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APPLICATION OF FINITE ELEMENT METHOD IN SOLVING BOUNDARY CONTROL PROBLEM GOVERNED BY PARABOLIC VARIATIONAL INEQUALITIES WITH AN INFINITE NUMBER OF VARIABLES

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Abstract. In this paper, we introduce the finite element method and solve the boundary control problem governed by parabolic variational inequalities with an infinite number of variables by using this method.

1. Introduction

Finite element and boundary element methods are major numerical tools for different types of boundary value problems and for studying partial differential equations modeling real-world problems, functional analysis plays a vital role in reducing the problem in a form amenable to computer analysis. It is a basic tool for error estimation between solutions of continuous and discrete problems and convergence of solutions of the latter to the original problem. The finite element method is a general technique to build finite-dimensional spaces of a Hilbert space of some classes of functions, such as Sobolev spaces of different orders, and their subspaces, in order to apply the Ritz and Galerkin methods to a variational problem. The technique is based on ideas like (i) Division of the domain Ω in which the problem is posed in a set of simple

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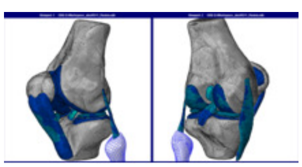
subdomains, called elements-often these elements are triangles, quadrilaterals tetrahedra. (ii) A space H of functions defined on Ω is then approximated by appropriate functions defined on each subdomain with suitable matching conditions at interfaces [16].

A systematic study of variational formulation of the boundary value problems and their discretization began in the early seventies [16].

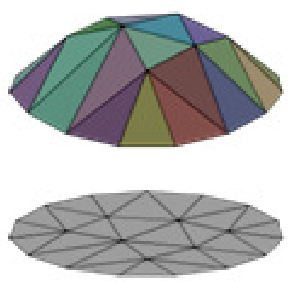
The finite element method has been applied in every conceivable area of engineering, such as structural analysis, semiconductor devices, meteorology, flow through porous media, heat conduction, wave propagation, electromagnetism, environmental studies, biomechanics [16].

The finite element method is a numerical method for solving problems of engineering and mathematical physics. Typical problem areas of interest include structural analysis, heat transfer, fluid flow, mass transport, and electromagnetic potential. The analytical solution of these problems generally require the solution to boundary value problems for partial differential equations. The finite element method formulation of the problem results in a system of algebraic equations. The method approximates the unknown function over the domain. To solve the problem, it subdivides a large system into smaller, simpler parts that are called finite elements. The simple equations that model these finite elements are then assembled into a larger system of equations that models the entire problem. FEM then uses variational methods from the calculus of variations to approximate a solution by minimizing an associated error function [17].

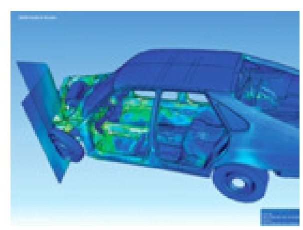
Let us introduce some examples of application of finite element method:



Finite element model of a human knee joint



A piecewise linear function in $2D$



Visualization of how a car deforms in crash using finite element analysis

A set of inequalities defining a control of a system governed by self-adjoint elliptic operators with an infinite number of variables are presented in Gali et al. ([11]). In Gali et al. ([10], [12], [13]) the optimal control problem for system described by elliptic and hyperbolic operators with an infinite number of variables have been discussed. El-Zahaby ([8]) presented the necessary conditions for control problems governed by elliptic variational inequalities with an infinite number of variables. Necessary conditions for optimality in distributed control problem governed by parabolic variational inequalities with an infinite number of variables are established by El-Zahaby et al. ([9]). Boundary control problem with nonlinear state equation with an infinite number of variables are established by El-Zahaby and Mostafa.

In this paper, we shall use the theory of Barbu ([2], [3], [4], [5]) to introduce boundary control problem govened by parabolic equation with nonlinear boundary value condition in the case of infinite number of variables and will apply finite element method.

This paper is organized as follows:

In section 2, some functional spaces with an infinite number of variables will be introduced. In section 3, we introduce the main results.

2. Preliminaries

We consider some function spaces of infinitely many variables (Berezankii et al. [6], [7]). For this this purpose, we introduce the infinite product

$$R^\infty = R^1 \times R^1 \times \cdots ,$$

with elements $x = (x_\mu)_{\mu=1}^\infty$, $x_\mu \in R^1$ and denote by $dg(x)$ the product of measures

$$g_1(x_1) \otimes g_2(x_2) \otimes \cdots ,$$

defined on the σ-hull of the cylindrical sets in R^∞ generated by the finite dimensional Borel sets $g(R^\infty) = 1$ with $0 < g_k(t) \in C^\infty(R^1)$ is fixed weight

$$\int_{R^1} g_k(t)dt = 1, \quad k = 1, 2, \cdots .$$

We have a Hilbert space of functions of infinitely many variables

$$L_2(R^\infty) = L_2(R^\infty, dg(x)).$$

In the following, we shall use the chain

$$W^l(R^\infty) \subseteq L_2(R^\infty) = W^0(R^\infty) \subseteq W^{-l}(R^\infty),$$

where $W^l(R^\infty)$ is a Sobolev space, which is the completion of the class $C_0^\infty(R^\infty)$ of infinitely differentiable functions of compact support with respect to the scalar product

$$(u, v)_{w^l(R^\infty)} = \sum_{|\alpha|\leq l} (D^\alpha u, D^\alpha v)_{L^2(R^\infty)},$$

where

$$D^\alpha = \frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1}(\partial x_2)^{\alpha_2}\dots}, \quad |\alpha| = \sum_{i=1}^{\infty} \alpha_i.$$

The differentiation is in the sense of a generalized function on R^∞. $W^{-l}(R^\infty)$ are the dual of $W^l(R^\infty)$. $L^2(0,T;W^l(R^\infty))$ denote the space of all measurable functions $t \to \varphi(t)$ of $]0,T[\to W^l(R^\infty)$ and the variable t denotes the time : $t \in]0,T[$, $T \prec \infty$ with the Lebesgue measure dt on $]0,T[$ such that

$$\|\varphi\|_{L^2(0,T;W^l(R^\infty))} = \left(\int_0^T \|\varphi(t)\|^2_{W^l(R^\infty)} dt \right)^{\frac{1}{2}} \prec \infty,$$

is endowed with the scalar product

$$(f,g)_{L^2(0,T;W^l(R^\infty))} = \int_0^T (f(t),g(t))_{W^l(R^\infty)} dt,$$

which is a Hilbert space ([15]).

Since $W^l(R^\infty)$ is a Hilbert space, the dual of $L^2(0,T;W^l(R^\infty))$ is the space $L^2(0,T;W^{-1}(R^\infty))$.

Similarly, we can define the spaces

$$L^2(0,T;L^2(\Omega)) = L^2(Q), \quad (Q = \Omega\times]0,T[),$$
$$L^2(0,T;L^2(\Gamma)) = L^2(\Sigma), \quad (\Sigma = \Gamma\times]0,T[),$$

where Ω is a bounded and open set in R^∞ with sufficiently smooth boundary Γ and Σ is the lateral boundary of Q.

Let us introduce the space $W^{1,p}([0,T];E)$ the space

$$\{y \in L^p(0,T;L^2(\Omega)); y' \in L^p(0,T;E)\},$$

where the derivative y' of y is taken in the sense of vectorial distribution on $]0,T[$, E is a Banach space, and the space

$$W_p^{2,1}(Q) = L^p(0,T;W^{1,2}(\Omega)) \cap W^{1,p}(0,T;L^p(\Omega)).$$

For $p = 2$ we set $W_2^{2,1}(Q) = H^{2,1}(Q)$.

Finally, let us introduce the space denoted by $W(Q)$ the space of all functions

$$y \in L^2(0,T;W^l(R^\infty))$$

such that

$$\left(\frac{d}{dt}\right) y \in L^2(0,T;W^{-l}(R^\infty)).$$

$W(Q)$ is a Banach space with the natural norm

$$\|y\|^2_{W(Q)} = \|y\|^2_{L^2(0,T;W^1(R^\infty))} + \left\|\frac{dy}{dt}\right\|^2_{L^2(0,T;W^{-1}(R^\infty))}.$$

The considered spaces are assumed to be real and $l = 1$.

3. Main results

We introduce convex control problem governed by boundary value problem of the form

$$\begin{aligned} &y_t + Ay = 0 \quad \text{in} \quad Q = \Omega\times]0,T[, \\ &\frac{\partial y}{\partial v} + \beta_i(y) \ni u_i + f_i \qquad \text{in} \quad \Sigma_i = \Gamma\times]0,T[, \quad i = 1,2, \\ &y(x,0) = y_0(x) \quad \text{in} \quad \Omega, \end{aligned} \tag{3.1}$$

where Ω is a bounded open set in R^∞ with boundary Γ consists of two parts Γ_1 and Γ_2, that is, $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \Phi$, and $\Sigma = \Gamma\times]0,T[$ is the lateral boundary of Q, $\frac{\partial}{\partial v}$ is the outward normal derivative corresponding to A, and β_i are maximal monotone graphs in $R\times R$, which satisfy the conditions

$$\beta_i(0) \ni 0, \quad i = 1,2, \tag{3.2}$$

the controls u_i are taken from the Hilbert spaces $L^2(\Sigma_i)$, $i = 1,2$. The functions y_0, f_i are fixed in $L^2(R^\infty)$ and $L^2(\Sigma_i)$, $i = 1,2$, respectively. A is a second order self-adjoint elliptic partial deferential operator with an infinite number of variables that maps $W^1(R^\infty)$ onto $W^{-1}(R^\infty)$ and take the form:

$$Ay(x) = -\sum_{k=1}^{\infty} \frac{1}{\sqrt{p_k(x_k,t)\partial x_k^2}} \frac{\partial^2}{\partial x_k^2} \sqrt{p_k(x_k,t)}y(x) + q(x,t)y(x),$$

$$Ay(x) = -\sum_{k=1}^{\infty} (D_x^2 y)(x) + q(x)y(x). \tag{3.3}$$

For each $t \in]0,T[$, the bilinear form

$$\pi(t,\varphi,\psi) = (A(t)\varphi,\psi)_{L^2(R^\infty)} \quad \varphi,\psi \in W^1(R^\infty) \quad \text{on} \quad W^1(R^\infty), \tag{3.4}$$

is coercive on $W^1(R^\infty)$, that is, there exists $\lambda \in R$, such that

$$\pi(t;\varphi,\varphi) \geq \lambda\|\varphi\|^2_{W^1(R^\infty)} \quad \lambda > 0. \tag{3.5}$$

Definition 3.1. A function $y \in W(Q)$ is a solution to (3.1) if there exist functions $\omega_i \in L^2(\Sigma_i)$, $i = 1,2$, such that

$$\omega_i(\sigma,t) \in \beta_i(y(\sigma,t)) \quad \text{a.e. } (\sigma,t) \in \Sigma_i, i = 1,2 \tag{3.6}$$

and

$$\int_Q yk_t dp(x)dt + \int_0^T \pi(y,k)dt + \sum_{i=1}^{2} \int_{\Sigma_i} (\omega_i - v_i)kd\Gamma dt = \int_{R^\infty} y_0(x)k(x,0)dp(x), \tag{3.7}$$

for all $k \in W(Q)$ such that $k(x,T) = 0$. Here $\pi(y,k)$ is bilinear functional has the form (3.4), condition (3.7) can be equivalently defined as

$$\frac{d}{dt}((y(t),\psi) + \pi(y(t),\psi)) + \sum_{i=1}^{2} \int_{\Gamma_i} (\omega_i - v_i)\psi d\Gamma = 0 \quad \text{a.e.} \quad t \in]0,T[,$$

$$y(0) = y_0 \quad \text{for all} \quad \psi \in W^1(R^\infty). \tag{3.7'}$$

Let ρ be a C_0^∞-mollifie function on R, satisfying $\rho(r) \succ 0$ for $r \in]-1,1[$, $\rho(r) = 0$ for $|r| \succ 1$, $\rho(r) = \rho(-r)$ for all $r \in R$ and $\int\limits_{-\infty}^{\infty} \rho(r)dr = 1$. We define, for $\varepsilon \succ 0$

$$\beta_i^\varepsilon(r) = \int_{-\infty}^{\infty} \beta_{i\varepsilon}(r - \varepsilon\theta)\rho(\theta)d\theta, \quad i = 1,2, \quad r \in R,$$

where

$$\beta_{i\varepsilon}(r) = \varepsilon^{-1}(r - (1 + \varepsilon\beta_i)^{-1}r).$$

It should be recalled that β_i^ε are monotonically increasing infinitely differentiable functions. Moreover, β_i^ε are Lipschitzian with Lipschitz constant ε^{-1}, and in a certain sense which will be explained below they approximate β_i, for $\varepsilon \to 0$. For each $\varepsilon \succ 0$, consider the approximating system:

$$y_t + Ay = 0 \quad \text{in } Q,$$

$$\frac{\partial y}{\partial v} + \beta_i^\varepsilon(y) = u_i + f_i \quad \text{in } \Sigma_i, \quad i = 1,2, \tag{3.8}$$

$$y(x,0) = y_0(x) \quad \text{in } \Omega.$$

According to a standard existence result due to Lions [15] the system (3.8) has a unique solution $y_\varepsilon \in W(Q)$.

Let $\mathcal{A}_\varepsilon : w^1(R^\infty) \to (w^{-1}(R^\infty))$ be the operator defined by

$$(\mathcal{A}_\varepsilon y, \psi) = \pi(y,\psi) + \sum_{i=1}^{2} \int_{\Gamma_1} \beta_i^\varepsilon(y)\psi d\sigma, \quad y,\psi \in w^1(\mathbb{R}^\infty). \tag{3.9}$$

and let $f \in L^2(0,T;(w^{-1}(R^\infty))$ be given by:

$$(f(t),\psi) = \sum_{i=1}^{2} \int_{\Gamma_i} u_i \psi d\sigma, \quad \psi \in w^1(R^\infty). \tag{3.10}$$

Then in the sense of Definition 3.1, (3.8) can be written as

$$\frac{dy}{dt} + \mathcal{A}_\varepsilon y = f, \quad t \in [0,T],$$
$$y(0) = y_0. \tag{3.11}$$

Let $j_i : R \to \bar{R}$, $i = 1, 2$, be two lower semi-continuous convex functions such that $\partial j_i = \beta_i$ (it is well known that such functions always exist).

Under the assumptions and the coerciveness condition (3.5), we have:

Theorem 3.2. *Let $y_0 \in L^2(R^\infty)$ and $u_i \in L^2(\Sigma_i)$, $f_i \in L^2(\Sigma_i)$, $i = 1, 2$. Then the system* (3.1) *has a unique solution $y \in W(Q)$. Furthermore, for $\varepsilon \to 0$ we have*

$$y_\varepsilon \to y \quad \text{strongly in} \quad C([0,T];L^2(\Omega)) \cap L^2(0,T;W^1(R^\infty)) \tag{3.12}$$

and weakly in W (Q). And there exists $c \succ 0$ independent of u_i such that

$$\|y\|_{W(Q)} + \sum_{i=1}^{2} \|\beta_i(y)\|_{L^2(\Sigma_i)} \leq C\left(\sum_{i=1}^{2} \|u_i\|_{L^2(\Sigma_i)} + 1\right). \tag{3.13}$$

Proof. We take the inner product of (3.11) with y_ε and integrate over $[0,t]$. From (3.9) and (3.10) it follows that

$$||y_\varepsilon(t)\|_{L^2(\Omega)} + \int_0^t \|y_\varepsilon(s)\|^2_{w^1(R^\infty)} ds \leq C(\|u_1\|^2_{L^2(\Sigma_1)} + \|u_2\|^2_{L^2(\Sigma_2)} + 1), \quad t \in [0,T], \tag{3.14}$$

where t is independent of ε.

Next we take the inner product of (3.11) with $\beta_i^\varepsilon(y_\varepsilon)$. In as much as $\pi(\psi, \beta_i^\varepsilon(\psi)) \geq 0$ for all $\psi \in w^1(R^\infty)$, we find, after some calculations,

$$\int_\Omega j_i^\varepsilon(y_\varepsilon) dx + \sum_{i=1}^{2} \int_{\Sigma_i} \beta_i^\varepsilon(y_\varepsilon - u_i) \beta_i^\varepsilon(y_\varepsilon) d\sigma dt \leq \int_\Omega j_i^\varepsilon(y_0) dx \quad \text{for} \quad i = 1, 2, \tag{3.15}$$

where

$$j_i^\varepsilon(r) = \int_0^r \beta_i^\varepsilon(s) ds, \quad i = 1, 2.$$

Along with assumption (3.2), (3.15) yields

$$\sum_{i=1}^{2} \|\beta_i^{\varepsilon}(y_\varepsilon)\||^2_{L^2(\Sigma_i)} \leq C\bigg(\sum_{i=1}^{2} \|u_i\|^2_{\mathrm{L}^2(\Sigma_i)} + 1\bigg). \tag{3.16}$$

And by (3.11), (3.14) we see that

$$\|y_\varepsilon\|^2_{W(Q)} + \sum_{i=1}^{2} \|\beta_i^{\varepsilon}(y_\varepsilon)\|^2_{L^2(\Sigma_i)} \leq C\bigg(\sum_{i=1}^{2} \|u_i\|^2_{L^2(\Sigma_i)} + 1\bigg), \tag{3.17}$$

where C is independent of ε. Now using (3.11), for ε, $\lambda > 0$ we get

$$\|y_\varepsilon(t) - y\lambda(t)\|^2_{L^2(\Omega)} + \|y_\varepsilon(t) - y_\lambda(t)\|^2_{L^2(0,T;w^1(R^\infty))}$$
$$+ C\sum_{i=1}^{2} \int_{\Sigma_i} \beta_i^{\varepsilon}(y_\varepsilon) - \beta_i^{\lambda}(y_\lambda))(y_\varepsilon - y_\lambda)d\sigma dt \leq 0.$$

If we take into account the following relations:

$$\beta_i^{\varepsilon}(r) = \int_{-\infty}^{\infty} \beta_{i\varepsilon}(r - \varepsilon\theta)\rho(\theta)d\theta, \quad i = 1, 2, \quad r \in R,$$

where

$$\beta_{i\varepsilon}(r) = \varepsilon^{-1}(r - (1 + \varepsilon\beta_i)^{-1}r).$$

And if we take into account (3.16) and the monotonicity of β_i, we find

$$\|y_\varepsilon - y_\lambda\|^2_{C([0,T];L^2(\Omega))} + \|y_\varepsilon - y_\lambda\|_{L^2([0,T];w^1(R^\infty))} \leq C(\varepsilon - \lambda). \tag{3.18}$$

Hence y exist in the strong topology of $L^2([0,T]; w^1(R^\infty)) \cap C([0,T]; L^2(\Omega))$, particular, this implies that

$$y_\varepsilon \to y \quad \text{strongly in} \quad L^2([0,T]; H^{\frac{1}{2}}(\Gamma)) \subset L^2(\Sigma).$$

To obtain (3.12), (3.13) we let $\varepsilon \to 0$ in (3.17) and let $\lambda \to 0$ in (3.18) and this completes the proof. □

We shall study the following control problem:

(P) Minimize

$$\frac{1}{2}\int_Q h(x,t)|y(x,t) - y_d(x,t)|^2 dp(x)dt + \psi_1(u_1) + \psi_2(u_2) + \phi(y(T)) \tag{3.19}$$
$$= J(y,u),$$

on the class of all $u_i \in L^2(\Sigma_i)$, $i = 1, 2$, and $y \in W(Q)$ subject to the state system (3.1).

We shall assume that the following conditions are satisfied:

(1) $U_i = L^2(\Sigma_i)$, $i = 1, 2$, are the spaces of controls u_i, $i = 1, 2$.
(2) The functions $\psi_i : L^2(\Sigma_i) \to \bar{R}$, u_i, $i = 1, 2$ are lower semi-continuous convex functions and not identically equal to infinity.
(3) The function $\phi : L^2(R^\infty) \to R$ is convex and continuous on $L^2(R^\infty)$.
(4) $h \in L^\infty(Q)$ and $y_d \in L^2(Q)$ are given; $h \geq 0$ a.e. on Q.
(5) A is the elliptic symmetric operator which is presented by (3.3) and β_i, $i = 1, 2$, are two maximal monotone graphs in $R \times R$ which satisfy condition (3.2).
(6) $y_0 \in L^2(R^\infty)$ and $f_i \in L^2(\Sigma_i)$, $i = 1, 2$, satisfy the assumptions of Theorem 3.2. Under our assumptions, the coerciveness condition (3.5), we may apply the result of Barbu ([2], [5]) for every pair $(u_1, u_2) \in L^2(\Sigma_1) \times L^2(\Sigma_2)$. Problem (P), has at least optimal (y^*, u_1^*, u_2^*) where $y^* \in W(Q)$, $u_1 \in L^2(\Sigma_i)$, $i = 1, 2$, for which the infimum of the functional (3.10) is attained for $y = y^*$ and $u_i = u_i^*$, $i = 1, 2$. The optimality result is given in the case in which β_i are single-valued and satisfy the following condition.
(7) The function β_i ae monotonically increasing and locally Lipschitzian on the real axis R. Moreover, there exists $c \succ 0$, such that

$$\beta_i'(r) \leq c(|\beta_i(r)| + |r| + 1) \quad \text{a.e.} \quad r \in R, \quad i = 1, 2. \tag{3.20}$$

In the following we shall introduce the finite element discretization of the state equation and optimal control problem ([1], [14]):

At first let us consider the finite element approximation of the state equation (3.1). For the spatial discretization we consider conforming Largrange triangle elements. We assume that Ω is a polygonal domain. Let Υ^h be a quasi-uniform partitioning of Ω into disjoint regular triangles τ, so that

$$\bar{\Omega} = \bigcup_{\tau \in \Upsilon^h} \bar{\tau}.$$

Associated with Υ^h is a finite dimensional subspace V^h of $C([0, T]; \bar{\Omega})$, such that for $\chi \in V^h$ and $\tau \in \Upsilon^h$, $\chi|_\tau$ are piecewise linear polynomials. We set

$$V_0^h = V^h \cap W_0^1(R^\infty).$$

Let Υ_U^h be a partitioning of Γ into disjoint regular segments s, so that

$$\Gamma = \bigcup_{s \in \Upsilon_U^h} \bar{s}.$$

Associated with Υ_U^h is another finite dimensional subspace U^h of $L^2([0, T]; \Gamma)$, such that for $\chi \in U^h$ and $s \in \Upsilon_U^h$, $\chi|_s$ are piecewise linear polynomials. Here we suppose that Υ_U^h is the restriction of Υ^h on the boundary Γ and $U^h = V^h(\Gamma)$, where $V^h(\Gamma)$ is the restriction of V^h on the boundary Γ.

Lagrange interpolation operator $I_h : C([0,T];\bar{\Omega}) \to V^h$, we have the following error estimate

$$\|w - I_h w\|_{l,r^\infty} \leq C h^{m-l} \|v\|_{m,R^\infty}, \quad 0 \leq l \leq 1 \leq m \leq \infty. \tag{3.21}$$

$Q_h : L^2(\Gamma) \to V^h(\Gamma)$ and $\widetilde{Q}_h : L^2(R^\infty) \to V_0^h$ denote orthogonal projection operators. Furthermore, $R_h : W^1([0,T];R^\infty) \to V_0^h$ denotes the Ritsz projection operator defined as:

$$\pi(R_h w, v_h) = \pi(w, v_h), \quad \forall\, v_h \in V_0^h. \tag{3.22}$$

It is well known that the Ritz projection satisfies:

$$\begin{aligned} &\|w - R_h w\|_{s,R^\infty} \leq C h^{l-s} \|w\|_{l,R^\infty}, \\ &w \in W_0^1(R^\infty) \cap W^l(R^\infty), \ \ \forall\, 0 \leq s \leq 1 \leq l \leq \infty. \end{aligned} \tag{3.23}$$

For the $L^2(\Gamma)$ projection operator Q_h we also have:

$$\|w - Q_h w\|_{0,\Gamma} \leq C h^{s-\frac{1}{2}} \|w\|_{s,R^\infty}, \quad w \in W^s(R^\infty), \quad \forall\, \frac{1}{2} \leq s \leq \infty \tag{3.24}$$

and

$$\|(I - Q_h)\partial_n w\|_{0,\Gamma} \leq C h^{\frac{1}{2}} \|w\|_{2,R^\infty}, \quad \text{for} \quad w \in W^2(R^\infty).$$

The semi-discrete finite element approximation of (3.1) reads :
find $y_h \in L^2(V^h)$ such that

$$\begin{aligned} -(y_h, \partial_t v_h)_Q + \pi(y_h, v_h)_Q = (f, v_h)_Q + (y_0^h, v_h(.,0)), \quad \forall\, v_h \in W^1(V_0^h), \\ y_h = Q_h(u) \quad \text{on} \quad \Sigma_i = \Gamma \times]0,T[. \end{aligned} \tag{3.25}$$

With y_0^h an approximation of y_0 the semi-discrete finite element approximation of (3.10), (3.1) reads as follows:

$$\text{Minimize} \quad J_h(y_h, u_h) \quad \text{over} \quad u_h \in U_{ad}^h,\ y_h \in L^2(V^h) \tag{3.26}$$

subject to

$$\begin{aligned} -\,(y_h, \partial_t v_h)_Q + \pi(y_h, v_h)_Q = (f, v_h)_Q + (y_0^h, v_h(.,0)), \\ \forall\, v_h \in W^1(V_0^h), \\ y_h = Q_h(u_h) \quad \text{on} \ \ \Sigma_i, \end{aligned} \tag{3.27}$$

where U_{ad}^h is an appropriate approximation to $U_i = L^2(\Sigma_i)$. It follows that (3.26), (3.27) has a unique solution (y_h, u_h).

We next consider the fully discrete approximation for the above semi-discrete problem by using the $dG(0)$ scheme in time. For simplicity we consider an equi-distant partition of the time interval.

Let $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$ with $k = \frac{T}{N}$ and $t_i = ik$, $i = 1, 2, \cdots, N$. We also set $I_i = (t_{i-1}, t_i]$, for $i = 1, 2, \cdots, N$. We construct the finite element spaces $V^h \in W^1(R^\infty)$ with the mesh Υ_U^h. Similarly we construct the finite element spaces U^h of $L^2([0,T];\Gamma)$, with the mesh Υ_U^h in our case we have $U^h = V^h(\Gamma)$, then we denote by V^h, U^h the finite element spaces defined on Υ^h, Υ_U^h on each time step.

Let V_k denote the space of piecewise constant functions on the time partion. We define the L^2 projection operator $P_k : L^2(0,T) \to V_k$ on I_i through

$$P_k(w)(t) = \frac{1}{k}\int w(s)ds \quad \text{for} \quad t \in I_i.$$

Then we have the following estimate:

$$\|(I - P_k)w\|_{L^2(0,T;H)} \leq Ck\|w_t\|_{L^2(0,T;H)}, \quad \forall\, w \in W^1(0,T;H), \tag{3.28}$$

where H denote some separable Hilbert space.

We consider a $dG(0)$ scheme for the time discretization and set

$$V_{hk} = \{\theta; \bar{\Omega} \times [0,T] \to R, \quad \theta(.,t)|_{\bar{\Omega}} \in V^h, \quad \theta(x,.)|_{I_n} \in P_0\}.$$

We introduce for $Y, \Phi \in V_{hk}$

$$A(Y_{hk};\Phi) = (f,\Phi)_Q + (y_0, \Phi_+^0), \quad \forall\, \Phi \in V_{hk}^0,$$

$$Y_{hk} = \Lambda(u) \quad \text{on} \;\; \Gamma, \tag{3.29}$$

where V_{hk}^0 denotes the subspace of V_{hk} with functions vanishing on Γ, and $\Lambda = P_k Q_h$.

As a result of the application of finite element method on boundary control problem governed by parabolic variational inequalities with an infinite number of variables. We estimate the error introduced by the discretization of the state equation, that is, The error between the solution of problem (3.1) and (3.29) by the following theorem :

Theorem 3.3. *Suppose that* $f \in L^2(L^2(R^\infty))$, $u \in L^2(L^2(\Gamma))$, *and* $y_0 \in L^2(R^\infty)$. *Let* $y \in L^2(L^2(R^\infty))$ *and* $Y_{hk} \in V_{hk}$ *with* $Y_{hk}|_\Sigma = \Lambda(u)$ *be the solution of problems* (3.1), (3.29), *respectively. Then we have*

$$\|y - Y_{hk}\|_{L^2(L^2(\Gamma))} \leq C\big(h^{\frac{1}{2}} + k^{\frac{1}{4}}\big)\big(\|f\|_{L^2(L^2(\Gamma))} + |\; y_0\|_{0,R^\infty} + \|u\|_{L^2(L^2(\Gamma))}\big).$$

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