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# POSITIVE SOLUTIONS OF DIFFERENTIAL SINGULAR EQUATION SYSTEMS ON THE HALF-LINE

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**Abstract.** In this paper, the author investigates the existence of positive solutions for singular boundary value problems for systems of differential equations involving parameter on the half-line by using the fixed point theory in the cone with a special norm and space. The results include, extend and improve many known results including singular and non-singular cases.

## 1. Introduction

In this paper, the author is concerned with the following system of second-order singular ordinary differential equations:

$$\begin{cases} (p(t)u'(t))' + \lambda(f(t,u(t),v(t)) - k^2u(t)) = 0, \quad 0 < t < +\infty, \\ (p(t)v'(t))' + \lambda(g(t,u(t),v(t)) - k^2v(t)) = 0, \\ \alpha_1 u(0) - \beta_1 \lim\limits_{t\to 0^+} p(t)u'(t) = 0, \\ \alpha_2 \lim\limits_{t\to+\infty} u(t) + \beta_2 \lim\limits_{t\to+\infty} p(t)u'(t) = 0, \\ \gamma_1 v(0) - \delta_1 \lim\limits_{t\to 0^+} p(t)v'(t) = 0, \\ \gamma_2 \lim\limits_{t\to+\infty} v(t) + \delta_2 \lim\limits_{t\to+\infty} p(t)v'(t) = 0. \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a parameter, $k \in (-\infty, +\infty)$, $\alpha_i,\ \beta_i,\ \gamma_i,\ \delta_i \geq 0,\ (i = 1, 2)$, the nonlinearities $f,\ g : (0, +\infty) \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ are continuous functions and may have singularity at $t = 0$.

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There has been increasing interest in the subject of differential equations due to its strong application back-ground, especially in the study of radial solutions of nonlinear elliptic equations and models of gas pressure in a semi-infinite porous medium [5,7-11]. The boundary value problem, for some special case where $f$ is continuous at $t = 0$ has been extensively studied by many authors. Zima[13] studied the existence of positive solutions on the half-line for the following second-order differential equations with no singularity: $x''(t) - k^2x(t) + f(t, x(t)) = 0,\ t \in (0, +\infty)$, and with boundary values condition $x(0) = 0,\ \lim_{t\to+\infty} x(t) = 0$, where $k > 0,\ f : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is a non-negative continuous function, and $f(t, x) \leq a(t) + b(t)x$ for $(t, x) \in [0, +\infty)\times[0, +\infty)$, in which $a, b : [0, +\infty) \to [0, +\infty)$ are continuous functions. Hao et al.[4] established the existence theorems of positive solutions for the following equations on the half-line: $x''(t) - k^2x(t) + m(t)f(t, x(t)) = 0,\ t \in (0, +\infty)$, and with boundary values condition $x(0) = 0,\ \ \lim_{t\to+\infty} x(t) = 0$, where $f : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is continuous and $\sup\{f(t, x) : (t, x) \in [0, +\infty) \times [0, +\infty)\} < +\infty$, $m : (0, +\infty) \to [0, +\infty)$ is continuous and may be singular at $t = 0$.

Recently, the authors [12] studied the boundary value problem

$$\begin{cases} (p(t)x'(t))' + \lambda(f(t, x(t)) - k^2x(t)) = 0,\ t \in (0, +\infty), \\ \alpha_1x(0) - \beta_1 \lim_{t\to0^+} p(t)x'(t) = 0, \\ \alpha_2 \lim_{t\to+\infty} x(t) + \beta_2 \lim_{t\to+\infty} p(t)x'(t) = 0, \end{cases} \tag{1.2}$$

and in [6], with $k = 0$, Lian and Ge obtained the existence of at least one positive solution for (1.2) by using the Krasnosel-skii fixed point theorem.

Motivated by the work of above papers, the aim of this paper is to consider the existence of positive solutions to the singular system (1.1). The problem we discuss is different from those in [4, 6, 12, 13]. Firstly, our study is on singular nonlinear differential equations system on the half-line with general boundary conditions. Secondly, the system (1.1) involves a parameter $\lambda$. Finally, the techniques used in this paper are the approximation method, and a special cone in a special space is established in order to overcome the difficulties caused by singularity and infinite interval and to apply the fixed point theorem in cone.

The main purpose of this paper is to obtain the existence of positive solutions for the system (1.1). Usually, we solve the positive fixed points of an integral operator $F$ instead of the positive solutions of system (1.1). In this paper we obtain the positive solutions on $[0, +\infty)$, which expands the domain of definition of $t$ from finite interval to infinite interval. The main difficulty is to testify that the operator $F$ is completely continuous, since we can not use the Ascoli- Arzela theorem in infinite interval $[0, +\infty)$. Some modification of the compactness criterion in infinite interval $[0, +\infty)$ (see lemma 2.2 ) can help to resolve this problem.

The rest of the paper is organized as follows. In Section 2, we present some necessary lemmas that will be used to prove our main results. In Section 3, first, we give Theorem 3.1 which is a result about completely continuous operator. Then we discuss the existence of at least one positive solution for the system (1.1).

## 2. Preliminaries and some lemmas

In this section, we present some notations and lemmas that will be used in the proof of our main results.

**Lemma 2.1.** [6] *Under the condition* $p \in C[0,+\infty) \cap C^1(0,+\infty)$ *with* $p > 0$ *on* $(0,+\infty)$, $\int_0^\infty \frac{1}{p(s)}ds < +\infty$; $\alpha_i,\ \beta_i \geq 0\ \ (i=1,2)$ *with* $\rho = \alpha_2\beta_1 + \alpha_1\beta_2 + \alpha_1\alpha_2 B(0,\infty) > 0$ *in which* $B(t,s) = \int_t^s \frac{1}{p(v)}dv$, *the boundary value problem*

$$\begin{cases} (p(t)u'(t))' + v(t) = 0,\ 0 < t < +\infty, \\ \alpha_1 u(0) - \beta_1 \lim_{t\to 0^+} p(t)u'(t) = 0, \\ \alpha_2 \lim_{t\to+\infty} u(t) + \beta_2 \lim_{t\to+\infty} p(t)u'(t) = 0, \end{cases} \tag{2.1}$$

*has a unique solution for any* $v \in L(0,+\infty)$. *Moreover, this unique solution can be expressed in the form*

$$u(t) = \int_0^\infty G(t,s)v(s)ds,$$

*where* $G(t,s)$ *is defined by*

$$G(t,s) = \frac{1}{\rho}\begin{cases} (\beta_1 + \alpha_1 B(0,s))(\beta_2 + \alpha_2 B(t,\infty)), & 0 \leq s \leq t < +\infty, \\ (\beta_1 + \alpha_1 B(0,t))(\beta_2 + \alpha_2 B(s,\infty)), & 0 \leq t \leq s < +\infty. \end{cases} \tag{2.2}$$

**Remark 2.1.** From (2.2), it is easy to get the following properties of $G(t,s)$:

(1) $G(t,s)$ is continuous on $[0,+\infty)\times[0,+\infty)$.
(2) For each $s \in [0,+\infty)$, $G(t,s)$ is continuously differentiable on $[0,+\infty)$ except $t = s$.
(3) $\frac{\partial G(t,s)}{\partial t}|_{t=s^+} = -\frac{\partial G(t,s)}{\partial t}|_{t=s^-} = -\frac{1}{p(s)}$.
(4) For each $s \in [0,+\infty)$, $G(t,s)$ satisfies the corresponding homogeneous BVP (i.e. the BVP(2.1) with $v(t)\equiv 0$) on $[0,+\infty)$ except $t = s$. In other words, $G(t,s)$ is the Green function of BVP (2.1) on the half-line.
(5) $G(t,s) \leq G(s,s) \leq \frac{1}{\rho}(\beta_1 + \alpha_1 B(0,s))(\beta_2 + \alpha_2 B(s,\infty)) < +\infty$.
(6) $\overline{G}(s) = \lim_{t\to+\infty} G(t,s) = \frac{1}{\rho}\beta_2(\beta_1 + \alpha_1 B(0,s)) \leq G(s,s) < +\infty$.
(7) For any $t \in [a,b] \subset (0,+\infty)$, and $s \in [0,+\infty)$, $G(t,s) \geq \omega G(s,s)$, where

$$\omega = \min\left\{\frac{\beta_2 + \alpha_2 B(b,\infty)}{\beta_2 + \alpha_2 B(0,\infty)}, \frac{\beta_1 + \alpha_1 B(0,a)}{\beta_1 + \alpha_1 B(0,\infty)}\right\}, \quad 0 < \omega < 1.$$

This paper, we consider the space $X = E \times E$,

$$E = \{x \in C[0, +\infty) : \lim_{t \to +\infty} x(t) \text{ exists}\}, \tag{2.3}$$

with the norm $\|(u, v)\| = \|u\| + \|v\|$ where $\|u\| = \sup_{t\in[0,+\infty)} |u(t)|$, $\|v\| = \sup_{t\in[0,+\infty)} |v(t)|$, then $(E, \|\cdot\|)$ is a Banach space (see [10]), and $(X, \|\cdot\|)$ is also a Banach space. Define

$$K = \{(u, v) \in X : \ u(t) \geq 0,\ v(t) \geq 0, u(t) \geq \omega\|u\|,\ v(t) \geq \omega\|v\|,\ t \in [a, b]\}.$$

which induces a partial order: $(u_1, v_1) \leq (u_2, v_2)$ if and only if $u_1 \leq u_2,\ v_1 \leq v_2$ if and only if $u_1(t) \leq u_2(t),\ v_1(t) \leq v_2(t)$ for $t \in [0, +\infty)$, and $|(u_1, u_2, ..., u_n)| = |u_1| + |u_2| + ...|u_n|$.

**Lemma 2.2.**[1] *Let $E$ be defined by (2.3) and $M \subset E$. Then $M$ is relatively compact in $E$ if the following conditions hold:*

(1) *$M$ is uniformly bounded in $E$;*
(2) *the functions from $M$ are equicontinuous on any compact interval of $[0, +\infty)$;*
(3) *the functions from $M$ are equiconvergent, that is, for any given $\varepsilon > 0$, there exists a $T = T(\varepsilon) > 0$ such that $|x(t) - x(+\infty)| < \varepsilon$, for any $t > T,\ x \in M$.*

**Lemma 2.3.** [2, 3] *Let $P$ be a positive cone in a real Banach space $E$. Denote $P_r = \{x \in P : \|x\| < r\}$, $\overline{P}_{r,R} = \{x \in P : r \leq \|x\| \leq R\}$, $0 < r < R < +\infty$. Let $A : \overline{P}_{r,R} \to P$ be a completely continuous operator. If the following conditions are satisfied:*

(1) $\|Ax\| \leq \|x\|$, $\forall x \in \partial P_R$.
(2) *there exists a $x_0 \in \partial P_1$, such that $x \neq Ax + mx_0$, $\forall x \in \partial P_r$, $m > 0$.*

then $A$ has fixed points in $\overline{P}_{r,R}$.

**Remark 2.2.** If (1) and (2) are satisfied for $x \in \partial P_r$ and $x \in \partial P_R$ respectively. Then Lemma 2.3 is still true.

## 3. Main results

Let us list the following assumptions:

**(H$_1$)** *The functions $f,\ g : (0, +\infty) \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ are continuous functions and satisfy*

$$k^2 x \leq f(t, x, y) \leq \phi_1(t) h_1(t, x, y),$$

*and*

$$k^2 y \leq g(t, x, y) \leq \phi_2(t) h_2(t, x, y)$$

*for any* $(t, x, y) \in (0, +\infty) \times [0, +\infty) \times [0, +\infty)$*, where* $\phi_i : (0, +\infty) \to [0, +\infty)$ *is continuous and singular at* $t = 0$*,* $\phi_i(t) \not\equiv 0$ *on* $[0, +\infty)$*,* $h_i : [0, +\infty) \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ *is continuous and bounded for* $0 \le t < +\infty$ *and* $x$*,* $y$ *in any bounded set of* $[0, +\infty)$*,* $(i = 1, 2)$*.*

($\mathbf{H}_2$) $0 < \int_0^\infty G(s, s)\phi_i(s)ds < +\infty$*,* $(i = 1, 2)$*.*

From the above assumptions ($\mathbf{H}_1$) ($\mathbf{H}_2$). Let

$$\begin{aligned} A(u, v)(t) &= \lambda \int_0^\infty G(t, s)(f(s, u(s), v(s)) - k^2 u(s))ds, \\ B(u, v)(t) &= \lambda \int_0^\infty G(t, s)(g(s, u(s), v(s)) - k^2 v(s))ds, \\ F(u, v) &= (A(u, v), B(u, v)), \quad t \in [0, +\infty). \end{aligned} \tag{3.1}$$

Obviously the system (1.1) has a solution $(u, v)$ if and only if $(u, v) \in K$ is a fixed point of the operator $F$ defined by (3.1), i.e. $F(u, v) = (u, v)$.

**Theorem 3.1.** *Assume* ($\boldsymbol{H}_1$) $-$ ($\boldsymbol{H}_2$) *hold, then* $F : K \to K$ *is a completely continuous operator.*

*Proof.* First, we show that $F : K \to K$ is well defined. Let $(u, v) \in K$. Then there exists $r^* > 0$, such that $|u(t)| + |v(t)| \le r^*$, also we have $|u(t)| \le r^*$, $|v(t)| \le r^*$, for any $t \in [0, +\infty)$. From ($\mathbf{H}_1$), it is easy to see that $F(u, v) \ge (0, 0)$ and we have $S_{ir^*} := \sup\{h_i(t, x, y) :\ 0 \le t < +\infty,\ 0 \le x \le r^*\ 0 \le y \le r^*\} < +\infty$, $(i = 1, 2)$. Thus, by ($\mathbf{H}_1$) and ($\mathbf{H}_2$), for any $t \in [0, +\infty)$, we have

$$\begin{aligned} &\lambda \int_0^\infty G(t, s)(f(s, u(s), v(s)) - k^2 u(s))ds \\ &\le \lambda \int_0^\infty G(s, s)\phi_1(s)h_1(s, u(s), v(s))ds \\ &\le \lambda S_{1r^*} \int_0^\infty G(s, s)\phi_1(s)ds < +\infty. \\ &\lambda \int_0^\infty G(t, s)(g(s, u(s), v(s)) - k^2 v(s))ds \\ &\le \lambda \int_0^\infty G(s, s)\phi_2(s)h_2(s, u(s), v(s))ds \\ &\le \lambda S_{2r^*} \int_0^\infty G(s, s)\phi_2(s)ds < +\infty. \end{aligned} \tag{3.2}$$

One the other hand, for any $t_1, t_2 \in [0, +\infty)$ and $s \in [0, +\infty)$, by (5) of Remark 2.1, we have

$$|G(t_1, s) - G(t_2, s)|\phi_i(s) \le 2G(s, s)\phi_i(s),\ i = (1, 2).$$

Hence, by ($\mathbf{H}_2$), the Lebesgue's dominated convergence theorem and the continuity of $G(t,s)$, for any $t_1, t_2 \in [0,+\infty)$, $(u,v) \in K$, we have

$$\begin{aligned}|A(u,v)(t_1) - A(u,v)(t_2)| \leq & \lambda \int_0^\infty |G(t_1,s) - G(t_2,s)| f(s,u(s),v(s))ds \\ \leq & \lambda \int_0^\infty |G(t_1,s) - G(t_2,s)| \phi_1(s) h_1(s,u(s),v(s))ds \\ \leq & \lambda S_{1r^*} \int_0^\infty |G(t_1,s) - G(t_2,s)| \phi_1(s) ds \\ \to & 0, \quad \text{as } t_1 \to t_2.\end{aligned} \tag{3.3}$$

It follows from (3.2) and (3.3) that $A(u,v) \in C[0,+\infty)$. By (6) of Remark 2.1, we also have

$$\lim_{t\to+\infty} A(u,v)(t) = \lambda \int_0^\infty \overline{G}(s)(f(s,u(s),v(s)) - k^2 u(s))ds < +\infty.$$

Similarly, we have $|B(u,v)(t_1) - B(u,v)(t_2)| \to 0$, as $t_1 \to t_2$ and

$$\lim_{t\to+\infty} B(u,v)(t) = \lambda \int_0^\infty \overline{G}(s)(g(s,u(s),v(s)) - k^2 v(s))ds < +\infty.$$

Hence, $F(u,v)$ is well defined for any $(u,v) \in K$.

For any $(u,v) \in K$, $t \in [0,+\infty)$, by (3.1), we have

$$\begin{aligned}A(u,v)(t) &= \lambda \int_0^\infty G(t,s)(f(s,u(s),v(s)) - k^2 u(s))ds \\ &\leq \lambda \int_0^\infty G(s,s)(f(s,u(s),v(s)) - k^2 u(s))ds.\end{aligned}$$

So

$$\|A(u,v)\| \leq \lambda \int_0^\infty G(s,s)(f(s,u(s),v(s)) - k^2 u(s))ds. \tag{3.4}$$

On the other hand, by the properties (7) of $G(t,s)$, for any $t \in [a,b]$, we obtain

$$A(u,v)(t) \geq \lambda\omega \int_0^\infty G(s,s)(f(s,u(s),v(s)) - k^2 u(s))ds. \tag{3.5}$$

From (3.4) and (3.5), we know that $A(u,v)(t) \geq \omega\|A(u,v)\|$ for any $t \in [a,b]$. In the same way, we can show that $B(u,v)(t) \geq \omega\|B(u,v)\|$ for any $t \in [a,b]$. Therefore, $F(K) \subseteq K$.

Next, for any positive integer $m$, we define an operator $F_m : K \to K$ by

$$\begin{aligned}
&A_m(u,v)(t) = \lambda \int_{\frac{1}{m}}^{\infty} G(t,s)(f(s,u(s),v(s)) - k^2u(s))ds, \\
&B_m(u,v)(t) = \lambda \int_{\frac{1}{m}}^{\infty} G(t,s)(g(s,u(s),v(s)) - k^2v(s))ds, \\
&F_m(u,v) = (A_m(u,v), B_m(u,v)), \quad t \in [0,+\infty),
\end{aligned} \tag{3.6}$$

and prove that $F_m : K \to K$ is completely continuous, for each $m \geq 1$. Firstly, we show that $F_m : K \to K$ is continuous. Let $(u_n, v_n), (u,v) \in K$ such that $\|(u_n, v_n) - (u,v)\| \to 0$ as $n \to +\infty$, i.e. $\|u_n - u\| \to 0$, $\|v_n - v\| \to 0$ as $n \to +\infty$. By (3.6) and $(\mathbf{H}_2)$, we know

$$\begin{aligned}
&|A_m(u_n,v_n)(t) - A_m(u,v)(t)| \\
=&|\lambda \int_{\frac{1}{m}}^{\infty} G(t,s)(f(s,u_n(s),v_n(s)) - k^2u_n(s))ds \\
&- \lambda \int_{\frac{1}{m}}^{\infty} G(t,s)(f(s,u(s),v(s)) - k^2u(s))ds| \\
\leq&\lambda \int_{\frac{1}{m}}^{\infty} G(s,s) \left(|f(s,u_n(s),v_n(s)) - k^2u_n(s)| + |f(s,u(s),v(s)) - k^2u(s)|\right) ds \\
\leq&\lambda \int_{\frac{1}{m}}^{\infty} G(s,s) \left(|f(s,u_n(s),v_n(s))| + |f(s,u(s),v(s))|\right) ds \\
\leq&\lambda \int_{\frac{1}{m}}^{\infty} G(s,s) \left(\phi_1(s)h_1(s,u_n(s),v(s)) + \phi_1(s)h_1(s,u(s),v(s))\right) ds \\
\leq&2\lambda S_{1r^*} \int_0^{\infty} G(s,s)\phi_1(s)ds < +\infty,
\end{aligned}$$

where $S_{1r^*} := \sup\{h_1(t,x,y) : \ 0 \leq t < +\infty, \ 0 \leq x \leq r^*, \ 0 \leq y \leq r^*\} < +\infty$ (by $(\mathbf{H}_1)$), $r^*$ is a real number such that $r^* \geq \max_{n\in\mathbb{N}}\{\|u\|, \|v\|, \|u_n\|, \|v_n\|\}$, $\mathbb{N}$ is a natural number set.

For any $\varepsilon > 0$, by $(\mathbf{H}_2)$, there exists a sufficiently large $A_0$ ($A_0 > 1/m$) such that

$$2\lambda S_{1r^*} \int_{A_0}^{\infty} G(s,s)\phi_1(s)ds < \frac{\varepsilon}{3}. \tag{3.7}$$

From $\|u_n - u\| \to 0$, $\|v_n - v\| \to 0$ as $n \to +\infty$, there exists a sufficiently large nature number $N_0$ such that when $n > N_0$, for all $s \in [0, +\infty)$, we have

$$\begin{aligned} |u_n(s) - u(s)| \leq \|u_n - u\| < \frac{\varepsilon}{3}\left(\lambda(k^2+1)\int_{\frac{1}{m}}^{A_0} G(s,s)ds\right)^{-1}, \\ |v_n(s) - v(s)| \leq \|v_n - v\| < \frac{\varepsilon}{3}\left(\lambda(k^2+1)\int_{\frac{1}{m}}^{A_0} G(s,s)ds\right)^{-1}. \end{aligned} \tag{3.8}$$

On the other hand, by the continuity of $f(t,x,y)$ on $[1/m, A_0] \times [0, r^*] \times [0, r^*]$, for the above $\varepsilon > 0$, there exists a $\delta > 0$ such that for any $s \in [1/m, A_0]$ and $x, x', y, y' \in [0, r^*]$, when $|x - y| < \delta$, $|x' - y'| < \delta$, we have

$$|f(s,x,y) - f(s,x',y')| < \frac{\varepsilon}{3}\left(\lambda\int_{\frac{1}{m}}^{A_0} G(s,s)ds\right)^{-1}. \tag{3.9}$$

From $\|u_n - u\| \to 0$, $\|v_n - v\| \to 0$ as $n \to +\infty$, there exists a sufficiently large nature number $N_1 > N_0$ such that when $n > N_1$, for all $s \in [1/m, A_0]$, we have

$$|u_n(s) - u(s)| \leq \|u_n - u\| < \delta, \quad |v_n(s) - v(s)| \leq \|v_n - v\| < \delta.$$

Hence, by (3.9), when $n > N_1$, for all $s \in [1/m, A_0]$, we obtain

$$|f(s, u_n(s), v_n(s)) - f(s, u(s), v(s))| < \frac{\varepsilon}{3}\left(\lambda\int_{\frac{1}{m}}^{A_0} G(s,s)ds\right)^{-1}. \tag{3.10}$$

Therefore, by (3.6)-(3.8) and (3.10), when $n > N_1$, for any $t \in [0, +\infty)$, we have

$$\begin{aligned} &|A_m(u_n, v_n)(t) - A_m(u,v)(t)| \\ =&\lambda\Big|\int_{\frac{1}{m}}^{\infty} G(t,s)(f(s,u_n(s),v_n(s)) - k^2u_n(s))ds \\ &- \int_{\frac{1}{m}}^{\infty} G(t,s)(f(s,u(s),v(s)) - k^2u(s))ds\Big| \\ \leq&\lambda\int_{\frac{1}{m}}^{A_0} G(s,s)(|f(s,u_n(s),v_n(s)) - f(s,u(s),v(s))| + k^2\,|u_n(s) - u(s)|)ds \\ &+ \lambda\int_{A_0}^{\infty} G(s,s)(f(s,u_n(s),v_n(s)) + (f(s,u(s),v(s)))ds \\ \leq&\frac{2\varepsilon}{3} + \lambda\int_{A_0}^{\infty} G(s,s)\phi_1(s)(h_1(s,u_n(s),v_n(s)) + h_1(s,u(s),v(s)))ds \\ \leq&\frac{2\varepsilon}{3} + 2\lambda S_{1r^*}\int_{A_0}^{\infty} G(s,s)\phi_1(s)ds < \varepsilon. \end{aligned}$$

This implies that the operator $A_m : K \to E$ is continuous for each natural number $m$.

It what follows, we need to prove that $A_m : K \to E$ is a compact operator for each natural number $m$. Let $M$ be any bounded subset of $K$. Then there exists a constant $r > 0$ such that $\|(u, v)\| \le r$ for any $(u, v) \in M$, also $u \le r,\ v \le r$. By (3.6), $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$, for any $(u, v) \in M$, we have

$$\begin{aligned}|A_m(u,v)(t)| &= \lambda \left| \int_{\frac{1}{m}}^{\infty} G(t,s)(f(s,u(s),v(s)) - k^2 u(s))ds \right| \\ &\le \lambda \int_{\frac{1}{m}}^{\infty} G(s,s)\phi_1(s)h_1(s,u(s),v(s))ds \\ &\le \lambda S_{1r} \int_0^{\infty} G(s,s)\phi_1(s)ds < +\infty,\end{aligned}$$

where $S_{1r} := \sup\{h_1(t,x,y) :\ 0 \le t < +\infty,\ 0 \le x \le r,\ 0 \le y \le r\}$. Therefore, $A_m M$ is uniformly bounded. By the similar proof as (3.3), we can prove that for any $t, t' \in [0, +\infty)$ and $x \in M$, we have

$$\begin{aligned}|A_m(u,v)(t) - A_m(u,v)(t')| \le& \lambda \int_{\frac{1}{m}}^{\infty} |G(t,s) - G(t',s)| f(s,u(s),v(s))ds \\ \le& \lambda \int_0^{\infty} |G(t,s) - G(t',s)|\phi_1(s)h_1(s,u(s),v(s))ds \\ \le& \lambda S_{1r^*} \int_0^{\infty} |G(t,s) - G(t',s)|\phi_1(s)ds \\ \to& 0,\ \text{as } t \to t'.\end{aligned}$$

This yields that $A_m M$ is equicontinuous on $[0, +\infty)$. It follows from

$$\begin{aligned}&|A_m(u,v)(t) - A_m(u,v)(+\infty)| \\ \le& \lambda \int_{\frac{1}{m}}^{\infty} |G(t,s) - \overline{G}(s)|(\phi_1(s)h_1(s,u(s),v(s)) - k^2 u(s))ds \\ \le& \lambda \int_{\frac{1}{m}}^{\infty} |G(t,s) - \overline{G}(s)|\phi_1(s)h_1(s,u(s),v(s))ds \\ \le& S_{1r}\lambda \int_0^{\infty} |G(t,s) - \overline{G}(s)|\phi_1(s)ds \\ \to& 0,\quad t \to +\infty,\end{aligned}$$

that $A_m M$ is equiconvergent. Therefore, by Lemma 2.2, we know that the operator $A_m : K \to E$ is completely continuous for each natural number $m$.

Finally, we show that $A : K \to E$ is a completely continuous operator. For any $t \in [0, +\infty)$ and $(x, y) \in K$ satisfying $\|x\| \le 1$, $\|y\| \le 1$, by (3.1) and

(3.6), we have

$$\begin{aligned}|A(u,v)(t)-A_m(u,v)(t)| =&\lambda\int_0^{\frac{1}{m}} G(t,s)(f(s,u(s),v(s))-k^2u(s))ds\\ \le&\lambda\int_0^{\frac{1}{m}} G(t,s)\phi_1(s)h_1(s,u(s),v(s))ds\\ \le&\lambda S_1\int_0^{\frac{1}{m}} G(s,s)\phi_1(s)ds,\end{aligned}$$

where $S_1 := \sup\{h_1(t,x,y):\ 0\le t<+\infty,\ 0\le x\le 1,\ 0\le y\le 1\}<+\infty$. This together with $(\mathbf{H}_2)$ implies that

$$\|A(u,v)-A_m(u,v)\|\le \lambda S_1\int_0^{\frac{1}{m}} G(s,s)\phi_1(s)ds\to 0,\ m\to+\infty.$$

Therefore, $A : K \to E$ is a completely continuous operator. By the same proof, we obtain that $B : K \to E$ is a completely continuous operator. That is $F : K \to K$ completely continuous. □

**Theorem 3.2.** *Assume that* $(\boldsymbol{H}_1)$ *and* $(\boldsymbol{H}_2)$ *hold,* $h_1$*,* $h_2$ *and* $f$*,* $g$ *satisfy the following condition:*

$(\boldsymbol{H}_3)$

$$0\le h_1^0=\overline{\lim_{x\to0^+}}\sup_{t\in[0,+\infty)}\frac{h_1(t,x,y)}{x},\quad h_2^0=\overline{\lim_{y\to0^+}}\sup_{t\in[0,+\infty)}\frac{h_2(t,x,y)}{y}<L,$$

*and*

$$k^2<l+k^2<f_\infty=\underline{\lim_{x\to+\infty}}\inf_{t\in[a,b]}\frac{f(t,x,y)}{x},\quad g_\infty=\underline{\lim_{y\to+\infty}}\inf_{t\in[a,b]}\frac{g(t,x,y)}{y}\le+\infty,$$

*where*

$$L=\left(\max\left\{\int_0^\infty G(s,s)\phi_1(s)ds,\int_0^\infty G(s,s)\phi_2(s)ds\right\}\right)^{-1},$$

$$l=\left(\min_{t\in[a,b]}\omega\int_a^b G(t,s)ds\right)^{-1}.$$

*Then system (1.1) has at least one positive solution for any*

$$\lambda\in\left(\frac{l}{\min\{f_\infty-k^2,g_\infty-k^2\}},\ \frac{L}{\max\{h_1^0,h_2^0\}}\right).$$

*Proof.* From the condition, there exists $\varepsilon > 0$, such that

$$\frac{l}{\min\{f_\infty - k^2, g_\infty - k^2\} - \varepsilon} \leq \lambda \leq \frac{L}{\max\{h_1^0, h_2^0\} + \varepsilon}.$$

By the first inequality of $(\mathbf{H}_3)$, there exists $r > 0$, for the above $\varepsilon$, such that

$$h_1(t,x,y) \leq (h_1^0 + \varepsilon)x, \quad 0 \leq x \leq r,\ t \in [0,+\infty)$$

and

$$h_2(t,x,y) \leq (h_2^0 + \varepsilon)y, \quad 0 \leq y \leq r,\ t \in [0,+\infty).$$

Set $K_{r_1} = \{(u,v) \in K : \|u\| < r_1,\ \|v\| < r_1\}$, $(r_1 \leq r)$, by the definition of $\|\cdot\|$, we know that

$$u(t) \leq \|u\| = r_1 \leq r,\ v(t) \leq \|v\| = r_1 \leq r, \quad \forall (u,v) \in \partial K_{r_1},\ t \in [0,+\infty).$$

Then for any $(u,v) \in \partial K_{r_1}$, $t \in [0,+\infty)$,

$$\begin{aligned}\|A(u,v)\| =&\lambda \sup_{t\in[0,+\infty)} \left|\int_0^\infty G(t,s)(f(s,u(s),v(s)) - k^2u(s))ds\right| \\ \leq&\lambda \sup_{t\in[0,+\infty)} \int_0^\infty G(t,s)f(s,u(s),v(s))ds \\ \leq&\lambda \sup_{t\in[0,+\infty)} \int_0^\infty G(t,s)\phi_1(s)h_1(s,u(s),v(s))ds \\ \leq&\lambda \sup_{t\in[0,+\infty)} \int_0^\infty G(t,s)\phi_1(s)(h_1^0 + \varepsilon)u(s)ds \\ \leq& \sup_{s\in[0,+\infty)} u(s)\lambda(h_1^0 + \varepsilon)\int_0^\infty G(s,s)\phi_1(s)ds \\ \leq& r_1 = \|u\|.\end{aligned}$$

Similarly, we have $\|B(u,v)\| \leq r_1 = \|v\|$. Thus,

$$\|F(u,v)\| = \|A(u,v)\| + \|B(u,v)\| \leq \|u\| + \|v\| = \|(u,v)\|. \tag{3.11}$$

On the other hand, by the second inequality of $(\mathbf{H}_3)$, there exists $r_0 > \omega r_1 > 0$, for the above $\varepsilon$ such that

$$\begin{aligned} f(t,x,y) \geq (f_\infty - \varepsilon)x, \quad x \geq r_0,\ t \in [a,b]. \\ g(t,x,y) \geq (g_\infty - \varepsilon)y, \quad y \geq r_0,\ t \in [a,b]. \end{aligned} \tag{3.12}$$

Write $r_2 \geq r_0/\omega > r_1$, $K_{r_2} = \{(u,v) \in K : \|u\| < r_2,\ \|v\| < r_2\}$.

Let $(u_0,v_0) = (1,1) \in \partial K_1 = \{(u,v) \in K : \|u\| = 1,\ \|v\| = 1\}$. Then

$$(u,v) \neq F(u,v) + \mu(u_0,v_0), \quad \forall (u,v) \in \partial K_{r_2},\ \forall \mu > 0. \tag{3.13}$$

Suppose that (3.13) is false, then there exists $(u_0, v_0) = (1, 1) \in \partial K_1, (u_2, v_2) \in \partial K_{r_2},\ \mu_2 > 0$ such that

$$(u_2, v_2) = F(u_2, v_2) + \mu_2(u_0, v_0).$$

(3.12) and the fact that

$$u_2(t) \geq \omega\|u_2\| = \omega r_2 \geq r_0,\ v_2(t) \geq \omega\|v_2\| = \omega r_2 \geq r_0,\ t \in [a, b],$$

we know that

$$f(t, u_2(t), v_2(t)) \geq (f_\infty - \varepsilon)u_2(t),\ t \in [a, b]$$

and

$$g(t, u_2(t), v_2(t)) \geq (g_\infty - \varepsilon)v_2(t),\ t \in [a, b].$$

For $(u_2, v_2) \in K_{r_2}$, $t \in [a, b]$, we have

$$\begin{aligned} r_2 \geq u_2(t) =& \lambda \int_0^\infty G(t, s)(f(s, u_2(s), v_2(s)) - k^2 u(s))ds + \mu_0 \\ \geq& \lambda \int_0^\infty G(t, s)((f_\infty - \varepsilon)u_2(s) - k^2 u(s))ds + \mu_0 \\ \geq& \lambda(f_\infty - \varepsilon - k^2)r_2 \int_a^b G(t, s)ds + \mu_0 \\ \geq& r_2 + \mu_0, \end{aligned}$$

which is a contradiction. This implies that (3.13) holds.

From (3.11), (3.13), Lemma 2.3, Theorem 3.1 and the fact that $\overline{K}_{r_1} \subset K_{r_2}$, we can obtain that the operator $F$ has fixed point $(u, v)$, which belongs to $K_{r_2} \backslash \overline{K}_{r_1}$, such that $0 < 2r_1 < \|(u, v)\| < 2r_2$. It is easy to see that $(u, v)$ is a positive solution of system (1.1). $\square$

**Remark 3.1.** Note that $\frac{l}{f_\infty - k^2} < 1$, $\frac{l}{g_\infty - k^2} < 1$ and $\frac{L}{h_1^0} > 1$, $\frac{L}{h_2^0} > 1$, so if $\lambda = 1$, then Theorem 3.2 also holds.

**Remark 3.2.** From Theorem 3.2, we can see $h_1(t, x, y)$, $h_2(t, x, y)$ and $f(t, x, y)$, $g(t, x, y)$ need not be superlinear or sublinear. In fact, Theorem 3.2 still holds, if one of the following conditions is satisfied

(1) If $f_\infty = g_\infty = +\infty$, $h_1^0 > 0$, $h_2^0 > 0$, then for each $\lambda \in (0, \frac{L}{\max\{h_1^0, h_2^0\}})$.

(2) If $f_\infty = g_\infty = +\infty$, $h_1^0 = h_2^0 = 0$, then for each $\lambda \in (0, +\infty)$.

(3) If $f_\infty > l + k^2 > 0$, $g_\infty > l + k^2 > 0$, $h_1^0 = h_2^0 = 0$, then for each $\lambda \in (\frac{l}{\min\{f_\infty - k^2, g_\infty - k^2\}}, +\infty)$.

**Theorem 3.3.** *Assume that* $(\boldsymbol{H}_1)$ *and* $(\boldsymbol{H}_2)$ *hold,* $h_1$*,* $h_2$ *and* $f$*,* $g$ *satisfy the following condition:*

($\boldsymbol{H}_4$)

$$0 \le h_1^\infty = \overline{\lim_{x\to+\infty}} \sup_{t\in[0,+\infty)} \frac{h_1(t,x,y)}{x}, \quad h_2^\infty = \overline{\lim_{y\to+\infty}} \sup_{t\in[0,+\infty)} \frac{h_2(t,x,y)}{y} < L,$$

*and*

$$k^2 < l + k^2 < f_0 = \underline{\lim_{x\to 0^+}} \inf_{t\in[a,b]} \frac{f(t,x,y)}{x}, \quad g_0 = \underline{\lim_{y\to 0^+}} \inf_{t\in[a,b]} \frac{g(t,x,y)}{y} \le +\infty,$$

*where constants $L$ and $l$ are defined by Theorem 3.2. Then system (1.1) has at least one positive solution for any*

$$\lambda \in \left( \frac{l}{\min\{f_0 - k^2, g_0 - k^2\}}, \ \frac{L}{\max\{h_1^\infty, h_2^\infty\}} \right).$$

*Proof.* The proof is similar to that of Theorem 3.2, and so we omit it. $\square$

**Remark 3.3.** Note that $\frac{l}{f_0-k^2} < 1$, $\frac{l}{g_0-k^2} < 1$ and $\frac{L}{h_1^\infty} > 1$, $\frac{L}{h_2^\infty} > 1$, so if $\lambda = 1$, then Theorem 3.3 also holds.

**Remark 3.4.** We can see that Theorem 3.3 still holds, if one of the following conditions is satisfied:

(1) If $h_1^\infty < L$, $h_2^\infty < L$, $f_0 = g_0 = +\infty$, then for each $\lambda \in (0, \frac{L}{\max\{h_1^\infty, h_2^\infty\}})$.
(2) If $h_1^\infty = h_2^\infty = 0$, $f_0 = g_0 = +\infty$, then for each $\lambda \in (0, +\infty)$.
(3) If $h_1^\infty = h_2^\infty = 0$, $f_0 > l + k^2 > 0$, $g_0 > l + k^2 > 0$, then for each $\lambda \in (\frac{l}{\min\{f_0-k^2, g_0-k^2\}}, +\infty)$.

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