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CHARACTERIZATIONS ON LINEAR N-NORMED SPACES

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Abstract. In this paper, we get if $f : X \to Y$ preserves equality of n-distance with X and Y be two linear n-normed spaces, f is n-continuous at some $\theta \neq x_0 \in X$, then f is affine.

1. Introduction

In 1989, Misiak [8,9] defined n-normed spaces and investigated the properties of these spaces. The concept of an n-normed space is a generalization of the concepts of a normed space and of a 2-normed space. In 2004, Chu et al. [3] defined the concept of n-isometry which is suitable for representing the notion of n-distance preserving mappings in linear n-normed spaces and studied the Aleksandrov problem in linear n-normed spaces.

S.Mazur and S.Ulam [10] proved the theorem: Every isometry of a real normed linear space onto a real normed linear space is a linear mapping up to translation. The property is not true for complex normed vector spaces. The hypothesis of surjectivity is essential. Without this assumption Baker [1] proved that every isometry from a real normed space into a strictly convex normed space is linear up to translation. Chu [2] proved that the Mazur-Ulam theorem holds when X and Y are linear 2-normed spaces. Chu et al. [4] proved that the n-isometry mapped to a linear n-normed space is affine.

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Weijia Jia[7] proved that under some conditions, a 2-continuous function is also a 2-isometry. In this paper, we replace n-isometry by the more general notion of equality of n-distance preserving mappings and give the definition of n-continuous. At last, we get under some conditions, a n-continuous function is also a n-isometry.

2. Definitions and Lemmas

In this section, we will give some definitions and lemmas in linear n-normed spaces.

Definition 2.1. [3] *Let X be a real linear space with dim $X \geq n$ and $\|\cdot, \ldots, \cdot\| : X^n \rightarrow R$ a function. Then $(X, \|\cdot, \ldots, \cdot\|)$ is called a linear n-normed space if*

(nN_1) $\|x_1, \ldots, x_n\| = 0 \Longleftrightarrow x_1, \ldots, x_n$ *are linearly dependent,*
(nN_2) $\|x_1, \ldots, x_n\| = \|x_{j1}, \ldots, x_{jn}\|$ *for every permutation $(j1, \ldots, jn)$ of $(1, \ldots, n)$,*
(nN_3) $\|\alpha x_1, \ldots, x_n\| = |\alpha| \|x_1, \ldots, x_n\|$,
(nN_4) $\|x+y, x_2, \ldots, x_n\| \leq \|x, x_2, \ldots, x_n\| + \|y, x_2, \ldots, x_n\|$
for all $\alpha \in R$ and $x, y, x_1, \ldots, x_n \in X$.

The function $\|\cdot, \ldots, \cdot\|$ is called an n-norm on X.

Definition 2.2 ([3]). *Let X and Y be linear n-normed spaces and $f : X \rightarrow Y$ be a mapping. We call f an n-isometry if*
$\|x_1 - x_0, \ldots, x_n - x_0\| = \|f(x_1) - f(x_0), \ldots, f(x_n) - f(x_0)\|$
for all $x_0, x_1, \ldots, x_n \in X$.

Definition 2.3. *Let X and Y be linear n-normed spaces, $x_0 \in X$ and $f : X \rightarrow Y$ be a mapping. Then f is said to be n-continuous at x_0 if for every $\varepsilon > 0$, there exists positive real number δ such that*
$\|x_1 - x_0, \ldots, x_n - x_0\| < \delta$ *implies* $\|f(x_1) - f(x_0), \ldots, f(x_n) - f(x_0)\| < \varepsilon$
and f is said to be n-continuous on X if f is n-continuous at x for all $x \in X$.

Definition 2.4. *Let X and Y be linear n-normed spaces and $f : X \rightarrow Y$ be a mapping. Then f is said to preserve equality of n-distance if and only if there exists a function $p : R_+^0 \rightarrow R_+^0$ such that for each $x_0, x_1, \ldots, x_n \in X$*
$\|f(x_1) - f(x_0), \ldots, f(x_n) - f(x_0)\| = p(\|x_1 - x_0, \ldots, x_n - x_0\|)$
The function p is called the gauge function for f.

Definition 2.5 ([3]). *The points $x_0, x_1, \ldots, x_n$ of X are said to be n-collinear if for every i, $\{x_j - x_i | 0 \leq j \neq i \leq n\}$ is linearly dependent.*

Remark 2.6 ([3]). *The points x_0, x_1 and x_2 are said to be 2-collinear if and only if $x_2 - x_0 = t(x_1 - x_0)$ for some real number t.*

Lemma 2.7 ([3]). *Let x_i be an element of a linear n-normed space X for every $i \in \{1, \ldots, n\}$ and γ be a real number. Then*
$\|x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n\| = \|x_1, \ldots, x_i, \ldots, x_j + \gamma x_i, \ldots, x_n\|$
for all $1 \leq i \neq j \leq n$.

From now on, unless otherwise specified, let X and Y be real linear n-normed spaces, $f : X \to Y$ be a mapping. X and Y have dimensions greater than $n - 1$. Let $x_0, x_1, \ldots, x_n$ be points of X.

3. Main Results

Lemma 3.1. *For x_0, $x_0' \in X$, if x_0 and x_0' are linearly dependent with the same direction, that is, $x_0' = \alpha x_0$ for some $\alpha > 0$, then*
$\|x_0 + x_0', x_1, \ldots, x_{n-1}\| = \|x_0, x_1, \ldots, x_{n-1}\| + \|x_0', x_1, \ldots, x_{n-1}\|$
for all $x_1, \ldots, x_{n-1} \in X$.

Proof. Let $x_0' = \alpha x_0$ for some $\alpha > 0$. Then we have

$$\begin{aligned}
\|x_0 + x_0', x_1, \ldots, x_{n-1}\| &= \|x_0 + \alpha x_0, x_1, \ldots, x_{n-1}\| \\
&= (1 + \alpha)\|x_0, x_1, \ldots, x_{n-1}\| \\
&= \|x_0, x_1, \ldots, x_{n-1}\| + \alpha\|x_0, x_1, \ldots, x_{n-1}\| \\
&= \|x_0, x_1, \ldots, x_{n-1}\| + \|x_0', x_1, \ldots, x_{n-1}\|
\end{aligned}$$

for all $x_1, \ldots, x_{n-1} \in X$ □

Remark 3.2. *For every $\gamma \in R_+^0$, $\theta \neq x_0 \in X$, there exist $x_1, \ldots, x_{n-1}$ such that*

$$||x_0, x_1, \ldots, x_{n-1}|| = \gamma.$$

Indeed, assume $x_1', \ldots, x_{n-1}'$ are linear independent to x_0. Then

$$||x_0, x_1', \ldots, x_{n-1}'|| = \rho > 0.$$

We obtain

$$||x_0, \sqrt[n-1]{\frac{\gamma}{\rho}} x_1', \ldots, \sqrt[n-1]{\frac{\gamma}{\rho}} x_{n-1}'|| = \gamma$$

Let

$$x_i = \sqrt[n-1]{\frac{\gamma}{\rho}} x_i' (i = 1, \ldots, n - 1).$$

Theorem 3.3. *If $x_0, x_1, \ldots, x_n \in X$ and $\|x_1 - x_0,\ x_2 - x_0,\ \ldots\ ,\ x_n - x_0\| \neq 0$, that is $x_0, x_1, \ldots, x_n$ are not collinear, then for any $0 \neq \gamma \in R^0_+$, there exists an element $\omega \in X$ such that*

$$\begin{aligned}&||x_0-\omega, x_1-\omega, x_2-\omega, \ldots, x_{n-1}-\omega||\\&=\gamma\\&=||x_1-\omega, x_2-\omega, x_3-\omega, \ldots, x_n-\omega||\end{aligned}$$

Proof. By hypothesis, we have

$$\beta \triangleq ||x_1-x_0, x_2-x_0, \ldots, x_n-x_0|| > 0.$$

Set $\omega = x_1 + \frac{2\gamma}{\beta}(\frac{x_0+x_n}{2} - x_1)$, we have

$$\begin{aligned}&||x_0-\omega, x_1-\omega, \ldots, x_{n-1}-\omega||\\&=||x_0-x_1-\frac{2\gamma}{\beta}(\frac{x_0+x_n}{2}-x_1), -\frac{2\gamma}{\beta}(\frac{x_0+x_n}{2}-x_1), \ldots,\\&\quad x_{n-1}-x_1-\frac{2\gamma}{\beta}(\frac{x_0+x_n}{2}-x_1)||\\&=||x_0-x_1, -\frac{2\gamma}{\beta}(\frac{x_0+x_n}{2}-x_1), x_2-x_1, \ldots, x_{n-1}-x_1||\\&=\frac{\gamma}{\beta}||x_0-x_1, x_0-x_1+x_n-x_1, x_2-x_1, \ldots, x_{n-1}-x_1||\\&=\frac{\gamma}{\beta}||x_0-x_1, x_n-x_1, x_2-x_1, \ldots, x_{n-1}-x_1||\\&=\frac{\gamma}{\beta}||x_1-x_0, x_2-x_0, \ldots, x_n-x_0||\\&=\gamma\end{aligned}$$

and

$$\begin{aligned}&||x_1-\omega, x_2-\omega, \ldots, x_n-\omega||\\&=||-\frac{2\gamma}{\beta}(\frac{x_0+x_n}{2}-x_1), x_2-x_1-\frac{2\gamma}{\beta}(\frac{x_0+x_n}{2}-x_1), \ldots,\\&\quad x_n-x_1-\frac{2\gamma}{\beta}(\frac{x_0+x_n}{2}-x_1)||\\&=\frac{\gamma}{\beta}||x_0-x_1+x_n-x_1, x_2-x_1, x_3-x_1, \ldots, x_n-x_1||\\&=\frac{\gamma}{\beta}||x_0-x_1, x_2-x_1, x_3-x_1, \ldots, x_n-x_1||\\&=\frac{\gamma}{\beta}||x_1-x_0, x_2-x_0, x_3-x_0, \ldots, x_n-x_0||\\&=\gamma\end{aligned}$$

□

Because of this theorem, some results in [3] can be simplified in condition.

Lemma 3.4 ([3]). *Assume that if x, y and z are 2-collinear then $f(x)$, $f(y)$ and $f(z)$ are 2-collinear and that f satisfies (nDOPP). Then f preserves the n-distance $\frac{1}{k}$ for any positive integer k.*

Theorem 3.5 ([3]). *Assume that if $x_0, x_1, \ldots, x_m$ are m-collinear, then $f(x_0), f(x_1), \ldots, f(x_m)$ are m-collinear, $m = 2, n$, and that if $y_1 - y_2 = \alpha(y_3 - y_2)$ for some $\alpha \in (0, 1]$ then $f(y_1) - f(y_2) = \beta(f(y_3) - f(y_2))$ for some $\beta \in (0, 1]$. If f satisfies (nDOPP), then f is an n-isometry.*

Lemma 3.6. *Let X and Y be two linear n-normed spaces, $f : X \to Y$ preserves equality of n-distance, p is the gauge function for f. Then*

(1) *$p(0) = 0$, that is, if $x_0, x_1, \ldots, x_n$ are n-collinear, then $f(x_0), f(x_1), \ldots, f(x_n)$ are n-collinear.*
(2) *If f is n-continuous at $\theta \neq x_0 \in X$, then p is continuous.*

Proof. (1) $p(0)=p(\|u, \ldots, u\|)=\|f(u) - f(\theta), \ldots, f(u) - f(\theta)\|=0$.
If $x_0, x_1, \ldots, x_n$ are n-collinear, then

$$||x_1 - x_0, \ldots, x_n - x_0|| = 0.$$

Because $p(0) = 0$, that is

$$||f(x_1) - f(x_0), \ldots, f(x_n) - f(x_0)|| = 0.$$

Then $f(x_1) - f(x_0), \ldots, f(x_n) - f(x_0)$ are linearly dependent. That is, $f(x_0), f(x_1), \ldots, f(x_n)$ are n-collinear.
(2) For any $\varepsilon > 0$, $t_0 \in R_+^0$,
(a) Let $t_0=0$. By Remark 3.2, select $x_1, \ldots, x_{n-1} \in X$ such that

$$||x_0, x_1, \ldots, x_{n-1}|| = 1.$$

On the other hand, we have

$$\begin{aligned}
&|p(t) - p(0)|=p(t)\\
&=p(t\|x_0, x_1, \ldots, x_{n-1}\|)\\
&=\|f(x_0) - f(\theta), f(\sqrt[n-1]{t}x_1) - f(\theta), \ldots, f(\sqrt[n-1]{t}x_{n-1}) - f(\theta)\|\\
&=\|f(\theta) - f(x_0), f(\sqrt[n-1]{t}x_1) - f(x_0), \ldots, f(\sqrt[n-1]{t}x_{n-1}) - f(x_0)\| .
\end{aligned}$$

For any $t \in R_+^0$, since f is n-continuous at x_0 and

$$\begin{aligned}
||\theta - x_0, \sqrt[n-1]{t}x_1 - x_0, \ldots, \sqrt[n-1]{t}x_{n-1} - x_0|| &= ||x_0, \sqrt[n-1]{t}x_1, \ldots, \sqrt[n-1]{t}x_{n-1}||\\
&= |t|||x_0, x_1, \ldots, x_{n-1}||\\
&= |t|.
\end{aligned}$$

When $|t|<\delta$, we have $|p(t)-p(0)|<\varepsilon$. That is, p is continuous at 0.
(b) If $t_0 \neq 0$. By Remark 3.2, there exist $x_1, \ldots, x_{n-1} \in$X, such that

$$||\frac{1}{t_0}x_0, x_1, \ldots, x_{n-1}|| = 1.$$

For any $t \in R_+^0$,

$$\begin{aligned} &| p(t) - p(t_0) | \\ &=| p\ (\mathrm{t}\|\tfrac{1}{t_0}x_0,\ x_1,\ \ldots\ ,\ x_{n-1}\|) - p(t_0\ \|\ \tfrac{1}{t_0}x_0,\ x_1,\ \ldots\ ,\ x_{n-1}\|)| \\ &=|p(\|\tfrac{t}{t_0}x_0,\ x_1,\ \ldots\ ,\ x_{n-1}\|)-p(\|x_0,\ x_1,\ \ldots\ ,\ x_{n-1}\|)| \\ &=|\|f(\tfrac{t}{t_0}x_0) - f(\theta),\ f(x_1) - f(\theta),\ \ldots\ ,\ f(x_{n-1}) - f(\theta)\| \\ &\quad -\|f(x_0) - f(\theta),\ f(x_1) - f(\theta),\ \ldots\ ,\ f(x_{n-1}) - f(\theta)\|| \\ &\leq\|f(\tfrac{t}{t_0}x_0) - f(x_0),\ f(x_1) - f(\theta),\ \ldots,\ f(x_{n-1}) - f(\theta)\| . \end{aligned}$$

Since $\frac{t}{t_0}x_0, x_0, \theta$ are 2-collinear, from (1), $f(\frac{t}{t_0}x_0), f(x_0), f(\theta)$ are 2-collinear. That is

$$f(\theta) - f(x_0) = k(f(\frac{t}{t_0}x_0) - f(x_0))$$

for some real number k. Thus we have

$$\begin{aligned} &\|f(\tfrac{t}{t_0}x_0) - f(x_0),\ f(x_1) - f(\theta),\ \ldots\ ,\ f(x_{n-1}) - f(\theta)\| \\ &=\|f(\tfrac{t}{t_0}x_0) - f(x_0),\ f(x_1) - f(\theta) + k(f(\tfrac{t}{t_0}x_0) - f(x_0)),\ \ldots\ , \\ &\quad f(x_{n-1}) - f(\theta)+k(f(\tfrac{t}{t_0}x_0) - f(x_0))\ \| \\ &=\|f(\tfrac{t}{t_0}x_0) - f(x_0),\ f(x_1) - f(x_0),\ \ldots\ ,\ f(x_{n-1}) - f(x_0)\| . \end{aligned}$$

By hypothesis, f is n-continuous at x_0. When

$$\begin{aligned} \|\frac{t}{t_0}x_0 - x_0, x_1 - x_0, \ldots, x_{n-1} - x_0\| &= | t - t_0 | \|\frac{1}{t_0}x_0, x_1, \ldots, x_{n-1}\| \\ &= | t - t_0 | < \delta. \end{aligned}$$

We have

$$||f(\frac{t}{t_0}x_0) - f(x_0), f(x_1) - f(x_0), \ldots, f(x_{n-1}) - f(x_0)|| < \varepsilon.$$

That is, if $|t - t_0|<\delta$, then

$$|p(t) - p(t_0)|\leq\|f(\tfrac{t}{t_0}x_0) - f(x_0),\ f(x_1) - f(x_0),\ \ldots\ ,\ f(x_{n-1}) - f(x_0)\|<\varepsilon$$

To sum up, if f is n-continuous at x_0, p is a continuous function. □

Lemma 3.7. *Let X, Y, f, p be as in Lemma 3.6. If $p\neq 0$, then f is injective.*

Proof. If f is not injective, assume there are $u, v \in$X and $u\neq v$, but $f(u) = f(v)$.
Since $p \neq 0$, there exists $0\neq\alpha\in R_+^0$, such that $p(\beta)=\alpha$.
Since $u - v \neq \theta$, by Remark 3.2, we could find $x_1, \ldots, x_{n-1} \in X$ such that

$$||u - v, x_1 - v, \ldots, x_{n-1} - v|| = \beta.$$

Thus

$$||f(u)-f(v), f(x_1)-f(v), \ldots, f(x_{n-1})-f(v)|| = p(\beta) = \alpha > 0,$$

which is a contradiction to

$$||f(u)-f(v), f(x_1)-f(v), \ldots, f(x_{n-1})-f(v)|| = 0,$$

as $f(u) = f(v)$. □

Lemma 3.8. *Let X, Y, f, p be as in Lemma 3.6. If p is injective, $f(\theta)=\theta$, then u and v are linearly dependent if and only if $f(u)$ and $f(v)$ are linearly dependent.*

Theorem 3.9. *Let X and Y be two linear n-normed spaces, $f : X \to Y$ preserves equality of n-distance, p is the gauge function for f. If p is injective, f is n-continuous at some point $\theta \neq x_0 \in X$, then $f = \sqrt[n]{\lambda} g$, where g is a n-isometry.*

Proof. Let $h(x) = f(x) - f(\theta)$. Then h is a mapping of preserving equality of n-distance with the same gauge function as f. And $h(\theta) = \theta$. Thus we may assume that $f(\theta)=\theta$. By Lemma 3.6, we know p is continuous. Thus, in order to show p is linear, we only need to prove that it is additive. For any $\gamma_1, \gamma_2 \in R_+^0$, assume $\gamma_1, \gamma_2 > 0$ (as γ_i=0, i=1,2 are trivial), select $\gamma_0 > 0$ such that $\gamma_i < \gamma_0$, i=1,2. Since $p(0) = 0$ and p is injective, p is monotone increasing function on R_+^0. So $p(\gamma_i) < p(\gamma_0)$, i=1,2. By Lemma 3.7, f is injective, thus there exists $\omega \neq \theta$ in Y such that $f^{-1}(\omega) \neq \theta$.
By Remark 3.2, we can find $x_1, \ldots, x_{n-1} \in X$, such that

$$||f^{-1}(\omega), x_1, \ldots, x_{n-1}|| = \gamma_0.$$

Thus, we have

$$||(\omega), f(x_1), \ldots, f(x_{n-1})|| = p(||f^{-1}(\omega), x_1, \ldots, x_{n-1}||) = p(\gamma_0) > 0.$$

Set $x_0 = f^{-1}(\frac{p(\gamma_1)}{p(\gamma_0)}\omega)$, then

$$\begin{aligned}
&p(\|x_0,\, x_1,\, \ldots,\, x_{n-1}\|)\\
&=\|f(x_0) - f(\theta),\, f(x_1) - f(\theta),\, \ldots,\, f(x_{n-1}) - f(\theta)\|\\
&=\|\tfrac{p(\gamma_1)}{p(\gamma_0)}\omega, f(x_1), \ldots, f(x_{n-1})\|\\
&=\tfrac{p(\gamma_1)}{p(\gamma_0)}\|\omega, f(x_1), \ldots, f(x_{n-1})\|\\
&=p(\gamma_1).
\end{aligned}$$

Since p is injective, so $\|x_0,\, x_1,\, \ldots,\, x_{n-1}\| = \gamma_1$.
Set $x_0' = \frac{x_0}{\gamma_1}$, we have

$$||x_0', x_1, \ldots, x_{n-1}|| = 1, \;\; f(\gamma_1 x_0') = \frac{p(\gamma_1)}{p(\gamma_0)}\omega.$$

Similarly, we could find $x_0^{''} \in X$ such that

$$||x_0^{''}, x_1, \ldots, x_{n-1}|| = 1, \;\; f(\gamma_2 x_0^{''}) = -\frac{p(\gamma_2)}{p(\gamma_0)}\omega.$$

Because $\gamma_1 x_0^{'}$ and $-\gamma_2 x_0^{'}$ are linearly dependent, by Lemma 3.8, so are $f(\gamma_1 x_0^{'})$ and $f(-\gamma_2 x_0^{'})$, $f(-\gamma_2 x_0^{'}){=}kf(\gamma_1 x_0^{'})$ for some real number k.
And also, we have $p(\gamma_1+\gamma_2){=}p(\|(\gamma_1+\gamma_2)x_0^{'}, x_1, \ldots, x_{n-1}\|)$

$$\begin{aligned}
&= p(\|\gamma_1 x_0^{'} - (-\gamma_2 x_0^{'}), x_1 - (-\gamma_2 x_0^{'}), \ldots, x_{n-1} - (-\gamma_2 x_0^{'})\|)\\
&= \|f(\gamma_1 x_0^{'}) - f(-\gamma_2 x_0^{'}), f(x_1) - f(-\gamma_2 x_0^{'}), \ldots, f(x_{n-1}) - f(-\gamma_2 x_0^{'})\|\\
&\le \|f(\gamma_1 x_0^{'}), f(x_1) - f(-\gamma_2 x_0^{'}), \ldots, f(x_{n-1}) - f(-\gamma_2 x_0^{'})\|\\
&\quad + \|f(-\gamma_2 x_0^{'}), f(x_1) - f(-\gamma_2 x_0^{'}), \ldots, f(x_{n-1}) - f(-\gamma_2 x_0^{'})\|\\
&= \|f(\gamma_1 x_0^{'}), f(x_1), \ldots, f(x_{n-1})\| + \|f(-\gamma_2 x_0^{'}), f(x_1), \ldots, f(x_{n-1})\|\\
&= p(\|\gamma_1 x_0^{'}, x_1, \ldots, x_{n-1}\|) + p(\|-\gamma_2 x_0^{'}, x_1, \ldots, x_{n-1}\|)\\
&= p(\gamma_1) + p(\gamma_2).
\end{aligned}$$

Since $f(\gamma_1 x_0^{'})$ and $f(\gamma_2 x_0^{''})$ are linearly dependent, from Lemma 3.8, $-\gamma_2 x_0^{''}{=}l(\gamma_1 x_0^{'} - \gamma_2 x_0^{''})$ for some real number l. Because $f(\gamma_1 x_0^{'})$ and $-f(\gamma_2 x_0^{''})$ are linearly dependent with the same direction, from Lemma 3.1, we have

$$\begin{aligned}
&p(\gamma_1) + p(\gamma_2) = p(\|\gamma_1 x_0^{'}, x_1, \ldots, x_{n-1}\|) + p(\|\gamma_2 x_0^{''}, x_1, \ldots, x_{n-1}\|)\\
&= \|f(\gamma_1 x_0^{'}), f(x_1), \ldots, f(x_{n-1})\| + \|-f(\gamma_2 x_0^{''}), f(x_1), \ldots, f(x_{n-1})\|\\
&= \|f(\gamma_1 x_0^{'}) - f(\gamma_2 x_0^{''}), f(x_1), \ldots, f(x_{n-1})\|\\
&= \|f(\gamma_1 x_0^{'}) - f(\gamma_2 x_0^{''}), f(x_1) - f(\gamma_2 x_0^{''}), \ldots, f(x_{n-1}) - f(\gamma_2 x_0^{''})\|\\
&= p(\|\gamma_1 x_0^{'} - \gamma_2 x_0^{''}, x_1 - \gamma_2 x_0^{''}, \ldots, x_{n-1} - \gamma_2 x_0^{''}\|)\\
&= p(\|\gamma_1 x_0^{'} - \gamma_2 x_0^{''}, x_1, \ldots, x_{n-1}\|)\\
&\le p(\|\gamma_1 x_0^{'}, x_1, \ldots, x_{n-1}\| + \|\gamma_2 x_0^{''}, x_1, \ldots, x_{n-1}\|)\\
&= p(\gamma_1 + \gamma_2).
\end{aligned}$$

That is $p(\gamma_1+\gamma_2){=}p(\gamma_1){+}p(\gamma_2)$.
Until now, we have got p is linear on R_+^0. Then

$$\begin{aligned}
\|f(x_1) - f(x_0), \ldots, f(x_n) - f(x_0)\| &= p(\|x_1 - x_0, \ldots, x_n - x_0\|)\\
&= \lambda\|x_1 - x_0, \ldots, x_n - x_0\|.
\end{aligned}$$

Set $g = \frac{f}{\sqrt[n]{\lambda}}$, then g is a n-isometry. □

A direct application of Theorem 3.3 in [4] yields the following corollary:

Corollary 3.10. *Let X and Y be two linear n-normed spaces, $f : X \to Y$ preserves equality of n-distance, p be the gauge function for f. If p is injective, f is n-continuous at some point $\theta \neq x_0 \in X$, then f is affine.*

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