int_s^t [2|g(x(r),r)-g(0,r)|^2+2|g(0,r)|^2]^{\frac{p}{2}}dr \\
&\le C_{13}(t-s)^{\frac{p-2}{2}} E\int_s^t [\kappa(|x(r)|^{2\alpha})+K]^{\frac{p}{2}}dr,
\end{aligned}
$$

where $C_{13}=2^{\frac{p}{2}}[2^{p-1}(T-t_0)^{\frac{p}{2}}+\frac{1}{2}(2p(p-1))^{\frac{p}{2}}]$.

If $\kappa(\cdot)$ is concave and $\kappa(0)=0$, then we can find the positive constants a and b such that $\kappa(u)\le a+bu$ for all $u\ge 0$. Therefore, we derive that

$$E|x(t)-x(s)|^p \le C_{13}(t-s)^{\frac{p-2}{2}} E\int_s^t [a+b|x(r)|^{2\alpha}+K]^{\frac{p}{2}}dr.$$

By the elemantary inequality $|y+z+w|^p\le 3^{p-1}(|y|^p+|z|^p+|w|^p)$, it is easy to see that

$$
\begin{aligned}
&E|x(t)-x(s)|^p \\
&\le 3^{\frac{p-2}{2}} C_{13}(t-s)^{\frac{p-2}{2}} E\int_s^t [a^{\frac{p}{2}}+b^{\frac{p}{2}}|x(r)|^{p\alpha}+K^{\frac{p}{2}}]dr \\
&\le 3^{\frac{p-2}{2}} C_{13}\left\{[a^{\frac{p}{2}}+K^{\frac{p}{2}}](t-s)^{\frac{p}{2}}+b^{\frac{p}{2}}(t-s)^{\frac{p-2}{2}} E\int_s^t [E\sup_{t_0\le u\le r}|x(u)|^p]^{\alpha}dr\right\}.
\end{aligned}
$$

By Theorem 4.3, we have that

$$
\begin{aligned}
&E|x(t)-x(s)|^p \\
&\le 3^{\frac{p-2}{2}} C_{13}\left\{[a^{\frac{p}{2}}+K^{\frac{p}{2}}](t-s)^{\frac{p}{2}}+b^{\frac{p}{2}}(t-s)^{\frac{p-2}{2}} E\int_s^t C_{12}^{\alpha}dr\right\} \\
&\le 3^{\frac{p-2}{2}} C_{13}[a^{\frac{p}{2}}+K^{\frac{p}{2}}+b^{\frac{p}{2}}C_{12}^{\alpha}](t-s)^{\frac{p}{2}},
\end{aligned}
$$

which is the required inequality. The proof is complete. □

Remark 4.5. Theorem 3.2 shown that the Picard iteration sequence $x_n(t)$ converge to the unique solution $x(t)$ of the SDEs (2.1). In Theorem 4.1, we gives that Picard sequence of the equation(2.1) has a continuity under the conditions. Theorem 4.2 shows that one can use the Picard iteration procedure to obtain the approximate solution of the systems give the estimate for the error of the approximation. Also in Theorem 4.4, we show that the accurate solution of the SDEs has a L^p-continuity.

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