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EXISTENCE, BLOW-UP AND EXPONENTIAL DECAY FOR A NONLINEAR KIRCHHOFF-CARRIER-LOVE EQUATION WITH DIRICHLET CONDITIONS

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Abstract. In this paper, we consider the initial boundary value problem for a nonlinear Kirchhoff-Carrier-Love equation. At first, by combining the linearization method, the Faedo-Galerkin method and the weak compactness method, we prove the local existence and uniqueness of a weak solution. Next, by constructing Lyapunov functional, we establish a blow-up result for solutions with a negative initial energy and give a sufficient condition to obtain the exponential decay of weak solutions.

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1. Introduction

In this paper, we consider the following initial boundary value problem:

$$\begin{aligned} & u_{tt}+\lambda u_t-\frac{\partial}{\partial x}\left[B_1\left(x,t,u,\|u\|^2,\|u_x\|^2,\|u_t\|^2,\|u_{xt}\|^2\right)u_x\right] \quad (1.1)\\ & -\frac{\partial}{\partial x}\left[B_2\left(x,t,u,\|u\|^2,\|u_x\|^2,\|u_t\|^2,\|u_{xt}\|^2\right)u_{xt}\right]\\ & -\frac{\partial}{\partial x}\left[B_3\left(x,t,u,\|u\|^2,\|u_x\|^2,\|u_t\|^2,\|u_{xt}\|^2\right)u_{xtt}\right]\\ = \; & F\left(x,t,u,u_x,u_t,u_{xt},\|u\|^2,\|u_x\|^2,\|u_t\|^2,\|u_{xt}\|^2\right)\\ & -\frac{\partial}{\partial x}\left[G\left(x,t,u,u_x,u_t,u_{xt},\|u\|^2,\|u_x\|^2,\|u_t\|^2,\|u_{xt}\|^2\right)\right]\\ & +f(x,t), \quad x\in\Omega=(0,1), 0<t<T, \end{aligned}$$

$$u(0,t)=u(1,t)=0, \tag{1.2}$$

$$u(x,0)=\tilde{u}_0(x),\ u_t(x,0)=\tilde{u}_1(x), \tag{1.3}$$

where $\lambda>0$ is constant, $\tilde{u}_0$, $\tilde{u}_1\in H_0^1\cap H^2$, and f, F, G are given functions under suitable assumptions later.

This problem has its origin in the model of Kirchhoff-Carrier-Love type because it connects Kirchhoff, Carrier and Love equations. For more details, Eq. (1.1) has its origin in the nonlinear vibration of an elastic string (Kirchhoff [5]), for which the associated equation is

$$\rho h u_{tt}=\left(P_0+\frac{Eh}{2L}\int_0^L\left|\frac{\partial u}{\partial y}(y,t)\right|^2dy\right)u_{xx}, \tag{1.4}$$

here u is the lateral deflection, L is the length of the string, h is the cross-sectional area, E is Young's modulus, ρ is the mass density, and P_0 is the initial tension. It is also related to the Carrier equation. In [1], Carrier established the equation which models vibrations of an elastic string when changes in tension are not small

$$\rho u_{tt}-\left(1+\frac{EA}{LT_0}\int_0^L u^2dx\right)u_{xx}=0, \tag{1.5}$$

where $u(x,t)$ is the x-derivative of the deformation, T_0 is the tension in the rest position, E is the Young modulus, A is the cross-section of a string, L is the length of a string and ρ is the density of a material. Clearly, if properties of a material vary with x and t, then there is a hyperbolic equation of the type

$$u_{tt}-B\left(x,t,\|u(t)\|^2\right)u_{xx}=0. \tag{1.6}$$

On the other hand, Eq.(1.1) arises from the Love equation

$$u_{tt} - \frac{E}{\rho} u_{xx} - 2\mu^2\omega^2 u_{xxtt} = 0, \tag{1.7}$$

presented by V. Radochová [15]. This equation describes the vertical oscillations of a rod, which was established from Euler's variational equation of an energy functional

$$\int_0^T dt \int_0^L \left[\frac{1}{2} F\rho \left(u_t^2 + \mu^2\omega^2 u_{tx}^2 \right) - \frac{1}{2} F \left(E u_x^2 + \rho\mu^2\omega^2 u_x u_{xtt} \right) \right] dx, \tag{1.8}$$

where u is the displacement, L is the length of the rod, F is the area of cross-section, ω is the cross-section radius, E is the Young modulus of the material and ρ is the mass density.

Note that Prob.(1.1)-(1.3), with the special case

$$B_i = B = B\left(x, t, u, \|u\|^2, \|u_x\|^2, \|u_t\|^2, \|u_{xt}\|^2\right),$$

$i = 1, 2, 3$, has just been considered in [18], where results relate to the existence, blow-up and exponential decay estimates have been proved. In case $B = B(x,t)$ and $F = F(u, u_x)$, $G = G(u, u_x)$ such that $(F, G) = (\frac{\partial \mathcal{F}}{\partial u}, \frac{\partial \mathcal{F}}{\partial u_x})$, the authors have proved that the solution blows up in finite time when $f(x,t) \equiv 0$ and the initial energy is negative. On the other hand, they have established a sufficient condition, in which the initial energy is positive and small, to guarantee the global existence and exponential decay of weak solutions.

It is well known that the existence, global existence, decay properties and blow-up of solutions to the initial boundary value problem for Kirchhoff type models under different types of hypotheses have been extensively studied by many authors, for example, we refer to [1]-[4], [7]-[21], and references therein.

In [2], the authors studied the existence of global solutions and exponential decay for a Kirchhoff-Carrier model with viscosity. In [14], the author investigated on the global existence, decay properties, and blow-up of solutions to the initial boundary value problem for the nonlinear Kirchhoff type. In [20], the viscoelastic equation of Kirchhoff type was considered and the authors established a new blow-up result for arbitrary positive initial energy, by using simple analysis techniques.

In this paper, motivated by [18], we also establish the linear recurrent sequence to prove that Prob.(1.1)-(1.3) has a solution. Furthermore, we try to consider the blow-up and decay properties of Prob.(1.1)-(1.3) with $B_1 = B_1\left(\|u_x(t)\|^2\right) \neq B_1(x,t)$ as in [18].

This paper is organized as follows. In the Section 2, because of the nonlinearities, we combine the linearization method for the nonlinear term, the Faedo-Galerkin method and the weak compactness method to prove Theorem 2.2 concerning the local existence of unique weak solution for Prob.(1.1)-(1.3), in case $F,\, G \in C^1([0,1]\times[0,T]\times\mathbb{R}^4\times\mathbb{R}^4_+)$; $B_i \in C^1([0,1]\times[0,T]\times\mathbb{R}\times\mathbb{R}^4_+)$ with $B_i(x,t,y,\vec{z}) \geq b_i > 0$, $i = 1,2,3$ $\forall (x,t) \in [0,1]\times[0,T]$, $\forall y \in \mathbb{R}$, $\forall \vec{z} \in \mathbb{R}^4_+$.

In Sections 3, 4, Prob.(1.1)-(1.3) is considered in case $B_1 = B_1\left(\|u_x(t)\|^2\right)$, $B_i = B_i(x,t)$, $i = 2,3$ and $F = F(u,u_x)$, $G = G(u,u_x)$ such that $(F,G) = (\frac{\partial \mathcal{F}}{\partial u}, \frac{\partial \mathcal{F}}{\partial u_x})$. Here, by constructing Lyapunov functional, we prove Theorem 3.1, Theorem 4.1 in order to obtain a blow-up result and the exponential decay of weak solutions. More precisely, in Section 3, with $f(x,t) \equiv 0$ and a negative initial energy, we prove that the solution of Prob.(1.1)-(1.3) blows up in finite time. In Section 4, we give a sufficient condition, where the initial energy is positive and small, any the global weak solution is exponential decaying.

By adopting and modifying the methods of [18], the results obtained in this study are superior to the results established in [18].

2. Existence of a weak solution

In this section, we consider the local existence for Prob.(1.1)-(1.3). Without loss of generality, we can suppose that $\lambda = 0$, $f(x,t) = 0$.

First, we set the preliminary as follows.

Let $\langle\cdot,\cdot\rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space, $\|\cdot\|$ be the norm in L^2 and $\|\cdot\|_X$ be the norm in the Banach space X. Let $L^p(0,T;X)$, $1 \leq p \leq \infty$ be the Banach space of the real functions $u : (0,T) \to X$ measurable, with

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, dt\right)^{1/p} < \infty \quad \text{for } 1 \leq p < \infty,$$

and

$$\|u\|_{L^\infty(0,T;X)} = ess\sup_{0<t<T} \|u(t)\|_X \quad \text{for } p = \infty.$$

Denote $u(t) = u(x,t)$, $u'(t) = u_t(t) = \frac{\partial u}{\partial t}(x,t)$, $u''(t) = u_{tt}(t) = \frac{\partial^2 u}{\partial t^2}(x,t)$, $u_x(t) = \frac{\partial u}{\partial x}(x,t)$, $u_{xx}(t) = \frac{\partial^2 u}{\partial x^2}(x,t)$.

With $F \in C^k([0,1]\times\mathbb{R}_+\times\mathbb{R}^4\times\mathbb{R}^4_+)$, $F = F(x,t,y_1,\cdots,y_4,z_1,\cdots,z_4)$, we put $D_1F = \frac{\partial F}{\partial x}$, $D_2F = \frac{\partial F}{\partial t}$, $D_{i+2}F = \frac{\partial F}{\partial y_i}$, $D_{i+6}F = \frac{\partial F}{\partial z_i}$, with $i = 1,\cdots,4$ and $D^\alpha F = D_1^{\alpha_1}\cdots D_{10}^{\alpha_{10}}F$, $\alpha = (\alpha_1,\cdots,\alpha_{10}) \in \mathbb{Z}^{10}_+$, $|\alpha| = \alpha_1+\cdots+\alpha_{10} \leq k$, $D^{(0,\cdots,0)}F = F$.

Similarly, with $B_i \in C^k([0,1] \times [0,T] \times \mathbb{R} \times \mathbb{R}_+^4)$, $B_i = B_i(x,t,y,z_1,\cdots,z_4)$, $i = 1,2,3$ we put $D_1 B_i = \frac{\partial B_i}{\partial x}$, $D_2 B_i = \frac{\partial B_i}{\partial t}$, $D_3 B_i = \frac{\partial B_i}{\partial y}$, $D_{j+3} B_i = \frac{\partial B_i}{\partial z_i}$, with $j = 1,\cdots,4$; $i = 1,2,3$ and $D^\beta B_i = D_1^{\beta_1}\cdots D_7^{\beta_7} B_i$, $\beta = (\beta_1,\cdots,\beta_7) \in \mathbb{Z}_+^7$, $|\beta| = \beta_1 + \cdots + \beta_7 \le k$, $D^{(0,\cdots,0)} B_i = B_i$, $i = 1,2,3$.

We recall the following properties related to the usual spaces $C([0,1])$, H^1, and

$$H_0^1 = \{v \in H^1 : v(1) = v(0) = 0\}.$$

We then have the following lemma.

Lemma 2.1. (i) *The imbedding $H^1 \hookrightarrow C([0,1])$ is compact and*

$$\|v\|_{C[0,1]} \le \sqrt{2}\left(\|v\|^2 + \|v_x\|^2\right)^{1/2} \text{ for all } v \in H^1. \tag{2.1}$$

(ii) *On H_0^1, $v \longmapsto \|v_x\|$ and $v \longmapsto \|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2\right)^{1/2}$ are equivalent norms. On the other hand*

$$\|v\|_{C([0,1])} \le \|v_x\| \text{ for all } v \in H_0^1. \tag{2.2}$$

The weak formulation of Prob.(1.1)-(1.3) can be stated in the following manner: Find

$$u \in W_T = \{u \in L^\infty\left(0,T; H_0^1 \cap H^2\right) : u', u'' \in L^\infty\left(0,T; H_0^1 \cap H^2\right)\},$$

such that u satisfies the following variational equation

$$\begin{aligned} &\langle u''(t), w\rangle + \langle B_1[u](t)u_x(t) + B_2[u](t)u_x'(t) + B_3[u](t)u_x''(t), w_x\rangle \\ &= \langle F[u](t), w\rangle + \langle G[u](t), w_x\rangle, \end{aligned} \tag{2.3}$$

for all $w \in H_0^1$, a.e., $t \in (0,T)$, together with the initial conditions

$$u(0) = \tilde{u}_0, \; u'(0) = \tilde{u}_1, \tag{2.4}$$

where

$$\begin{aligned} B_i[u](x,t) &= B_i\left(x,t,u(x,t), \|u(t)\|^2, \|u_x(t)\|^2, \left\|u'(t)\right\|^2, \left\|u_x'(t)\right\|^2\right) \\ &\quad i = 1,2,3, \\ F[u](x,t) &= F\Big(x,t,u(x,t),u_x(x,t),u'(x,t),u_x'(x,t), \|u(t)\|^2, \\ &\qquad \|u_x(t)\|^2, \left\|u'(t)\right\|^2, \left\|u_x'(t)\right\|^2\Big), \\ G[u](x,t) &= G\Big(x,t,u(x,t),u_x(x,t),u'(x,t),u_x'(x,t), \|u(t)\|^2, \\ &\qquad \|u_x(t)\|^2, \left\|u'(t)\right\|^2, \left\|u_x'(t)\right\|^2\Big). \end{aligned} \tag{2.5}$$

Let $T^* > 0$. Then we make the following assumptions:

(H_1) $\tilde{u}_0,\, \tilde{u}_1 \in H_0^1 \cap H^2$;

(H_2) $B_i \in C^1([0,1] \times [0,T^*] \times \mathbb{R} \times \mathbb{R}_+^4)$
and there exist three constants $b_{i*} > 0,\ i = 1,2,3$ such that
$B_i(x,t,y,\vec{z}) \geq b_{i*},\ \forall (x,t) \in [0,1] \times [0,T^*],\ \forall y \in \mathbb{R},\ \forall \vec{z} \in \mathbb{R}_+^4$;

(H_3) $F \in C^1([0,1] \times [0,T^*] \times \mathbb{R}^4 \times \mathbb{R}_+^4)$;

(H_4) $G \in C^1([0,1] \times [0,T^*] \times \mathbb{R}^4 \times \mathbb{R}_+^4)$.

In order to obtain unique and existence results of weak solution, we use a linearization method, which consists of two steps.

Step 1: This Step is devoted to establishing a linear recurrent sequence $\{u_m\}$.

For $M > 0$, we put

$$\begin{aligned} \bar{F}_M &= \|F\|_{C^1(A_M)} = \|F\|_{C^0(A_M)} + \sum\nolimits_{j=1}^{10} \|D_j F\|_{C^0(A_M)}, \qquad (2.6) \\ \bar{G}_M &= \|G\|_{C^1(A_M)} = \|G\|_{C^0(A_M)} + \sum\nolimits_{j=1}^{10} \|D_j G\|_{C^0(A_M)}, \\ B_i(M) &= \|B_i\|_{C^1(\tilde{A}_M)} = \|B_i\|_{C^0(\tilde{A}_M)} + \sum\nolimits_{j=1}^{7} \|D_j B_1\|_{C^0(\tilde{A}_M)},\ i = 1,2,3, \end{aligned}$$

where

$$\begin{aligned} \|F\|_{C^0(A_M)} &= \sup_{(x,t,y_1,\cdots,y_4,z_1,\cdots,z_4) \in A_M} |F(x,t,y_1,\cdots,y_4,z_1,\cdots,z_4)|, \\ \|B_i\|_{C^0(\tilde{A}_M)} &= \sup_{(x,t,y,z_1,\cdots,z_4) \in \tilde{A}_M} |B_i(x,t,y,z_1,\cdots,z_4)|, i = 1,2,3, \end{aligned}$$

with

$$A_M = [0,1] \times [0,T^*] \times [-M,M]^4 \times [0,M^2]^4$$

and

$$\tilde{A}_M = [0,1] \times [0,T^*] \times [-M,M] \times [0,M^2]^4.$$

For each $T \in (0,T^*]$ and $M > 0$, we put

$$\begin{cases} W(M,T) = \{v \in L^\infty(0,T;H_0^1 \cap H^2) : v' \in L^\infty(0,T;H_0^1 \cap H^2), \\ \qquad v'' \in L^\infty(0,T;H_0^1), \text{with } \|v\|_{L^\infty(0,T;H_0^1 \cap H^2)}, \\ \qquad \|v'\|_{L^\infty(0,T;H_0^1 \cap H^2)},\ \|v''\|_{L^\infty(0,T;H_0^1)} \leq M\}, \\ W_1(M,T) = \{v \in W(M,T) : v'' \in L^\infty(0,T;H_0^1 \cap H^2)\}. \end{cases} \qquad (2.7)$$

We now start to establish the linear recurrent sequence $\{u_m\}$. We shall choose the first term $u_0 \equiv 0$, and suppose that

$$u_{m-1} \in W_1(M,T), \qquad (2.8)$$

and then we find $u_m \in W_1(M,T)$ $(m \geq 1)$ such that u_m satisfies the linear variational problem:

$$\begin{cases} \langle u_m''(t), w\rangle + \langle B_{1m}(t)u_{mx}(t) + B_{2m}(t)u_{mx}'(t) + B_{3m}(t)u_{mx}''(t), w_x\rangle \\ = \langle F_m(t), w\rangle + \langle G_m(t), w_x\rangle,\ \forall w \in H_0^1, \\ u_m(0) = \tilde{u}_0,\ u_m'(0) = \tilde{u}_1, \end{cases} \tag{2.9}$$

where

$$\begin{aligned} B_{im}(x,t) &= B_i[u_{m-1}](x,t),\ i = 1,2,3, \\ F_m(x,t) &= F[u_{m-1}](x,t), \\ G_m(x,t) &= G[u_{m-1}](x,t). \end{aligned} \tag{2.10}$$

Now, we have the important result in Step 1 as follows.

Lemma 2.2. *Let* $(H_1)-(H_4)$ *hold. Then there exist positive constants* $M, T > 0$ *such that, for* $u_0 \equiv 0$, *there exists a recurrent sequence* $\{u_m\} \subset W_1(M,T)$ *defined by* (2.8)-(2.10).

Proof. The proof consists of several steps.

(i) *The Faedo-Galerkin approximation* (introduced by Lions [6]). Consider a special orthonormal basis $\{w_j\}$ on $H_0^1 : w_j(x) = \sqrt{2}\sin(j\pi x)$, $j \in \mathbb{N}$, formed by the eigenfunctions of the Laplacian $-\Delta = -\frac{\partial^2}{\partial x^2}$. Obviously, there exists $c_{mj}^{(k)}(t)$, $1 \leq j \leq k$, on interval $[0,T]$ such that if there is the expression of $u_m^{(k)}(t)$ in form

$$u_m^{(k)}(t) = \sum\nolimits_{j=1}^{k} c_{mj}^{(k)}(t)w_j, \tag{2.11}$$

then $u_m^{(k)}(t)$ satisfies

$$\begin{cases} \langle \ddot{u}_m^{(k)}(t), w_j\rangle + \langle B_{1m}(t)u_{mx}^{(k)}(t) + B_{2m}(t)\dot{u}_{mx}^{(k)}(t) + B_{3m}(t)\ddot{u}_{mx}^{(k)}(t), w_{jx}\rangle \\ = \langle F_m(t), w_j\rangle + \langle G_m(t), w_{jx}\rangle,\ 1 \leq j \leq k, \\ u_m^{(k)}(0) = \tilde{u}_{0k},\ \dot{u}_m^{(k)}(0) = \tilde{u}_{1k}, \end{cases} \tag{2.12}$$

in which

$$\begin{cases} \tilde{u}_{0k} = \sum_{j=1}^{k} \alpha_j^{(k)} w_j \to \tilde{u}_0 \text{ strongly in } H_0^1 \cap H^2, \\ \tilde{u}_{1k} = \sum_{j=1}^{k} \beta_j^{(k)} w_j \to \tilde{u}_1 \text{ strongly in } H_0^1 \cap H^2. \end{cases} \tag{2.13}$$

Indeed, it implies from (2.12) that an equivalent form of system (2.12) as below

$$\begin{cases} \ddot{c}_{mi}^{(k)}(t) + \sum_{j=1}^{k} \left[B_{1ij}^{(m)}(t)c_{mj}^{(k)}(t) + B_{2ij}^{(m)}(t)\dot{c}_{mj}^{(k)}(t) + B_{3ij}^{(m)}(t)\ddot{c}_{mj}^{(k)}(t)\right] = f_{mi}(t), \\ c_{mi}^{(k)}(0) = \alpha_i^{(k)},\ \dot{c}_{mi}^{(k)}(0) = \beta_i^{(k)},\ 1 \leq i \leq k, \end{cases} \tag{2.14}$$

with

$$\begin{cases} f_{mj}(t) = \langle F_m(t), w_j \rangle + \langle G_m(t), w_{jx} \rangle, \\ B_{sij}^{(m)}(t) = \langle B_{sm}(t) w_{ix}, w_{jx} \rangle, \ 1 \le i, \ j \le k, \ s = 1, 2, 3. \end{cases} \tag{2.15}$$

Then we have the following property:

For fixed $M > 0$ and $T > 0$, the system (2.14)-(2.15) has a unique solution $c_m^{(k)} = (c_{m1}^{(k)}, \cdots, c_{mk}^{(k)})$ on an interval $[0, T]$.

In fact, if we omit indexs m, k, the system (2.14)-(2.15) is written as follows

$$\begin{cases} (E + B_3(t))\, c''(t) + B_1(t) c(t) + B_2(t) c'(t) = f(t), \\ c(0) = \alpha, \ c'(0) = \beta, \end{cases} \tag{2.16}$$

where

$$\begin{aligned} c(t) &= (c_1(t), \cdots, c_k(t))^T, \\ f(t) &= (f_1(t), \cdots, f_k(t))^T, \\ f_j(t) &= \langle F_m(t), w_j \rangle + \langle G_m(t), w_{jx} \rangle, \ 1 \le j \le k, \\ \alpha &= (\alpha_1, \cdots, \alpha_k)^T, \ \alpha_i = \alpha_i^{(k)}, \ 1 \le i \le k, \\ \beta &= (\beta_1, \cdots, \beta_k)^T, \ \beta_i = \beta_i^{(k)}, \ 1 \le i \le k, \\ E &= [\delta_{ij}] = diag(1, \cdots, 1) \text{ is the identity matrix of size } k, \\ B_s(t) &= [B_{sij}(t)], \ s = 1, 2, 3, \\ B_{sij}(t) &= \langle B_{sm}(t) w_{ix}, w_{jx} \rangle, \ 1 \le i, j \le k, \ s = 1, 2, 3. \end{aligned} \tag{2.17}$$

By

$$\begin{aligned} y^T (E + B_3(t))\, y &= \|y\|^2 + \sum\nolimits_{i=1}^k y_i \sum\nolimits_{j=1}^k B_{3ij}(t) y_j \\ &= \|y\|^2 + \int_0^1 B_{3m}(x, t) \left| \sum\nolimits_{j=1}^k y_j w_{jx} \right|^2 dx \\ &\ge \|y\|^2 + b_{3*} \sum\nolimits_{j=1}^k \lambda_j y_j^2 \\ &\ge \left(1 + b_{3*} \pi^2\right) \|y\|^2 > 0, \ \forall y \in \mathbb{R}^k, \ y \ne 0, \end{aligned}$$

$E + B_3(t)$ is invertible, it implies that (2.16) can be rewritten as follows

$$\begin{cases} c''(t) + \bar{B}_1(t) c(t) + \bar{B}_2(t) c'(t) = \bar{f}(t), \\ c(0) = \alpha, \ c'(0) = \beta, \end{cases} \tag{2.18}$$

where

$$\begin{aligned} \bar{B}_s(t) &= (E + B_3(t))^{-1} B_s(t), \ s = 1, 2, \\ \bar{f}(t) &= (E + B_3(t))^{-1} f(t). \end{aligned} \tag{2.19}$$

Integrating (2.18) over $(0,t)$, it gives

$$c(t)=(Uc)(t), \tag{2.20}$$

where

$$\begin{aligned}(Uc)(t) &= h(t)+(Lc)(t), \qquad (2.21)\\ h(t) &= \alpha+t\left(\beta+\bar{B}_2(0)\alpha\right)+\int_0^t d\tau\int_0^\tau \bar{f}(s)ds,\\ (Lc)(t) &= \int_0^t\left[(t-s)\left(\bar{B}_2'(s)-\bar{B}_1(s)\right)-\bar{B}_2(s)\right]c(s)ds.\end{aligned}$$

Applying the contraction principle, system (2.20) has a unique solution $c(t)$ in $[0,T]$. The proof is given below:

Let

$$\gamma>B_{\max}\equiv T\sup_{0\le s\le T}\left\|\bar{B}_2'(s)-\bar{B}_1(s)\right\|_1+\sup_{0\le s\le T}\left\|\bar{B}_2(s)\right\|_1,$$

where we denote $\left\|\bar{B}_r(s)\right\|_1=\max\limits_{1\le j\le k}\sum_{i=1}^k\left|\bar{B}_{rij}(s)\right|$, $r=1,2$.

It is well known that $X=C^0\left([0,T];\mathbb{R}^k\right)$ is a Banach space with respect to the norm

$$\|c\|_{\gamma,X}=\sup_{0\le t\le T}e^{-\gamma t}\left|c(t)\right|_1,\ \ |c(t)|_1=\sum\nolimits_{j=1}^k|c_j(t)|,\ c\in X.$$

Clearly, $U:X\to X$. We will prove that $U:X\to X$ is contractive as follows.

First we note that, for all $c=(c_1,\cdots,c_k)$, $d=(d_1,\cdots,d_k)\in X$, $q=c-d$,

$$\begin{aligned}|(Uc)(t)-(Ud)(t)|_1 &= |(Lq)(t)|_1\\ &\le \int_0^t\left|\left[(t-s)\left(\bar{B}_2'(s)-\bar{B}_1(s)\right)-\bar{B}_2(s)\right]q(s)\right|_1 ds\\ &\le \int_0^t\left\|(t-s)\left(\bar{B}_2'(s)-\bar{B}_1(s)\right)-\bar{B}_2(s)\right\|_1|q(s)|_1ds\\ &\le \int_0^t\left[(t-s)\left\|\bar{B}_2'(s)-\bar{B}_1(s)\right\|_1+\left\|\bar{B}_2(s)\right\|_1\right]|q(s)|_1ds\\ &\le B_{\max}\|q\|_{\gamma,X}\int_0^t e^{\gamma s}ds\le B_{\max}\|c-d\|_{\gamma,X}\frac{e^{\gamma t}}{\gamma}.\end{aligned}$$

It follows that

$$e^{-\gamma t}\left|(Uc)(t)-(Ud)(t)\right|_1\le\frac{B_{\max}}{\gamma}\|c-d\|_{\gamma,X},$$

it leads to

$$\|Uc - Ud\|_{\gamma,X} \leq \frac{B_{\max}}{\gamma} \|c - d\|_{\gamma,X}. \tag{2.22}$$

Since $0 < \dfrac{B_{\max}}{\gamma} < 1$, $U : X \to X$ is contractive. Then, (2.20) has a unique solution $c \in X$. Thus, system (2.12) has a unique solution $u_m^{(k)}(t)$ in $[0, T]$. So, for fixed $M > 0$ and $T > 0$, the system (2.14)-(2.15) has a unique solution $c_m^{(k)} = (c_{m1}^{(k)}, \cdots, c_{mk}^{(k)})$ on an interval $[0, T]$.

(ii) *A priori estimates.* In what follows, we shall give a priori estimates to show that there exist positive constants M, $T > 0$ such that $u_m^{(k)} \in W(M, T)$, for all m and k.

Put

$$\begin{aligned} S_m^{(k)}(t) &= \left\|\dot{u}_m^{(k)}(t)\right\|^2 + \left\|\sqrt{B_{1m}(t)}u_{mx}^{(k)}(t)\right\|^2 + \left\|\sqrt{B_{3m}(t)}\dot{u}_{mx}^{(k)}(t)\right\|^2 \\ &+ \left\|\ddot{u}_m^{(k)}(t)\right\|^2 + \left\|\sqrt{B_{1m}(t)}\dot{u}_{mx}^{(k)}(t)\right\|^2 + \left\|\sqrt{B_{3m}(t)}\ddot{u}_{mx}^{(k)}(t)\right\|^2 \\ &+ \left\|\dot{u}_{mx}^{(k)}(t)\right\|^2 + \left\|\sqrt{B_{1m}(t)}\Delta u_m^{(k)}(t)\right\|^2 + \left\|\sqrt{B_{3m}(t)}\Delta \dot{u}_m^{(k)}(s)\right\|^2 \\ &+2\int_0^t \left[\left\|\sqrt{B_{2m}(s)}\dot{u}_{mx}^{(k)}(s)\right\|^2 + \left\|\sqrt{B_{2m}(s)}\ddot{u}_{mx}^{(k)}(s)\right\|^2 \right. \\ &\qquad \left. + \left\|\sqrt{B_{2m}(s)}\Delta \dot{u}_m^{(k)}(s)\right\|^2\right] ds. \end{aligned} \tag{2.23}$$

Then, from (2.12) and (2.23), we get

$$\begin{aligned} S_m^{(k)}(t) &= S_m^{(k)}(0) + 2\int_0^t \langle F_m(s), \dot{u}_m^{(k)}(s)\rangle ds - 2\int_0^t \langle F_m(s), \Delta\dot{u}_m^{(k)}(s)\rangle ds \\ &+ 2\int_0^t \langle G_m(s), \dot{u}_{mx}^{(k)}(s)\rangle ds + 2\int_0^t \langle F'_m(s), \ddot{u}_m^{(k)}(s)\rangle ds \\ &+ 2\int_0^t \langle G'_m(s), \ddot{u}_{mx}^{(k)}(s)\rangle ds + 2\int_0^t \langle G_{mx}(s), \triangle\dot{u}_m^{(k)}(s)\rangle ds \\ &+ \int_0^t ds \int_0^1 \left[B'_{1m}(x,s)\left(\left|u_{mx}^{(k)}(x,s)\right|^2 + \left|\dot{u}_{mx}^{(k)}(x,s)\right|^2 + \left|\Delta u_m^{(k)}(x,s)\right|^2\right)\right. \\ &\qquad \left. + B'_{3m}(x,s)\left(\left|\dot{u}_{mx}^{(k)}(x,s)\right|^2 + \left|\Delta\dot{u}_m^{(k)}(x,s)\right|^2 - \left|\ddot{u}_{mx}^{(k)}(x,s)\right|^2\right)\right] dx \\ &- 2\int_0^t \langle B'_{1m}(s)u_{mx}^{(k)}(s) + B'_{2m}(s)\dot{u}_{mx}^{(k)}(s), \ddot{u}_{mx}^{(k)}(s)\rangle ds \end{aligned} \tag{2.24}$$

$$- 2\int_0^t \langle B_{1mx}(s)u_{mx}^{(k)}(s) + B_{2mx}\dot{u}_{mx}^{(k)}(s) + B_{3mx}(s)\ddot{u}_{mx}^{(k)}(s), \Delta\dot{u}_m^{(k)}(s)\rangle ds$$
$$= S_m^{(k)}(0) + \sum\nolimits_{j=1}^{9} I_j.$$

Estimate $\xi_m^{(k)} = \left\|\ddot{u}_m^{(k)}(0)\right\|^2 + \left\|\sqrt{B_{3m}(0)}\ddot{u}_{mx}^{(k)}(0)\right\|^2$:

Letting $t \to 0_+$ in Eq.(2.12), multiplying the result by $\ddot{c}_{mj}^{(k)}(0)$, we have

$$\begin{aligned} &\left\|\ddot{u}_m^{(k)}(0)\right\|^2 + \left\|\sqrt{B_{3m}(0)}\ddot{u}_{mx}^{(k)}(0)\right\|^2 \\ &+ \left\langle B_{2m}(0)\tilde{u}_{1kx} + B_{1m}(0)\tilde{u}_{0kx}, \ddot{u}_{mx}^{(k)}(0)\right\rangle \\ &= \left\langle F_m(0), \ddot{u}_m^{(k)}(0)\right\rangle + \left\langle G_m(0), \ddot{u}_{mx}^{(k)}(0)\right\rangle. \end{aligned} \tag{2.25}$$

Then

$$\begin{aligned} \xi_m^{(k)} &= \left\|\ddot{u}_m^{(k)}(0)\right\|^2 + \left\|\sqrt{B_{3m}(0)}\ddot{u}_{mx}^{(k)}(0)\right\|^2 \\ &\le (\|B_{1m}(0)\tilde{u}_{0kx}\| + \|B_{2m}(0)\tilde{u}_{1kx}\|\ )\left\|\ddot{u}_{mx}^{(k)}(0)\right\| \\ &\quad + \|F_m(0)\|\left\|\ddot{u}_m^{(k)}(0)\right\| + \|G_m(0)\|\left\|\ddot{u}_{mx}^{(k)}(0)\right\| \\ &\le [\|B_{1m}(0)\tilde{u}_{0kx}\| + \|B_{2m}(0)\tilde{u}_{1kx}\| + \|F_m(0)\| + \|G_m(0)\|]\sqrt{\frac{\xi_m^{(k)}}{b_{3*}}} \\ &\le \frac{1}{b_{3*}}\left[\left\|\sqrt{B_{2m}(0)}\tilde{u}_{1kx}\right\| + \left\|\sqrt{B_{1m}(0)}\tilde{u}_{0kx}\right\| + \|F_m(0)\| + \|G_m(0)\|\right]^2. \end{aligned} \tag{2.26}$$

It is clear to see that the functions

$$B_{im}(x,0) = B_i\left(x, 0, \tilde{u}_0, \|\tilde{u}_0\|^2, \|\tilde{u}_{0x}\|^2, \|\tilde{u}_1\|^2, \|\tilde{u}_{1x}\|^2\right),$$

$i = 1, 2, 3$, are independent of m and the constant $\|F_m(0)\| + \|G_m(0)\|$ is also independent of m, by the fact that

$$\begin{aligned} &\|F_m(0)\| + \|G_m(0)\| \\ &= \left\|F\left(\cdot, 0, \tilde{u}_0, \tilde{u}_{0x}, \tilde{u}_1, \tilde{u}_{1x}, \|\tilde{u}_0\|^2, \|\tilde{u}_{0x}\|^2, \|\tilde{u}_1\|^2, \|\tilde{u}_{1x}\|^2\right)\right\| \\ &\quad + \left\|G\left(\cdot, 0, \tilde{u}_0, \tilde{u}_{0x}, \tilde{u}_1, \tilde{u}_{1x}, \|\tilde{u}_0\|^2, \|\tilde{u}_{0x}\|^2, \|\tilde{u}_1\|^2, \|\tilde{u}_{1x}\|^2\right)\right\|. \end{aligned}$$

Therefore

$$\xi_m^{(k)} \le \bar{S}_0, \text{ for all } m,\ k, \tag{2.27}$$

where $\bar{S}_0$ is a constant depending only on $\tilde{u}_0$, $\tilde{u}_1$, B_i, F, G, $i = 1, 2, 3$.

It implies from (2.13), (2.23) and (2.27) that

$$\begin{aligned} S_m^{(k)}(0) &= \|\tilde{u}_{1k}\|^2 + \left\|\sqrt{B_{1m}(0)}\tilde{u}_{0kx}\right\|^2 + \left\|\sqrt{B_{3m}(0)}\tilde{u}_{1kx}\right\|^2 \\ &\quad + \left\|\sqrt{B_{1m}(0)}\tilde{u}_{1kx}\right\|^2 + \|\tilde{u}_{1kx}\|^2 \\ &\quad + \left\|\sqrt{B_{1m}(0)}\Delta\tilde{u}_{0k}\right\|^2 + \left\|\sqrt{B_{3m}(0)}\Delta\tilde{u}_{1k}\right\|^2 + \xi_m^{(k)} \\ &\leq S_0, \end{aligned} \tag{2.28}$$

for all $m, k \in \mathbb{N}$, where S_0 is also a constant depending only on $\tilde{u}_0$, $\tilde{u}_1$, B_i, F, G, $i = 1, 2, 3$.

Estimate I_j:

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} I_1 &= 2\int_0^t \langle F_m(s), \dot{u}_m^{(k)}(s)\rangle ds \leq T\bar{F}_M^2 + \int_0^t \left\|\dot{u}_m^{(k)}(s)\right\|^2 ds, \\ I_2 &= -2\int_0^t \langle F_m(s), \Delta\dot{u}_m^{(k)}(s)\rangle ds \leq T\bar{F}_M^2 + \int_0^t \left\|\Delta\dot{u}_m^{(k)}(s)\right\|^2 ds, \\ I_3 &= 2\int_0^t \langle G_m(s), \dot{u}_{mx}^{(k)}(s)\rangle ds \leq T\bar{G}_M^2 + \int_0^t \left\|\dot{u}_{mx}^{(k)}(s)\right\|^2 ds. \end{aligned} \tag{2.29}$$

From

$$\begin{aligned} S_m^{(k)}(t) &\geq \left\|\dot{u}_m^{(k)}(t)\right\|^2 + \left\|\ddot{u}_m^{(k)}(t)\right\|^2 + \left\|\dot{u}_{mx}^{(k)}(t)\right\|^2 \\ &\quad + b_{1*}\left\|u_{mx}^{(k)}(t)\right\|^2 + b_{3*}\left\|\dot{u}_{mx}^{(k)}(t)\right\|^2 + b_{1*}\left\|\dot{u}_{mx}^{(k)}(t)\right\|^2 \\ &\quad + b_{3*}\left\|\ddot{u}_{mx}^{(k)}(t)\right\|^2 + b_{1*}\left\|\Delta u_m^{(k)}(t)\right\|^2 + b_{3*}\left\|\Delta\dot{u}_m^{(k)}(s)\right\|^2 \\ &\geq b_*\Big(2\left\|\dot{u}_m^{(k)}(t)\right\|^2 + \left\|\ddot{u}_m^{(k)}(t)\right\|^2 + 2\left\|\dot{u}_{mx}^{(k)}(t)\right\|^2 + \left\|u_{mx}^{(k)}(t)\right\|^2 \\ &\quad + \left\|\ddot{u}_{mx}^{(k)}(t)\right\|^2 + \left\|\Delta u_m^{(k)}(t)\right\|^2 + \left\|\Delta\dot{u}_m^{(k)}(s)\right\|^2\Big) \\ &\geq b_*\Big(\left\|\dot{u}_m^{(k)}(t)\right\|^2 + \left\|\ddot{u}_m^{(k)}(t)\right\|^2 + \left\|\dot{u}_{mx}^{(k)}(t)\right\|^2 + \left\|u_{mx}^{(k)}(t)\right\|^2 \\ &\quad + \left\|\ddot{u}_{mx}^{(k)}(t)\right\|^2 + \left\|\Delta u_m^{(k)}(t)\right\|^2 + \left\|\Delta\dot{u}_m^{(k)}(s)\right\|^2\Big), \end{aligned}$$

with $b_* = \min\{1, b_{1*}, b_{2*}, b_{3*}\}$, we get

$$I_1 + I_2 + I_3 \leq 2T\left(\bar{F}_M^2 + \bar{G}_M^2\right) + \frac{1}{b_*}\int_0^t S_m^{(k)}(s)ds. \tag{2.30}$$

Note that

$$\begin{aligned} F_m'(t) \;=\; & D_2F[u_{m-1}] + D_3F[u_{m-1}]u_{m-1}' \\ & + D_4F[u_{m-1}]\nabla u_{m-1}' \\ & + D_5F[u_{m-1}]u_{m-1}'' + D_6F[u_{m-1}]\nabla u_{m-1}'' \\ & + 2D_7F[u_{m-1}]\langle u_{m-1}(t), u_{m-1}'(t)\rangle \\ & + 2D_8F[u_{m-1}]\langle \nabla u_{m-1}(t), \nabla u_{m-1}'(t)\rangle \\ & + 2D_9F[u_{m-1}]\langle u_{m-1}'(t), u_{m-1}''(t)\rangle \\ & + 2D_{10}F[u_{m-1}]\langle \nabla u_{m-1}'(t), \nabla u_{m-1}''(t)\rangle, \end{aligned} \tag{2.31}$$

so

$$\left\|F_m'(t)\right\| \le \left(1 + 4M + 8M^2\right) \bar{F}_M \equiv \tilde{F}_M. \tag{2.32}$$

Thus

$$I_4 = 2\int_0^t \langle F_m'(s), \ddot{u}_m^{(k)}(s)\rangle ds \le T\tilde{F}_M^2 + \int_0^t \left\|\ddot{u}_m^{(k)}(s)\right\|^2 ds. \tag{2.33}$$

Similarly, we also obtain to the following estimate

$$I_5 = 2\int_0^t \langle G_m'(s), \ddot{u}_{mx}^{(k)}(s)\rangle ds \le T\tilde{G}_M^2 + \int_0^t \left\|\ddot{u}_{mx}^{(k)}(s)\right\|^2 ds, \tag{2.34}$$

with $\tilde{G}_M = \left(1 + 4M + 8M^2\right) \bar{G}_M$. By

$$\begin{aligned} G_{mx}(t) = {} & D_1G[u_{m-1}] + D_3G[u_{m-1}]\nabla u_{m-1} + D_4G[u_{m-1}]\Delta u_{m-1} \\ & + D_5G[u_{m-1}]\nabla u_{m-1}' + D_6G[u_{m-1}]\Delta u_{m-1}', \end{aligned} \tag{2.35}$$

we get

$$\|G_{mx}(t)\| \le (1 + 4M)\, \bar{G}_M \le \tilde{G}_M. \tag{2.36}$$

Hence

$$I_6 = 2\int_0^t \langle G_{mx}(s), \triangle\dot{u}_m^{(k)}(s)\rangle ds \le T\tilde{G}_M^2 + \int_0^t \left\|\triangle\dot{u}_m^{(k)}(s)\right\|^2 ds. \tag{2.37}$$

Consequently, we have

$$\begin{aligned} I_4 + I_5 + I_6 \;\le\; & 2T\left(\tilde{F}_M^2 + \tilde{G}_M^2\right) \\ & + \int_0^t \left[\left\|\ddot{u}_m^{(k)}(s)\right\|^2 + \left\|\ddot{u}_{mx}^{(k)}(s)\right\|^2 + \left\|\triangle\dot{u}_m^{(k)}(s)\right\|^2\right] ds \\ \le\; & 2T\left(\tilde{F}_M^2 + \tilde{G}_M^2\right) + \frac{1}{b_*}\int_0^t S_m^{(k)}(s) ds. \end{aligned} \tag{2.38}$$

By the fact that

$$
\begin{aligned}
B_{im}'(t) = {} & D_2 B_i[u_{m-1}] + D_3 B_i[u_{m-1}] u_{m-1}' \\
& + 2D_4 B_i[u_{m-1}] \langle u_{m-1}(t), u_{m-1}'(t) \rangle \\
& + 2D_5 B_i[u_{m-1}] \langle \nabla u_{m-1}(t), \nabla u_{m-1}'(t) \rangle \\
& + 2D_6 B_i[u_{m-1}] \langle u_{m-1}'(t), u_{m-1}''(t) \rangle \\
& + 2D_7 B_i[u_{m-1}] \langle \nabla u_{m-1}'(t), \nabla u_{m-1}''(t) \rangle, \ i = 1,2,3,
\end{aligned} \tag{2.39}
$$

so

$$
\begin{aligned}
\left| B_{im}'(x,t) \right| & \leq \left(1 + M + 8M^2\right) B_i(M) \equiv \bar{B}_{iM} \\
& \leq \bar{B}_M = \max_{i=1,2,3} \bar{B}_{iM}, \ i = 1,2,3.
\end{aligned} \tag{2.40}
$$

Hence

$$
\begin{aligned}
|I_7| = {} & \left| \int_0^t ds \int_0^1 \left[B_{1m}'(x,s) \left(\left| u_{mx}^{(k)}(x,s) \right|^2 + \left| \Delta u_m^{(k)}(x,s) \right|^2 + \left| \dot{u}_{mx}^{(k)}(x,s) \right|^2 \right) \right. \right. \\
& \left. \left. + B_{3m}'(x,s) \left(\left| \dot{u}_{mx}^{(k)}(x,s) \right|^2 - \left| \ddot{u}_{mx}^{(k)}(x,s) \right|^2 + \left| \Delta \dot{u}_m^{(k)}(x,s) \right|^2 \right) \right] dx \right| \\
\leq {} & \bar{B}_M \int_0^t \left[\left\| u_{mx}^{(k)}(s) \right\|^2 + 2 \left\| \dot{u}_{mx}^{(k)}(s) \right\|^2 + \left\| \Delta u_m^{(k)}(s) \right\|^2 \right. \\
& \left. + \left\| \ddot{u}_{mx}^{(k)}(s) \right\|^2 + \left\| \Delta \dot{u}_m^{(k)}(s) \right\|^2 \right] ds \\
\leq {} & \frac{\bar{B}_M}{b_*} \int_0^t S_m^{(k)}(s) ds.
\end{aligned} \tag{2.41}
$$

We deduce that

$$
\begin{aligned}
|I_8| & = \left| 2 \int_0^t \langle B_{1m}'(s) u_{mx}^{(k)}(s) + B_{2m}'(s) \dot{u}_{mx}^{(k)}(s), \ddot{u}_{mx}^{(k)}(s) \rangle ds \right| \\
& \leq 2\bar{B}_M \int_0^t \left(\left\| u_{mx}^{(k)}(s) \right\| + \left\| \dot{u}_{mx}^{(k)}(s) \right\| \right) \left\| \ddot{u}_{mx}^{(k)}(s) \right\| ds \\
& \leq 4\bar{B}_M \int_0^t S_m^{(k)}(s) ds.
\end{aligned} \tag{2.42}
$$

Furthermore,

$$
\begin{aligned}
B_{imx}(x,t) & = D_1 B_i[u_{m-1}] + D_3 B_i[u_{m-1}] \nabla u_{m-1}, \ i = 1,2,3, \\
|B_{imx}(x,t)| & \leq B_i(M)(1 + 2M) \equiv \hat{B}_{iM} \leq \hat{B}_M = \max_{i=1,2,3} \hat{B}_{iM}, \ i = 1,2,3,
\end{aligned} \tag{2.43}
$$

the following estimation is true

$$\begin{aligned}I_9 &= 2\int_0^t \Big\langle B_{1mx}(s)u_{mx}^{(k)}(s) + B_{2mx}(s)\dot{u}_{mx}^{(k)}(s) \\ &\quad + B_{3mx}(s)\ddot{u}_{mx}^{(k)}(s), \Delta\dot{u}_m^{(k)}(s)\Big\rangle ds \\ &\le 2\hat{B}_M \int_0^t \left(\left\|u_{mx}^{(k)}(s)\right\| + \left\|\dot{u}_{mx}^{(k)}(s)\right\| + \left\|\ddot{u}_{mx}^{(k)}(s)\right\|\right)\left\|\Delta\dot{u}_m^{(k)}(s)\right\| ds \\ &\le 6\hat{B}_M \int_0^t S_m^{(k)}(s)ds.\end{aligned} \tag{2.44}$$

Consequently, the estimations (2.24), (2.28), (2.30), (2.38), (2.41), (2.42) and (2.44) show that

$$\begin{aligned}S_m^{(k)}(t) &\le S_0 + 2T\left(\bar{F}_M^2 + \bar{G}_M^2 + \tilde{F}_M^2 + \tilde{G}_M^2\right) \\ &\quad + \left(\frac{2+\bar{B}_M}{b_*} + 4\bar{B}_M + 6\hat{B}_M\right)\int_0^t S_m^{(k)}(s)ds.\end{aligned} \tag{2.45}$$

We choose $M > 0$ sufficiently large such that

$$S_0 \le \frac{1}{2}M^2, \tag{2.46}$$

and then choose $T \in (0, T^*]$ small enough such that

$$\begin{aligned}&\left(\frac{1}{2}M^2 + 2T\left(\bar{F}_M^2 + \bar{G}_M^2 + \tilde{F}_M^2 + \tilde{G}_M^2\right)\right) \\ &\qquad \times \exp\left[T\left(\frac{2+\bar{B}_M}{b_*} + 4\bar{B}_M + 6\hat{B}_M\right)\right] \\ &\le M^2,\end{aligned} \tag{2.47}$$

where

$$k_T = \frac{6}{b_*}\sqrt{T}(1+3M)\left(\bar{F}_M + \bar{G}_M + \tilde{B}_M\right)\exp\left[T\left(1 + \frac{\bar{B}_M}{2b_*}\right)\right] < 1 \tag{2.48}$$

and

$$\tilde{B}_M = \max_{i=1,2,3}(1+4M)B_i(M).$$

From (2.45)-(2.48), we have

$$\begin{aligned}S_m^{(k)}(t) &\le M^2 \exp\left[-T\left(\frac{2+\bar{B}_M}{b_*} + 4\bar{B}_M + 6\hat{B}_M\right)\right] \\ &\quad + \left(\frac{2+\bar{B}_M}{b_*} + 4\bar{B}_M + 6\hat{B}_M\right)\int_0^t S_m^{(k)}(s)ds.\end{aligned} \tag{2.49}$$

Using Gronwall's Lemma, (2.49) leads to

$$S_m^{(k)}(t) \leq M^2 \exp\left[-(T-t)\left(\frac{2+\bar{B}_M}{b_*} + 4\bar{B}_M + 6\hat{B}_M\right)\right] \leq M^2, \tag{2.50}$$

for all $t \in [0,T]$, for all m and k, so

$$u_m^{(k)} \in W(M,T), \text{ for all } m \text{ and } k. \tag{2.51}$$

(iii) *Limiting process.* By (2.50), there exists a subsequence of $\{u_m^{(k)}\}$ with a same notation, such that

$$\begin{cases} u_m^{(k)} \to u_m & \text{in} \quad L^\infty(0,T;H_0^1 \cap H^2) \text{ weakly*}, \\ \dot{u}_m^{(k)} \to u_m' & \text{in} \quad L^\infty(0,T;H_0^1 \cap H^2) \text{ weakly*}, \\ \ddot{u}_m^{(k)} \to u_m'' & \text{in} \quad L^\infty(0,T;H_0^1) \text{ weakly*}, \\ u_m \in W(M,T). \end{cases} \tag{2.52}$$

Passing to limit in (2.12), (2.13), it is clear that u_m is satisfying (2.9), (2.10) in $L^2(0,T)$.

Furthermore, (2.9) and (2.52) imply that

$$\begin{aligned} &B_{3m}(t)\Delta u_m''(t) \\ &= -B_{1m}(t)\Delta u_m(t) - B_{2m}(t)\Delta u_m'(t) - B_{1mx}(t)u_{mx}(t) \\ &\quad -B_{2mx}(t)u_{mx}'(t) - B_{3mx}(t)u_{mx}''(t) + u_m''(t) - F_m(t) + G_{mx}(t) \\ &\equiv \Psi_m \in L^\infty(0,T;L^2). \end{aligned} \tag{2.53}$$

We have

$$b_* \left\|\Delta u_m''(t)\right\| \leq \left\|B_{3m}(t)\Delta u_m''(t)\right\| = \|\Psi_m(t)\| \leq \|\Psi_m\|_{L^\infty(0,T;L^2)}.$$

Hence $u_m'' \in L^\infty(0,T;H_0^1 \cap H^2)$, so we obtain $u_m \in W_1(M,T)$, Lemma 2.2 is proved. □

Then, **Step 1** is done from the Lemma 2.2.

Step 2: This Step will show that $\{u_m\}$ converges to u and u is exactly a unique local solution of Prob.(1.1)-(1.3). We have the following lemma.

Lemma 2.3. *Let* (H_1)-(H_4) *hold. Then*

(i) *Prob.*(1.1)-(1.3) *has a unique weak solution* $u \in W_1(M,T)$, *where* $M > 0$ *and* $T > 0$ *are chosen constants as in Lemma* 2.2.

(ii) *The linear recurrent sequence* $\{u_m\}$ *defined by* (2.8)-(2.10) *converges to the solution* u *of Prob.*(1.1)-(1.3) *strongly in the space*

$$W_1(T) = C^1\left([0,T];H_0^1\right). \tag{2.54}$$

Proof. We use the result obtained in Lemma 2.2 and the compact imbedding theorems to prove Lemma 2.3. It means that the existence and uniqueness of a weak solution of Prob.(1.1)-(1.3) is proved.

(**i**) *Existence.* It is well known that $W_1(T)$ is a Banach space (see Lions [6]), with respect to the norm

$$\|v\|_{W_1(T)} = \|v\|_{C\left([0,T];H_0^1\right)} + \left\|v'\right\|_{C\left([0,T];H_0^1\right)}. \tag{2.55}$$

It is clear that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Indeed, let

$$w_m = u_{m+1} - u_m.$$

Then, we have

$$\begin{cases} \langle w_m''(t), w\rangle + \langle B_{1,m+1}(t)w_{mx}(t) + B_{2,m+1}(t)w_{mx}'(t) + B_{3,m+1}(t)w_{mx}''(t), w_x\rangle \\ = \langle F_{m+1}(t) - F_m(t), w\rangle + \langle G_{m+1}(t) - G_m(t), w_x\rangle \\ \quad - \langle (B_{1,m+1}(t) - B_{1m}(t))\, u_{mx}(t) + (B_{2,m+1}(t) - B_{2m}(t))\, u_{mx}'(t), w_x\rangle \\ \quad - \langle (B_{3,m+1}(t) - B_{3m}(t))\, u_{mx}''(t), w_x\rangle\,,\ \forall w \in H_0^1, \\ w_m(0) = w_m'(0) = 0. \end{cases} \tag{2.56}$$

Consider (2.56) with $w = w_m'$, and then integrating in t, we get

$$\begin{aligned} Z_m(t) \quad = \quad & 2\int_0^t \left\langle F_{m+1}(s) - F_m(s), w_m'(s)\right\rangle ds \qquad (2.57) \\ & +2\int_0^t \left\langle G_{m+1}(s) - G_m(s), w_{mx}'(s)\right\rangle ds \\ & +\int_0^t ds \int_0^1 \left[B_{1,m+1}'(x,s)w_{mx}^2(x,s) + B_{3,m+1}'(x,s)\left|w_{mx}'(x,s)\right|^2\right] dx \\ & -2\int_0^t \left\langle (B_{1,m+1}(s) - B_{1m}(s))\, u_{mx}(s), w_{mx}'(s)\right\rangle ds \\ & -2\int_0^t \left\langle (B_{2,m+1}(s) - B_{2m}(s))\, u_{mx}'(s), w_{mx}'(s)\right\rangle ds \\ & -2\int_0^t \left\langle (B_{3,m+1}(s) - B_{3m}(s))\, u_{mx}''(s), w_{mx}'(s)\right\rangle ds \\ = \quad & J_1 + J_2 + J_3 + J_4 + J_5 + J_6 \end{aligned}$$

with

$$Z_m(t) \quad = \quad \left\|w_m'(t)\right\|^2 + \left\|\sqrt{B_{3,m+1}(t)}w_{mx}'(t)\right\|^2 \tag{2.58}$$

$$
\begin{aligned}
&+\left\|\sqrt{B_{1,m+1}(t)}w_{mx}(t)\right\|^2 \\
&+2\int_0^t \left\|\sqrt{B_{2,m+1}(s)}w'_{mx}(s)\right\|^2 ds \\
\geq\;& b_*\left(\left\|w'_m(t)\right\|^2+\left\|w'_{mx}(t)\right\|^2+\|w_{mx}(t)\|^2\right).
\end{aligned}
$$

By the fact that

$$
\begin{aligned}
\|F_{m+1}(s)-F_m(s)\| &\leq 2(1+2M)\bar{F}_M\|w_{m-1}\|_{W_1(T)}, \\
\|G_{m+1}(s)-G_m(s)\| &\leq 2(1+2M)\bar{G}_M\|w_{m-1}\|_{W_1(T)}, \\
\left|B'_{i,m+1}(x,s)\right| &\leq \left(1+M+8M^2\right)B_i(M)\equiv \bar{B}_{iM},\ i=1,3, \\
|B_{i,m+1}(x,s)-B_{im}(x,s)| &\leq (1+4M)B_i(M)\|w_{m-1}\|_{W_1(T)} \\
&\leq \tilde{B}_M\|w_{m-1}\|_{W_1(T)},\ i=1,2,3, \\
\left\|u_{mx}(s)+u'_{mx}(s)+u''_{mx}(s)\right\| &\leq 3M,
\end{aligned}
\tag{2.59}
$$

we obtain the estimations

$$
\begin{aligned}
J_1 &= 2\int_0^t \left\langle F_{m+1}(s)-F_m(s),w'_m(s)\right\rangle ds \\
&\leq \frac{4}{b_*}(1+2M)^2\bar{F}_M^2T\|w_{m-1}\|^2_{W_1(T)}+b_*\int_0^t\left\|w'_m(s)\right\|^2ds; \\
J_2 &= 2\int_0^t \left\langle G_{m+1}(s)-G_m(s),w'_{mx}(s)\right\rangle ds \\
&\leq \frac{4}{b_*}(1+2M)^2\bar{G}_M^2T\|w_{m-1}\|^2_{W_1(T)}+b_*\int_0^t\|w_{mx}(s)\|^2ds; \\
J_1+J_2 &\leq \frac{4}{b_*}(1+2M)^2\left(\bar{F}_M+\bar{G}_M\right)^2T\|w_{m-1}\|^2_{W_1(T)}+\int_0^t Z_m(s)ds; \\
J_3 &= \int_0^t ds\int_0^1 B'_{1,m+1}(x,s)w^2_{mx}(x,s)+B'_{3,m+1}(x,s)\left|w'_{mx}(x,s)\right|^2dx \\
&\leq \bar{B}_M\int_0^t\left(\|w_{mx}(s)\|^2+\left\|w'_{mx}(s)\right\|^2\right)ds \\
&\leq \frac{\bar{B}_M}{b_*}\int_0^t Z_m(s)ds
\end{aligned}
\tag{2.60}
$$

and

$$\begin{aligned}
|J_4+J_5+J_6| &\leq 2\int_0^t \left|\left\langle (B_{1,m+1}(s)-B_{1m}(s))\, u_{mx}(s), w'_{mx}(s)\right\rangle\right| ds \\
&\quad +2\int_0^t \left|\left\langle (B_{2,m+1}(s)-B_{2m}(s))\, u'_{mx}(s), w'_{mx}(s)\right\rangle\right| ds \\
&\quad +2\int_0^t \left|\left\langle (B_{3,m+1}(s)-B_{3m}(s))\, u''_{mx}(s), w'_{mx}(s)\right\rangle\right| ds \\
&\leq 6M\tilde{B}_M \left\|w_{m-1}\right\|_{W_1(T)} \int_0^t \left\|w'_{mx}(s)\right\| ds \\
&\leq \frac{9}{b_*} M^2 \tilde{B}_M^2 T \left\|w_{m-1}\right\|_{W_1(T)}^2 + \int_0^t Z_m(s) ds.
\end{aligned}$$

It follows from (2.57) and (2.60) that

$$\begin{aligned}
Z_m(t) &\leq \frac{9T}{b_*}(1+3M)^2 \left(\bar{F}_M+\bar{G}_M+\tilde{B}_M\right)^2 \left\|w_{m-1}\right\|_{W_1(T)}^2 \\
&\quad + \left(2+\frac{\bar{B}_M}{b_*}\right)\int_0^t Z_m(s) ds.
\end{aligned} \tag{2.61}$$

Using Gronwall's Lemma, (2.61) leads to

$$\left\|w_m\right\|_{W_1(T)} \leq k_T \left\|w_{m-1}\right\|_{W_1(T)}, \quad \forall m\in\mathbb{N}, \tag{2.62}$$

so

$$\left\|u_m-u_{m+p}\right\|_{W_1(T)} \leq M(1-k_T)^{-1} k_T^m, \quad \forall m,p\in\mathbb{N}. \tag{2.63}$$

It means that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$, so there exists $u\in W_1(T)$ such that

$$u_m \to u \text{ strongly in } W_1(T). \tag{2.64}$$

Note that $u_m\in W_1(M,T)$, so there exists a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$\begin{cases}
u_{m_j}\to u & \text{in} \quad L^\infty(0,T;H_0^1\cap H^2) \text{ weakly*}, \\
u'_{m_j}\to u' & \text{in} \quad L^\infty(0,T;H_0^1\cap H^2) \text{ weakly*}, \\
u''_{m_j}\to u'' & \text{in} \quad L^\infty(0,T;H_0^1) \text{ weakly*}, \\
u\in W(M,T). &
\end{cases} \tag{2.65}$$

On the other hand, by (2.6), (2.8), (2.10) and $(2.65)_4$, we obtain

$$\begin{aligned}
\left\|F_m(t)-F[u](t)\right\| &\leq 2(1+2M)\bar{F}_M \left\|u_{m-1}-u\right\|_{W_1(T)}, \\
\left\|G_m(t)-G[u](t)\right\| &\leq 2(1+2M)\bar{G}_M \left\|u_{m-1}-u\right\|_{W_1(T)}, \\
\left|B_{i,m+1}(x,t)-B_i[u](x,t)\right| &\leq \tilde{B}_M \left\|u_{m-1}-u\right\|_{W_1(T)}, \ i=1,2,3.
\end{aligned} \tag{2.66}$$

It implies from (2.64) and (2.66) that

$$\begin{cases} F_m \to F[u] \text{ strongly in } L^\infty(0,T;L^2), \\ G_m \to G[u] \text{ strongly in } L^\infty(0,T;L^2), \\ B_{im} \to B_i[u] \text{ strongly in } L^\infty(Q_T),\ i=1,2,3. \end{cases} \tag{2.67}$$

Passing to limit in (2.9), (2.10) as $m = m_j \to \infty$, by (2.64), (2.65) and (2.67), there exists $u \in W(M,T)$ satisfying the equation

$$\begin{aligned} &\langle u''(t), w\rangle + \langle B_3[u](t)u''_x(t) + B_2[u](t)u'_x(t) \\ &+B_1[u](t)u_x(t), w_x\rangle \\ &= \langle F[u](t), w\rangle + \langle G[u](t), w_x\rangle,\ \forall w \in H_0^1, \end{aligned} \tag{2.68}$$

and satisfying the initial conditions

$$u(0) = \tilde{u}_0,\ u'(0) = \tilde{u}_1. \tag{2.69}$$

Furthermore, by the assumptions (H_2)-(H_4), it implies from (2.65) and (2.68) that

$$\begin{aligned} B_3[u]\Delta u'' &= -B_1[u]\Delta u - B_2[u]\Delta u' - \frac{\partial}{\partial x}\left(B_1[u]\right)u_x \\ &\quad -\frac{\partial}{\partial x}\left(B_3[u]\right)u''_x + u'' - F[u] + \frac{\partial}{\partial x}G[u] \\ &\equiv \Psi \in L^\infty(0,T;L^2). \end{aligned} \tag{2.70}$$

Because of

$$b_*\left\|\Delta u''(t)\right\| \le \left\|B_3[u](t)\Delta u''(t)\right\| = \|\Psi(t)\| \le \|\Psi\|_{L^\infty(0,T;L^2)},$$

we obtain $u'' \in L^\infty(0,T;H_0^1 \cap H^2)$, and so $u \in W_1(M,T)$. The existence is proved.

(**ii**) *Uniqueness*. Let u_1, u_2 be two weak solutions of Prob.(1.1)-(1.3), such that

$$u_i \in W_1(M,T),\ i=1,2. \tag{2.71}$$

Then $w = u_1 - u_2$ verifies

$$\begin{cases} \langle w''(t), w\rangle + \langle B_{11}(t)w_x(t) + B_{21}(t)w'_x(t) + B_{31}(t)w''_x(t), w_x\rangle \\ = \langle F_1(t) - F_2(t), w\rangle + \langle G_1(t) - G_2(t), w_x\rangle \\ \quad -\langle \left(B_{11}(t) - B_{12}(t)\right)\ u_{2x}(t) + \left(B_{21}(t) - B_{22}(t)\right)u'_{2x}(t), w_x\rangle \\ \quad -\langle \left(B_{31}(t) - B_{32}(t)\right)u''_{2x}(t), w_x\rangle,\ \forall w \in H_0^1, \\ w(0) = w'(0) = 0, \end{cases} \tag{2.72}$$

where

$$B_{ij} = B_i[u_j],\ F_j = F[u_j],\ G_j = G[u_j],\ i=1,2,3,\ j=1,2. \tag{2.73}$$

Taking $v = w = u_1 - u_2$ in (2.72) and integrating with respect to t, we obtain

$$\begin{aligned}\rho(t) &= 2\int_0^t \left\langle F_1(s) - F_2(s), w'(s)\right\rangle ds \\ &+2\int_0^t \left\langle G_1(s) - G_2(s), w'_x(s)\right\rangle ds \\ &+\int_0^t ds \int_0^1 \left(B'_{11}(x,s)w_x^2(x,s) + B'_{31}(x,s)\left|w'_x(x,s)\right|^2\right) dx \\ &+2\int_0^t \left\langle (B_{11}(s)-B_{12}(s))\, u_{2x}(s) + (B_{21}(s)-B_{22}(s))\, u'_{2x}(s), w'_x(s)\right\rangle ds \\ &+2\int_0^t \left\langle (B_{31}(s) - B_{32}(s))\, u''_{2x}(s), w'_x(s)\right\rangle ds,\end{aligned} \tag{2.74}$$

where

$$\begin{aligned}\rho(t) &= \left\|w'(t)\right\|^2 + \left\|\sqrt{B_{31}(t)}w'_x(t)\right\|^2 + \left\|\sqrt{B_{11}(t)}w_x(t)\right\|^2 \\ &+2\int_0^t \left\|\sqrt{B_{21}(s)}w'_x(s)\right\|^2 ds \\ &\geq b_*\left(\left\|w'(t)\right\|^2 + \left\|w'_x(t)\right\|^2 + \|w_x(t)\|^2\right).\end{aligned} \tag{2.75}$$

On the other hand, by $(H_3) - (H_5)$, we deduce from (2.6), (2.75), that

$$\begin{aligned}\left|B'_{i1}(x,s)\right| &\leq \left(1 + M + 8M^2\right) B_i(M) \\ &\equiv \bar{B}_{iM} \leq \bar{B}_M,\ i = 1,3, \\ |B_{i1}(x,s) - B_{i2}(x,s)| &\leq \frac{2}{\sqrt{b_*}}(1+4M)B_i(M)\sqrt{\rho(s)} \\ &\leq \frac{2}{\sqrt{b_*}}\tilde{B}_M\sqrt{\rho(s)},\ i = 1,2,3, \\ \|F_1(s) - F_2(s)\| &\leq \frac{4}{\sqrt{b_*}}(1+2M)\bar{F}_M\sqrt{\rho(s)}, \\ \|G_1(s) - G_2(s)\| &\leq \frac{4}{\sqrt{b_*}}(1+2M)\bar{G}_M\sqrt{\rho(s)}, \\ \left\|u_{2x}(s) + u'_{2x}(s) + u''_{2x}(s)\right\| &\leq 3M.\end{aligned} \tag{2.76}$$

Combining (2.75) and (2.76), it implies to

$$\rho(t) \leq \frac{1}{b_*}\left[8(1+2M)\left(\bar{F}_M + \bar{G}_M\right) + \bar{B}_M + 6M\tilde{B}_M\right]\int_0^t \rho(s)ds. \tag{2.77}$$

By Gronwall's Lemma, (2.77) gives $\rho \equiv 0$, i.e., $u_1 \equiv u_2$. This completes the proof. □

Now we are in a position to give a main theorem of this section.

Theorem 2.4. *Let* (H_1)-(H_4) *hold. Then Prob.*(1.1)-(1.3) *has a unique local solution*

$$\begin{aligned} u \in L^\infty\left(0,T;H_0^1\cap H^2\right),\ u' \in L^\infty\left(0,T;H_0^1\cap H^2\right),\\ u'' \in L^\infty\left(0,T;H_0^1\cap H^2\right), \end{aligned} \tag{2.78}$$

for $T>0$ *small enough.*

Proof. We know that the proof of theorem will be obtained from Step 1 and Step 2. □

Remark 2.5. Based on the regularity obtained by (2.78), Prob.(1.1)-(1.3) has a unique strong solution

$$u \in C^1\left([0,T];H_0^1\cap H^2\right),\ u'' \in L^\infty\left(0,T;H_0^1\cap H^2\right). \tag{2.79}$$

3. Blow-up

In this section, we consider Prob. (1.1)-(1.3) with $\lambda>0$; $B_1 = B_1\left(\|u_x\|^2\right)$, $B_1(y)\geq b_{1*}>0$, $\forall y\geq 0$, $B_i = B_i(x,t)\in C^1([0,1]\times\mathbb{R}_+)$, $B_i(x,t)\geq b_{i*}>0$, $i=2,3$; $F=F(u,u_x)$, $G=G(u,u_x)$, F, $G\in C^1(\mathbb{R}^2;\mathbb{R})$, f, $f_t\in L^2(\mathbb{R}_+;L^2)$, as follows

$$\begin{cases} u_{tt} - B_1\left(\|u_x\|^2\right)u_{xx} - \frac{\partial}{\partial x}\left(B_2(x,t)\,u_{xt}\right) - \frac{\partial}{\partial x}\left(B_3(x,t)\,u_{xtt}\right) + \lambda u_t \\ = F(u,u_x) - \frac{\partial}{\partial x}\left(G(u,u_x)\right) + f(x,t),\ 0<x<1,\ 0<t<T, \\ u(0,t)=u(1,t)=0, \\ u(x,0)=\tilde{u}_0(x),\ u_t(x,0)=\tilde{u}_1(x). \end{cases} \tag{3.1}$$

By the same method as in the proof of Theorem 2.5, (3.1) has a weak solution $u(x,t)$ such that

$$u\in C^1([0,T];H^2\cap H_0^1),\ u''\in L^\infty(0,T;H^2\cap H_0^1), \tag{3.2}$$

for $T>0$ small enough. Furthermore, if the following assumptions hold, then a blow-up result is obtained.

$(\hat{H}_2)$ $f=0$;
$(\hat{H}_3)$ $B_1\in C^1(\mathbb{R}_+)$ and there exist the positive constants b_{1*}, χ_1 such that
  (i) $B_1(y)\geq b_{1*}>0$, $\forall y\geq 0$,
  (ii) $yB_1(y)\leq \chi_1\int_0^y B_1(z)dz$, $\forall y\geq 0$;

$(\hat{H}_4)$ $B_i \in C^1([0,1] \times \mathbb{R}_+)$, $i = 2, 3$ and there exist the positive constants b_{i*}, b_i^*, σ_i, such that

(i) $b_{i*} \leq B_i(x,t) \leq b_i^*$, $\forall (x,t) \in [0,1] \times \mathbb{R}_+$, $i = 2, 3$,

(ii) $-\sigma_i \leq B_i'(x,t) \leq 0$, $\forall (x,t) \in [0,1] \times \mathbb{R}_+$, $i = 2, 3$;

$(\hat{H}_5)$ There exist $\mathcal{F} \in C^2(\mathbb{R}^2; \mathbb{R})$ and the constants p, $q > 2$; d_1, $\bar{d}_1 > 0$, such that

(i) $\frac{\partial \mathcal{F}}{\partial u}(u,v) = F(u,v)$, $\frac{\partial \mathcal{F}}{\partial v}(u,v) = G(u,v)$,

(ii) $uF(u,v) + vG(u,v) \geq d_1 \mathcal{F}(u,v)$, for all $(u,v) \in \mathbb{R}^2$,

(iii) $\mathcal{F}(u,v) \geq \bar{d}_1 \left(|u|^q + |v|^p \right)$, for all $(u,v) \in \mathbb{R}^2$;

$(\hat{H}_6)$ $d_1 > 2\chi_1 + \dfrac{\sigma_2}{b_{1*}}$ and $\sigma_3 > 0$ is small enough with d_1, χ_1, σ_2, σ_3 as in $(\hat{H}_3)(ii)$, $(\hat{H}_4)(i), (ii)$.

Example 3.1. We give an example of the functions F, G satisfying $(\hat{H}_5)$ as below

$$\begin{aligned} F(u,v) &= \alpha\gamma_2 |u|^{\alpha-2} u |v|^\beta + q\gamma_3 |u|^{q-2} u, \\ G(u,v) &= p\gamma_1 |v|^{p-2} v + \beta\gamma_2 |u|^\alpha |v|^{\beta-2} v, \end{aligned}$$

where α, β, p, $q > 2$; γ_1, γ_2, $\gamma_3 > 0$ are the constants, with

$$\min\{p, q, \alpha+\beta\} > 2\chi_1 + \frac{\sigma_2}{b_{1*}},$$

with b_{1*}, χ_1 as in $(\hat{H}_3)$, $(\hat{H}_4)$. It is obvious that $(\hat{H}_5)$ holds, because there exists a $\mathcal{F} \in C^2(\mathbb{R}^2; \mathbb{R})$ defined by

$$\mathcal{F}(u,v) = \gamma_1 |v|^p + \gamma_2 |u|^\alpha |v|^\beta + \gamma_3 |u|^q,$$

such that

$$\begin{cases} \frac{\partial \mathcal{F}}{\partial u}(u,v) = \alpha\gamma_2 |u|^{\alpha-2} u |v|^\beta + q\gamma_3 |u|^{q-2} u = F(u,v), \\ \frac{\partial \mathcal{F}}{\partial v}(u,v) = p\gamma_1 |v|^{p-2} v + \beta\gamma_2 |u|^\alpha |v|^{\beta-2} v = G(u,v), \\ uF(u,v) + vG(u,v) \geq d_1 \mathcal{F}(u,v), \text{ for all } (u,v) \in \mathbb{R}^2, \end{cases}$$

in which $d_1 = \min\{p, q, \alpha+\beta\} > 2\chi_1 + \dfrac{\sigma_2}{b_{1*}}$;

$$\mathcal{F}(u,v) \geq \bar{d}_1 \left(|v|^p + |u|^q \right), \text{ for all } (u,v) \in \mathbb{R}^2,$$

with $\bar{d}_1 = \min\{\gamma_1, \gamma_3\}$. Put

$$\begin{aligned} H(0) &= -\frac{1}{2} \|\tilde{u}_1\|^2 - \frac{1}{2} \int_0^{\|\tilde{u}_{0x}\|^2} B_1(y) dy - \frac{1}{2} \left\| \sqrt{B_3(0)} \tilde{u}_{1x} \right\|^2 \\ &\quad + \int_0^1 \mathcal{F}\left(\tilde{u}_0(x), \tilde{u}_{0x}(x) \right) dx. \end{aligned} \tag{3.3}$$

First, we need the following lemma.

Lemma 3.1. *Let* $2 \le r_1 \le q$, $2 \le r_2$, $r_3 \le \min\{p,q\}$. *Then, for any* $v \in H_0^1$, *we have*

$$\|v\|^{r_1} + \|v_x\|^{r_2} + \|v_x\|^{r_3} \le 3\left(\|v\|_{L^q}^q + \|v_x\|_{L^p}^p + \|v_x\|^2\right). \tag{3.4}$$

Proof. The proof of Lemma 3.1 is not difficult, so we omit the details. □

Theorem 3.2. *Let* $(\hat{H}_2)$-$(\hat{H}_6)$ *hold. Then, for any* $\tilde{u}_0$, $\tilde{u}_1 \in H_0^1 \cap H^2$ *such that* $H(0) > 0$, *the weak solution* $u = u(x,t)$ *of Prob.*(3.1) *blows up in finite time.*

Proof. It consists of two steps.
Step 1: First, we prove that the Problem (3.1) has not a global weak solution. Indeed, by contradiction, we assume that

$$u \in C^1(\mathbb{R}_+; H^2 \cap H_0^1),\ u'' \in L^\infty(0,T; H^2 \cap H_0^1),\ \forall T > 0, \tag{3.5}$$

is a global weak solution of Prob. (3.1). We define the energy associated with (3.1) by

$$\begin{aligned} E(t) \ = \ & \frac{1}{2}\left\|u'(t)\right\|^2 + \frac{1}{2}\int_0^{\|u_x(t)\|^2} B_1(y)dy + \frac{1}{2}\left\|\sqrt{B_3(t)}u_x'(t)\right\|^2 \\ & - \int_0^1 \mathcal{F}\left(u(x,t), u_x(x,t)\right)dx, \end{aligned} \tag{3.6}$$

and we put $H(t) = -E(t)$, for all $t \ge 0$. Multiplying (3.1) by $u'(x,t)$ and integrating the resulting equation over $[0,1]$, we have

$$H'(t) = \lambda\left\|u'(t)\right\|^2 + \left\|\sqrt{B_2(t)}u_x'(t)\right\|^2 - \frac{1}{2}\int_0^1 B_3'(x,t)\left|u_x'(x,t)\right|^2 dx \ge 0. \tag{3.7}$$

It implies that

$$0 < H(0) \le H(t),\ \forall t \ge 0, \tag{3.8}$$

so

$$\begin{cases} 0 < H(0) \le H(t) \le \int_0^1 \mathcal{F}\left(u(x,t), u_x(x,t)\right)dx; \\ \|u'(t)\|^2 + \int_0^{\|u_x\|^2} B_1(y)dy + \left\|\sqrt{B_3(t)}u_x'(t)\right\|^2 \\ \le 2\int_0^1 \mathcal{F}\left(u(x,t), u_x(x,t)\right)dx,\ \forall t \ge 0. \end{cases} \tag{3.9}$$

Now, we define the functional

$$L(t) = H^{1-\eta}(t) + \varepsilon\Psi(t), \tag{3.10}$$

where

$$\begin{aligned}\Psi(t) &= \langle u'(t), u(t)\rangle + \langle B_3(t)u'_x(t), u_x(t)\rangle \\ &\quad +\tfrac{\lambda}{2}\|u(t)\|^2 + \tfrac{1}{2}\left\|\sqrt{B_2(t)}u_x(t)\right\|^2,\end{aligned} \tag{3.11}$$

for ε small enough and

$$0 < \eta < 1,\ 2/(1-2\eta) \le \min\{p, q\}. \tag{3.12}$$

In what follows, we show that, there exists a constant $\gamma > 0$ such that

$$L'(t) \geq \gamma\left[H(t) + \|u(t)\|_{L^q}^q + \|u_x(t)\|_{L^p}^p + \left\|u'(t)\right\|^2 + \left\|u'_x(t)\right\|^2 + \|u_x(t)\|^2\right]. \tag{3.13}$$

Multiplying (3.1) by $u(x,t)$ and integrating over $[0,1]$, it lead

$$\begin{aligned}\Psi'(t) &= \left\|u'(t)\right\|^2 + \left\|\sqrt{B_3(t)}u'_x(t)\right\|^2 - B_1\left(\|u_x(t)\|^2\right)\|u_x(t)\|^2 \\ &\quad +\langle B'_3(t)u'_x(t), u_x(t)\rangle + \frac{1}{2}\int_0^1 B'_2(x,t)u_x^2(x,t)dx \\ &\quad +\langle F\left(u(t), u_x(t)\right), u(t)\rangle + \langle G\left(u(t), u_x(t)\right), u_x(t)\rangle.\end{aligned} \tag{3.14}$$

Therefore

$$L'(t) = (1-\eta)H^{-\eta}(t)H'(t) + \varepsilon\Psi'(t) \geq \varepsilon\Psi'(t). \tag{3.15}$$

By $(\hat{H}_5)$, we obtain

$$\begin{cases}\langle F\left(u(t), u_x(t)\right), u(t)\rangle + \langle G\left(u(t), u_x(t)\right), u_x(t)\rangle \\ \geq d_1\displaystyle\int_0^1 \mathcal{F}\left(u(x,t), u_x(x,t)\right)dx, \\ \displaystyle\int_0^1 \mathcal{F}\left(u(x,t), u_x(x,t)\right)dx \geq \bar{d}_1\left(\|u_x(t)\|_{L^p}^p + \|u(t)\|_{L^q}^q\right).\end{cases} \tag{3.16}$$

On the other hand, by $(\hat{H}_3)$, $(\hat{H}_4)$, we get

$$\begin{aligned}-B_1\left(\|u_x(t)\|^2\right)\|u_x(t)\|^2 &\geq -\chi_1\int_0^{\|u_x(t)\|^2} B_1(y)dy, \\ \frac{1}{2}\int_0^1 B'_2(x,t)u_x^2(x,t)dx &\geq -\frac{\sigma_2}{2}\|u_x(t)\|^2, \\ \left|\langle B'_3(t)u'_x(t), u_x(t)\rangle\right| &\leq \left\|B'_3(t)u'_x(t)\right\|\|u_x(t)\| \\ &\leq \frac{1}{2}\left(\delta_1\|u_x(t)\|^2 + \frac{\sigma_3^2}{\delta_1}\left\|u'_x(t)\right\|^2\right),\ \forall\delta_1 > 0.\end{aligned} \tag{3.17}$$

It implies from (3.14), (3.16), (3.17) that

$$
\begin{aligned}
\Psi'(t) \geq\ & \left\|u'(t)\right\|^2 + \left\|\sqrt{B_3(t)}u'_x(t)\right\|^2 - B_1\left(\|u_x(t)\|^2\right)\|u_x(t)\|^2 \qquad (3.18)\\
& + \frac{1}{2}\int_0^1 B'_2(x,t)u_x^2(x,t)dx + \langle B'_3(t)u'_x(t), u_x(t)\rangle \\
& + \delta d_1 \int_0^1 \mathcal{F}\left(u(x,t), u_x(x,t)\right)dx \\
& + (1-\delta)d_1\left[H(t) + \frac{1}{2}\left\|u'(t)\right\|^2 + \frac{1}{2}\int_0^{\|u_x(t)\|^2} B_1(y)dy \right.\\
& \left. \qquad + \frac{1}{2}\left\|\sqrt{B_3(t)}u'_x(t)\right\|^2\right] \\
\geq\ & \left[1 + (1-\delta)\frac{d_1}{2}\right]\left\|u'(t)\right\|^2 + \left[1 + (1-\delta)\frac{d_1}{2}\right] b_{3*}\left\|u'_x(t)\right\|^2 \\
& + d_1\delta\int_0^1 \mathcal{F}\left(u(x,t), u_x(x,t)\right)dx + d_1(1-\delta)H(t) \\
& - \frac{\sigma_2}{2}\|u_x(t)\|^2 - \frac{1}{2}\left(\delta_1\|u_x(t)\|^2 + \frac{\sigma_3^2}{\delta_1}\left\|u'_x(t)\right\|^2\right) \\
& + \left[(1-\delta)\frac{d_1}{2} - \chi_1\right] b_{1*}\|u_x(t)\|^2 \\
\geq\ & \left[1 + (1-\delta)\frac{d_1}{2}\right]\left\|u'(t)\right\|^2 + d_1(1-\delta)H(t) \\
& + d_1\delta\bar{d}_1\left(\|u(t)\|_{L^q}^q + \|u_x(t)\|_{L^p}^p\right) \\
& + \left[\left(1 + (1-\delta)\frac{d_1}{2}\right)b_{3*} - \frac{\sigma_3^2}{2\delta_1}\right]\left\|u'_x(t)\right\|^2 \\
& + \frac{1}{2}\left[\left((1-\delta)d_1 - 2\chi_1\right)b_{1*} - \sigma_2 - \delta_1\right]\|u_x(t)\|^2,
\end{aligned}
$$

for all $\delta \in (0,1)$. By $d_1 > 2\chi_1 + \dfrac{\sigma_2}{b_{1*}}$, we have

$$
\begin{aligned}
\lim_{\delta\to 0_+,\ \delta_1\to 0_+}\left[\left((1-\delta)d_1 - 2\chi_1\right)b_{1*} - \sigma_2 - \delta_1\right] &= \left(d_1 - 2\chi_1\right)b_{1*} - \sigma_2 \\
&> 0,
\end{aligned}
$$

then, we can choose δ, $\delta_1 \in (0,1)$ with δ, δ_1 are small enough such that

$$
\left[(1-\delta)d_1 - 2\chi_1\right]b_{1*} - \sigma_2 - \delta_1 > 0. \qquad (3.19)
$$

Then, if $\sigma_3 > 0$ satisfies

$$\left(1 + (1-\delta)\frac{d_1}{2}\right) b_{3*} - \frac{\sigma_3^2}{2\delta_1} > 0, \tag{3.20}$$

we deduce from (3.18), (3.19) and (3.20) that there exists a constant $\gamma > 0$ such that (3.13) holds. From the formula of $L(t)$ and (3.13), we can choose $\varepsilon > 0$ small enough such that

$$L(t) \geq L(0) > 0, \ \forall t \geq 0. \tag{3.21}$$

Using the inequality

$$\left(\sum\nolimits_{i=1}^{5} x_i\right)^r \leq 5^{r-1} \sum\nolimits_{i=1}^{5} x_i^r, \text{ for all } r > 1, \text{ and } x_1, \cdots, x_5 \geq 0, \tag{3.22}$$

we deduce from (3.10)-(3.12) that

$$\begin{aligned} L^{1/(1-\eta)}(t) \quad \leq \quad & Const \ \Big[H(t) + \left|\langle u(t), u'(t)\rangle\right|^{1/(1-\eta)} \\ & + \left|\langle B_3(t)u_x'(t), u_x(t)\rangle\right|^{1/(1-\eta)} \\ & + \|u(t)\|^{2/(1-\eta)} + \left\|\sqrt{B_2(t)}u_x(t)\right\|^{2/(1-\eta)}\Big]. \end{aligned} \tag{3.23}$$

Using Young's inequality, we have

$$\begin{aligned} \left|\langle u(t), u'(t)\rangle\right|^{1/(1-\eta)} \quad & \leq \quad \|u(t)\|^{1/(1-\eta)} \left\|u'(t)\right\|^{1/(1-\eta)} \\ & \leq \quad \frac{1-2\eta}{2(1-\eta)} \|u(t)\|^s + \frac{1}{2(1-\eta)} \left\|u'(t)\right\|^2 \\ & \leq \quad Const \left(\|u_x(t)\|^s + \left\|u'(t)\right\|^2\right), \end{aligned} \tag{3.24}$$

where $s = 2/(1-2\eta) \leq \min\{p, q\}$ as in (3.12). Similarly, we also obtain

$$\begin{aligned} \left|\langle B_3(t)u_x'(t), u_x(t)\rangle\right|^{\frac{1}{1-\eta}} \quad & \leq \quad (b_3^*)^{\frac{1}{1-\eta}} \|u_x(t)\|^{\frac{1}{1-\eta}} \left\|u_x'(t)\right\|^{\frac{1}{1-\eta}} \\ & \leq \quad Const \left(\|u_x(t)\|^s + \left\|u_x'(t)\right\|^2\right), \\ \left\|\sqrt{B_2(t)}u_x(t)\right\|^{2/(1-\eta)} \quad & \leq \quad (b_2^*)^{1/(1-\eta)} \|u_x(t)\|^{2/(1-\eta)}. \end{aligned} \tag{3.25}$$

Combining (3.23) - (3.25), it implies that

$$\begin{aligned} L^{1/(1-\eta)}(t) \quad \leq \quad & Const \Big[H(t) + \left\|u'(t)\right\|^2 + \left\|u_x'(t)\right\|^2 \\ & + \|u(t)\|^{2/(1-\eta)} + \|u_x(t)\|^{2/(1-\eta)} + \|u_x(t)\|^s \ \Big]. \end{aligned} \tag{3.26}$$

Using (3.26) and Lemma 3.1 with $r_1 = \frac{2}{1-\eta}$, $r_2 = s$, $r_3 = 2/(1-\eta)$, we obtain

$$L^{1/(1-\eta)}(t) \leq Const\left[H(t) + \left\|u'(t)\right\|^2 + \left\|u'_x(t)\right\|^2 + \|u_x(t)\|^2 + \|u(t)\|_{L^q}^q + \|u_x(t)\|_{L^p}^p \right], \tag{3.27}$$

for all $t \geq 0$. It follows from (3.13) and (3.27) that

$$L'(t) \geq \bar{\lambda} L^{1/(1-\eta)}(t), \ \forall t \geq 0, \tag{3.28}$$

where $\bar{\lambda}$ is a positive constant. Integrating (3.28) over $(0, t)$, it leads to

$$L^{\eta/(1-\eta)}(t) \geq \frac{1-\eta}{\bar{\lambda}\eta} \frac{1}{T_* - t}, \ 0 \leq t < T_* = \frac{1-\eta}{\bar{\lambda}\eta} L^{-\eta/(1-\eta)}(0). \tag{3.29}$$

Therefore $\lim_{t \to T_*^-} L(t) = +\infty$. This is a contradiction with (3.27) and $u \in C^1([0, T_*]; H^2 \cap H_0^1)$. Thus, the Problem (3.1) has not a global weak solution

Step 2: Next, we put

$$T_\infty = \sup\{T > 0 : \text{Prob.(3.1) has a unique solution, } u \in C^1([0,T]; H^2 \cap H_0^1), \ u'' \in L^\infty(0, T; H^2 \cap H_0^1)\}.$$

Since Problem (3.1) has not a global weak solution, we have $T_\infty < +\infty$. We now prove that

$$\lim_{t \to T_\infty^-} \left(\|u(t)\|_{H^2 \cap H_0^1} + \left\|u'(t)\right\|_{H^2 \cap H_0^1} \right) = +\infty. \tag{3.30}$$

Indeed, assume that (3.30) is not true, then there exists a constant $M > 0$ and there exists a sequence $\{t_m\}$ with $\{t_m\} \subset (0, T_\infty)$, $t_m \to T_\infty$ such that

$$\|u(t_m)\|_{H^2 \cap H_0^1} + \left\|u'(t_m)\right\|_{H^2 \cap H_0^1} \leq M, \ \forall m \in \mathbb{N}.$$

Following the argument as above, for each $m \in \mathbb{N}$, there exists a unique weak solution

$$u_* \in C^1([t_m, t_m + \sigma]; H^2 \cap H_0^1), \ u'' \in L^\infty(t_m, t_m + \sigma; H^2 \cap H_0^1)$$

of Prob.(3.1) with the initial data

$$u_*(t_m) = u(t_m), \ u'_*(t_m) = u'(t_m),$$

for $\sigma > 0$ independent of $m \in \mathbb{N}$. By $t_m \to T_\infty$, we can get $t_m + \sigma > T_\infty$ for $m \in \mathbb{N}$ sufficiently large. It is clear to see that the function $\tilde{u}(t)$ with

$$\tilde{u}(t) = \begin{cases} u(t), & 0 \leq t \leq t_m, \\ u_*(t), & t_m \leq t \leq t_m + \eta, \end{cases}$$

is a weak solution of Prob.(3.1) on $[0, t_m + \sigma]$, $t_m + \sigma > T_\infty$, we obtain a contradiction to the maximality of T_∞. Thus, (3.30) holds. This completes the proof. $\square$

4. Exponential decay

In this section, we continue to consider Prob.(3.1) and we make the following assumptions.

$(\bar{H}_2)$ $f \in L^\infty\left(\mathbb{R}_+; L^2\right) \cap L^1\left(\mathbb{R}_+; L^2\right)$;

$(\bar{H}_3)$ $B_1 \in C^1(\mathbb{R}_+)$ and there exist two positive constants b_{1*}, χ_{1*} with $\chi_{1*} > \dfrac{d_2}{p}$ such that

(i) $B_1(y) \geq b_{1*} > 0$, $\forall y \geq 0$,

(ii) $yB_1(y) \geq \chi_{1*} \int_0^y B_1(z)dz$, $\forall y \geq 0$;

$(\bar{H}_4)$ $B_i \in C^1([0,1] \times \mathbb{R}_+)$, $i = 2, 3$ and there exist the positive constants b_{i*}, b_i^*, σ_3 such that

(i) $b_{i*} \leq B_i(x,t) \leq b_i^*$, $\forall (x,t) \in [0,1] \times \mathbb{R}_+$, $i = 2, 3$,

(ii) $-\sigma_3 \leq B_3'(x,t) \leq 0$, $\forall (x,t) \in [0,1] \times \mathbb{R}_+$,

(iii) $B_2'(x,t) \leq 0$, $\forall (x,t) \in [0,1] \times \mathbb{R}_+$;

$(\bar{H}_5)$ There exist $\mathcal{F} \in C^2(\mathbb{R}^2; \mathbb{R})$ and the constants p, q, α, β, d_2, $\tilde{d}_1$, $\bar{d}_2 > 0$, α, β, $q > 2$, $p > \max\{q, \alpha + \beta, d_2\}$ such that

(i) $\frac{\partial \mathcal{F}}{\partial u}(u,v) = F(u,v)$, $\frac{\partial \mathcal{F}}{\partial v}(u,v) = G(u,v)$, for all $(u,v) \in \mathbb{R}^2$,

(ii) $\mathcal{F}_1(u,v) \equiv \mathcal{F}(u,v) + \tilde{d}_1 |v|^p \leq \bar{d}_2 \left(|u|^\alpha |v|^\beta + |u|^q\right)$, for all $(u,v) \in \mathbb{R}^2$,

(iii) $uF(u,v) + vG(u,v) \leq d_2 \mathcal{F}(u,v)$, for all $(u,v) \in \mathbb{R}^2$;

$(\bar{H}_6)$ $\chi_{1*} > \dfrac{d_2}{p}$ with d_2 as in $(\bar{H}_5)$.

Example 4.1. We give an example of the functions F, G satisfying $(\bar{H}_5)$, as follows

$$\begin{aligned} F(u,v) &= \alpha\gamma_2 |u|^{\alpha-2} u |v|^\beta + q\gamma_3 |u|^{q-2} u, \\ G(u,v) &= -p\gamma_1 |v|^{p-2} v + \beta\gamma_2 |u|^\alpha |v|^{\beta-2} v, \end{aligned}$$

where α, β, p, $q > 2$; γ_1, γ_2, $\gamma_3 > 0$ are the constants, with α, β, $q > 2$ and $p > q$, $p > \alpha + \beta$.

We see that $(\bar{H}_5)$ holds, indeed, we consider $\mathcal{F} \in C^2(\mathbb{R}^2; \mathbb{R})$ defined by

$$\mathcal{F}(u,v) = -\gamma_1 |v|^p + \gamma_2 |u|^\alpha |v|^\beta + \gamma_3 |u|^q,$$

then we have

$$\begin{cases} \frac{\partial \mathcal{F}}{\partial u}(u,v) = \alpha\gamma_2 |u|^{\alpha-2} u |v|^\beta + q\gamma_3 |u|^{q-2} u = F(u,v), \\ \frac{\partial \mathcal{F}}{\partial v}(u,v) = -p\gamma_1 |v|^{p-2} v + \beta\gamma_2 |u|^\alpha |v|^{\beta-2} v = G(u,v), \\ \mathcal{F}_1(u,v) \equiv \mathcal{F}(u,v) + \gamma_1 |v|^p \leq \bar{d}_2 \left(|u|^\alpha |v|^\beta + |u|^q\right), \text{ for all } (u,v) \in \mathbb{R}^2, \end{cases}$$

where $\tilde{d}_1 = \gamma_1$, $\bar{d}_2 = \max\{\gamma_2, \gamma_3\}$.

On the other hand, $(\bar{H}_6)$ also holds, because of

$$\begin{aligned} uF(u,v)+vG(u,v) &= (p-\varepsilon)\mathcal{F}(u,v)-\varepsilon\gamma_1\left|v\right|^p \\ &\quad -(p-\alpha-\beta-\varepsilon)\gamma_2\left|u\right|^\alpha\left|v\right|^\beta-(p-q-\varepsilon)\gamma_3\left|u\right|^q \\ &\leq (p-\varepsilon)\mathcal{F}(u,v)=d_2\mathcal{F}(u,v),\ \text{for all } (u,v)\in\mathbb{R}^2, \end{aligned}$$

where $d_2=p-\varepsilon<p$, with $\varepsilon>0$ small enough such that

$$0<\varepsilon<p,\ p-\alpha-\beta-\varepsilon>0,\ p-q-\varepsilon>0.$$

Now, we show the main result of this section. That is, the solution u of Prob.(3.1) is exactly global and exponential decay provided that $E(0)$ is small enough and

$$I(0)=\left\|\sqrt{B(0)}\tilde{u}_{0x}\right\|-p\int_0^1\mathcal{F}_1(\tilde{u}_0(x),\tilde{u}_{0x}(x))dx>0,$$

where $p>\max\{q,\,\alpha+\beta,\,d_2\}$ with d_2 given in $(\bar{H}_5)(iii)$.

First, we construct the following Lyapunov functional. Let $u=u(x,t)$ be a global weak solution of Prob.(3.1) satisfying (3.5) as in Section 3. In order to obtain the decay result, we construct the functional

$$\mathcal{L}(t)=E(t)+\delta\Psi(t), \tag{4.1}$$

with $\delta>0$, and $\Psi(t)$ as in Section 3 and

$$\begin{aligned} \tilde{E}(t) &= \frac{1}{2}(g*u')(t)+\frac{1}{2}\left\|u'(t)\right\|^2+\frac{1}{2}\int_0^{\|u_x(t)\|^2}B_1(y)dy \\ &\quad +\frac{1}{2}\left\|\sqrt{B_3(t)}u_x'(t)\right\|^2 \\ &\quad +\tilde{d}_1\left\|u_x(t)\right\|_{L^p}^p-\int_0^1\mathcal{F}_1\left(u(x,t),u_x(x,t)\right)dx, \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} (g*u')(t) &= \int_0^t g(t-s)\left\|u'(s)\right\|^2ds, \\ g(t) &= 2\bar{\lambda}e^{-2\bar{k}t},\ 0<\bar{\lambda}<\lambda,\ \bar{k}>0. \end{aligned} \tag{4.3}$$

We rewrite $\tilde{E}(t)$ as follows

$$\begin{aligned}\tilde{E}(t) &= \frac{1}{2}\left\|u'(t)\right\|^2+\frac{1}{2}\left\|\sqrt{B_3(t)}u'_x(t)\right\|^2 \\ &+\left(\frac{1}{2}-\frac{1}{p}\right)\left[(g*u')(t)+\int_0^{\|u_x(t)\|^2}B_1(y)dy\right] \\ &+\tilde{d}_1\left\|u_x(t)\right\|_{L^p}^p+\frac{1}{p}I(t),\end{aligned}\tag{4.4}$$

where

$$I(t)=(g*u')(t)+\int_0^{\|u_x(t)\|^2}B_1(y)dy-p\int_0^1\mathcal{F}_1\left(u(x,t),u_x(x,t)\right)dx.\tag{4.5}$$

Now, we have the following theorem.

Theorem 4.1. *Assume that* $(\bar{H}_2)$-$(\bar{H}_6)$ *hold. Let* $\tilde{u}_0,\ \tilde{u}_1\in H_0^1\cap H^2$ *such that* $I(0)>0$ *and the initial energy* $\tilde{E}(0)$ *satisfy*

$$\eta^*=b_{1*}-p\bar{d}_2\left[R_*^{\alpha-2}\left(\frac{E_*}{\tilde{d}_1}\right)^{\frac{\beta}{p}}+R_*^{q-2}\right]>\frac{p\sigma_3}{2d_2},\tag{4.6}$$

where

$$\begin{aligned}E_* &= \left(E(0)+\frac{1}{2}\left\|f\right\|_{L^1(\mathbb{R}_+;L^2)}\right)\exp\left(\left\|f\right\|_{L^1(\mathbb{R}_+;L^2)}\right),\\ R_*^2 &= \frac{2pE_*}{(p-2)\,b_{1*}}.\end{aligned}\tag{4.7}$$

Assume that

$$\left\|f(t)\right\|^2\le\bar{C}_1\exp(-\bar{\eta}_1t),\ \ \forall t\ge0,\tag{4.8}$$

where $\bar{C}_1$, $\bar{\eta}_1$ *are two positive constants. Then, for all the global weak solution of Prob.*(3.1) *is exponential decaying, that is, there exist positive constants* $\bar{C}$, $\bar{\gamma}$ *such that*

$$\left\|u'(t)\right\|^2+\left\|u'_x(t)\right\|^2+\left\|u_x(t)\right\|^2+\left\|u_x(t)\right\|_{L^p}^p\le\bar{C}\exp(-\bar{\gamma}t),\ \ \forall t\ge0.\tag{4.9}$$

Proof. It consists of three steps.

Step 1: The estimate of $E'(t)$.

We have

$$\begin{aligned}\text{(i)}\ \ \tilde{E}'(t) &\le \frac{1}{2}\left\|f(t)\right\|+\frac{1}{2}\left\|f(t)\right\|\left\|u'(t)\right\|^2, &(4.10)\\ \text{(ii)}\ \ \tilde{E}'(t) &\le -\left(\lambda-\bar{\lambda}-\frac{\varepsilon_1}{2}\right)\left\|u'(t)\right\|^2-\bar{k}(g*u')(t)-b_{2*}\left\|u'_x(t)\right\|^2\\ &+\frac{1}{2}\int_0^1B'_3(x,t)\left|u'_x(x,t)\right|^2dx+\frac{1}{2\varepsilon_1}\left\|f(t)\right\|^2,\end{aligned}$$

for all $\varepsilon_1 > 0$. Indeed, multiplying (3.1) by $u'(x,t)$ and integrating over $[0,1]$, we get

$$\begin{aligned}\tilde{E}'(t) &= -\left(\lambda-\bar{\lambda}\right)\left\|u'(t)\right\|^2-\bar{k}(g*u')(t)-\left\|\sqrt{B_2(t)}u_x'(t)\right\|^2 \\ &\quad +\frac{1}{2}\int_0^1 B_3'(x,t)\left|u_x'(x,t)\right|^2dx+\langle f(t),u'(t)\rangle.\end{aligned} \tag{4.11}$$

On the other hand

$$\left|\langle f(t),u'(t)\rangle\right| \le \frac{1}{2}\|f(t)\|+\frac{1}{2}\|f(t)\|\left\|u'(t)\right\|^2. \tag{4.12}$$

By $B_3'(x,t)\le 0$, it follows from (4.11), (4.12) that (4.10)(i) holds.

Similarly,

$$\left|\langle f(t),u'(t)\rangle\right| \le \frac{1}{2\varepsilon_1}\|f(t)\|_0^2+\frac{\varepsilon_1}{2}\left\|u'(t)\right\|^2,\ \text{for all } \varepsilon_1>0. \tag{4.13}$$

By $B_3'(x,t)\le 0$, (4.11) and (4.13), that (4.10)(ii) is valid.

Step 2: The estimate of $I(t)$.

By the continuity of $I(t)$ and $I(0)>0$, there exists $T_1>0$ such that

$$I(t)>0,\ \forall t\in[0,T_1], \tag{4.14}$$

it implies that for all $t\in[0,T_1]$,

$$\begin{aligned}\tilde{E}(t) &\ge \frac{1}{2}\left\|u'(t)\right\|^2+\left(\frac{p-2}{2p}\right)b_{1*}\left\|u_x(t)\right\|^2 \\ &\quad +\frac{b_{3*}}{2}\left\|u_x'(t)\right\|^2+\tilde{d}_1\left\|u_x(t)\right\|_{L^p}^p.\end{aligned} \tag{4.15}$$

Combining $(4.10)_i$, (4.16) and using Gronwall's inequality, then we obtain

$$\begin{aligned}&\frac{1}{2}\left\|u'(t)\right\|^2+\left(\frac{p-2}{2p}\right)b_{1*}\left\|u_x(t)\right\|^2 \\ &\quad +\frac{b_{3*}}{2}\left\|u_x'(t)\right\|^2+\tilde{d}_1\left\|u_x(t)\right\|_{L^p}^p \\ &\le \tilde{E}(t)\le E_*,\ \forall t\in[0,T_1].\end{aligned} \tag{4.16}$$

Hence, it follows from $(\bar{H}_5)$(iii), (4.7), (4.16) that

$$\begin{aligned} p\int_0^1 \mathcal{F}_1\left(u(x,t),u_x(x,t)\right)dx &\leq p\bar{d}_2\int_0^1\left(|u(x,t)|^{\alpha}|u_x(x,t)|^{\beta}+|u(x,t)|^{q}\right)dx \\ &\leq p\bar{d}_2\left(\|u_x(t)\|^{\alpha}\int_0^1|u_x(x,t)|^{\beta}dx+\|u_x(t)\|^{q}\right) \\ &= p\bar{d}_2\left(\|u_x(t)\|^{\alpha}\|u_x(t)\|_{L^{\beta}}^{\beta}+\|u_x(t)\|^{q}\right) \\ &\leq p\bar{d}_2\left(\|u_x(t)\|^{\alpha}\|u_x(t)\|_{L^{p}}^{\beta}+\|u_x(t)\|^{q}\right) \\ &\leq p\bar{d}_2\left(\|u_x(t)\|^{\alpha-2}\|u_x(t)\|_{L^{p}}^{\beta}+\|u_x(t)\|^{q-2}\right) \\ &\quad\times\|u_x(t)\|^2 \qquad (4.17) \\ &\leq p\bar{d}_2\left[R_*^{\alpha-2}\left(\frac{E_*}{\tilde{d}_1}\right)^{\frac{\beta}{p}}+R_*^{q-2}\right]\|u_x(t)\|^2. \end{aligned}$$

Consequently, we have

$$I(t)\geq(g*u')(t)+\eta^*\|u_x(t)\|^2\geq 0,\forall t\in[0,T_1].$$

Put $T_\infty=\sup\{T>0: I(t)>0,\ \forall t\in[0,T]\}$. If $T_\infty<+\infty$, then the continuity of $I(t)$ leads to $I(T_\infty)\geq 0$. If $I(T_\infty)>0$, by the same arguments as in the above part we can deduce that there exists $\tilde{T}_\infty>T_\infty$ such that $I(t)>0$, for all $t\in[0,\tilde{T}_\infty]$. We obtain a contradiction to the definition of T_∞.

If $I(T_\infty)=0$, it follows that

$$0=I(T_\infty)\geq(g*u')(T_\infty)+\eta^*\|u_x(T_\infty)\|^2\geq 0.$$

Therefore, we have

$$u(T_\infty)=(g*u')(T_\infty)=0.$$

By the fact that the function $s\longmapsto g(T_\infty-s)\|u'(s)\|^2$ is continuous on $[0,T_\infty]$ and $g(T_\infty-s)>0$, for all $s\in[0,T_\infty]$, we have

$$\begin{aligned}(g*u')(T_\infty) &= \int_0^{T_\infty}g(T_\infty-s)\left\|u'(s)\right\|^2ds \\ &= 0,\end{aligned}$$

it follows that $\|u'(s)\|=0$, for all $s\in[0,T_\infty]$, it means that u is a constant function on $[0,T_\infty]$. Then, $u(0)=u(T_\infty)=0$. It leads to $I(0)=0$. We get a contradiction with $I(0)>0$. Consequently, $T_\infty=+\infty$, that is, $I(t)>0$, for all $t\geq 0$.

Step 3: Decay result.

At first, we verify that there exist the positive constants $\bar{\beta}_1$, $\bar{\beta}_2$ such that

$$\bar{\beta}_1E_1(t)\leq\mathcal{L}(t)\leq\bar{\beta}_2E_1(t),\ \forall t\geq 0, \qquad (4.18)$$

where

$$E_1(t) = \left\|u'(t)\right\|^2 + \left\|u_x'(t)\right\|^2 + \int_0^{\|u_x(t)\|^2} B_1(y)dy + \|u_x(t)\|_{L^p}^p + I(t). \quad (4.19)$$

Indeed, we have

$$\begin{aligned} \mathcal{L}(t) &= \frac{1}{2}\left\|u'(t)\right\|^2 + \frac{1}{2}\left\|\sqrt{B_3(t)}u_x'(t)\right\|^2 \\ &+\frac{p-2}{2p}\left[(g*u')(t) + \int_0^{\|u_x(t)\|^2} B_1(y)dy\right] \\ &+\tilde{d}_1\|u_x(t)\|_{L^p}^p + \frac{1}{p}I(t) \\ &+\delta\langle u'(t), u(t)\rangle + \delta\langle B_3(t)u_x'(t), u_x(t)\rangle \\ &+\frac{\delta\lambda}{2}\|u(t)\|^2 + \frac{\delta}{2}\left\|\sqrt{B_2(t)}u_x(t)\right\|^2. \end{aligned} \quad (4.20)$$

On the other hand

$$\begin{aligned} \langle u(t), u'(t)\rangle &\le \frac{1}{2}\|u_x(t)\|^2 + \frac{1}{2}\left\|u'(t)\right\|^2, \\ \langle B_3(t)u_x'(t), u_x(t)\rangle &\le \frac{1}{2}b_3^*\left(\left\|u_x'(t)\right\|^2 + \|u_x(t)\|^2\right). \end{aligned} \quad (4.21)$$

Then we have

$$\begin{aligned} \mathcal{L}(t) &\ge \frac{1}{2}\left\|u'(t)\right\|^2 + \frac{1}{2}b_{3*}\left\|u_x'(t)\right\|^2 \\ &+\frac{p-2}{2p}\left[(g*u')(t) + \int_0^{\|u_x(t)\|^2} B_1(y)dy\right] \\ &+\tilde{d}_1\|u_x(t)\|_{L^p}^p + \frac{1}{p}I(t) - \frac{\delta}{2}\left(\|u_x(t)\|^2 + \left\|u'(t)\right\|^2\right) \\ &-\frac{\delta b_3^*}{2}\left(\left\|u_x'(t)\right\|^2 + \|u_x(t)\|^2\right) + \frac{\delta b_{2*}}{2}\|u_x(t)\|^2 \\ &= \frac{1-\delta}{2}\left\|u'(t)\right\|^2 + \frac{1}{2}\left(b_{3*} - \delta b_3^*\right)\left\|u_x'(t)\right\|^2 \\ &+\tilde{d}_1\|u_x(t)\|_{L^p}^p + \frac{1}{p}I(t) + \frac{p-2}{2p}\left[(g*u')(t) + \int_0^{\|u_x(t)\|^2} B_1(y)dy\right] \\ &-\frac{\delta}{2}\left(1 + b_3^*\right)\|u_x(t)\|^2 + \frac{\delta b_{2*}}{2}\|u_x(t)\|^2 \end{aligned} \quad (4.22)$$

$$
\begin{aligned}
&\geq \frac{1-\delta}{2}\left\|u'(t)\right\|^2+\frac{1}{2}\left(b_{3*}-\delta b_3^*\right)\left\|u_x'(t)\right\|^2+\tilde{d}_1\left\|u_x(t)\right\|_{L^p}^p \\
&\quad+\frac{1}{p}I(t)+\frac{p-2}{2p}(g*u')(t)+\left[\frac{p-2}{2p}-\frac{\delta\left(1+b_3^*\right)}{2b_{1*}}\right]\int_0^{\|u_x(t)\|^2}B_1(y)dy \\
&\geq \bar{\beta}_1E_1(t),
\end{aligned}
$$

where δ is small enough and

$$
\begin{aligned}
\bar{\beta}_1 &= \min\left\{\frac{1-\delta}{2},\ \frac{b_{3*}-\delta b_3^*}{2},\ \frac{p-2}{2p}-\frac{\delta}{2b_{1*}}\left(1+b_3^*\right),\ \tilde{d}_1,\ \frac{1}{p}\right\}>0, \quad (4.23)\\
0 &< \delta<\min\left\{1,\ \frac{b_{3*}}{b_3^*},\ \frac{(p-2)b_{1*}}{p\left(1+b_3^*\right)}\right\}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathcal{L}(t) &\leq \frac{1+\delta}{2}\left\|u'(t)\right\|^2+\frac{b_3^*}{2}(1+\delta)\left\|u_x'(t)\right\|^2+\tilde{d}_1\left\|u_x(t)\right\|_{L^p}^p \quad (4.24)\\
&\quad+\frac{1}{p}I(t)+\frac{p-2}{2p}(g*u')(t) \\
&\quad+\left[\frac{p-2}{2p}+\frac{\delta}{2b_{1*}}\left(1+\lambda+b_2^*+b_3^*\right)\right]\int_0^{\|u_x(t)\|^2}B_1(y)dy \\
&\leq \bar{\beta}_2E_1(t),
\end{aligned}
$$

where

$$
\bar{\beta}_2=\max\left\{\frac{1+\delta}{2},\ \frac{b_3^*}{2}(1+\delta),\ \frac{p-2}{2p}+\frac{\delta}{2b_{1*}}\left(1+\lambda+b_2^*+b_3^*\right),\ \tilde{d}_1\right\}>0. \quad (4.25)
$$

Next, we show that the functional $\Psi(t)$ satisfies

$$
\begin{aligned}
\Psi'(t) &\leq \left\|u'(t)\right\|^2+\left(\frac{\sigma_3}{2}+b_3^*\right)\left\|u_x'(t)\right\|^2 \quad (4.26)\\
&\quad-\left(\chi_{1*}-\frac{d_2}{p}-\frac{\varepsilon_2}{2b_{1*}}\right)\int_0^{\|u_x(t)\|^2}B_1(y)dy \\
&\quad-\left(\frac{d_2(1-\delta_2)}{p}\eta^*-\frac{\sigma_3}{2}\right)\left\|u_x(t)\right\|^2+\frac{d_2}{p}(g*u')(t) \\
&\quad-\frac{d_2\delta_2}{p}I(t)-d_2\tilde{d}_1\left\|u_x(t)\right\|_{L^p}^p+\frac{1}{2\varepsilon_2}\left\|f(t)\right\|^2,
\end{aligned}
$$

for all $\varepsilon_2>0$, $\delta_2\in(0,1)$. Its proof is as follows.

By multiplying (3.1) by $u(x,t)$ and integrating over $[0,1]$, we obtain

$$\begin{aligned}\Psi'(t) &= \left\|u'(t)\right\|^2 - B_1\left(\|u_x(t)\|^2\right)\|u_x(t)\|^2 \\ &+\frac{1}{2}\int_0^1 B_2'(x,t)u_x^2(x,t)dx \\ &+\langle B_3'(t)u_x'(t), u_x(t)\rangle + \left\|\sqrt{B_3(t)}u_x'(t)\right\|^2 \\ &+\langle F\left(u(t),u_x(t)\right), u(t)\rangle + \langle G\left(u(t),u_x(t)\right), u_x(t)\rangle + \langle f(t), u(t)\rangle.\end{aligned} \tag{4.27}$$

Furthermore, by (H_4)(iii), we get

$$\begin{aligned}&\langle F\left(u(t),u_x(t)\right), u(t)\rangle + \langle G\left(u(t),u_x(t)\right), u_x(t)\rangle \\ &\le d_2\int_0^1 \mathcal{F}\left(u(x,t),u_x(x,t)\right)dx \\ &= d_2\left[\int_0^1 \mathcal{F}_1\left(u(x,t),u_x(x,t)\right)dx - \tilde{d}_1\|u_x(t)\|_{L^p}^p\right] \\ &= \frac{d_2}{p}\left[(g*u')(t) + \int_0^{\|u_x(t)\|^2} B_1(y)dy - I(t)\right] - d_2\tilde{d}_1\|u_x(t)\|_{L^p}^p \\ &= \frac{d_2}{p}\left[(g*u')(t) + \int_0^{\|u_x(t)\|^2} B_1(y)dy\right] - \frac{d_2\delta_2}{p}I(t) \\ &\quad -\frac{d_2(1-\delta_2)}{p}I(t) - d_2\tilde{d}_1\|u_x(t)\|_{L^p}^p.\end{aligned} \tag{4.28}$$

We also have

$$-B_1\left(\|u_x(t)\|^2\right)\|u_x(t)\|^2 \le -\chi_{1*}\int_0^{\|u_x(t)\|^2} B_1(y)dy, \tag{4.29}$$

$$\left\|\sqrt{B_3(t)}u_x'(t)\right\|^2 \le b_3^*\left\|u_x'(t)\right\|^2,$$

$$\frac{1}{2}\int_0^1 B_2'(x,t)u_x^2(x,t)dx \le 0,$$

$$\langle B_3'(t)u_x'(t), u_x(t)\rangle \le \frac{\sigma_3}{2}\left(\left\|u_x'(t)\right\|^2 + \|u_x(t)\|^2\right),$$

$$\langle f(t), u(t)\rangle \le \frac{\varepsilon_2}{2b_{1*}}\int_0^{\|u_x(t)\|^2} B_1(y)dy + \frac{1}{2\varepsilon_2}\|f(t)\|^2$$

and

$$-\frac{d_2(1-\delta_2)}{p}I(t) \le -\frac{d_2(1-\delta_2)\eta^*}{p}\|u_x(t)\|^2,$$

for all $\varepsilon_2 > 0$, $\delta_2 \in (0,1)$. Combining (4.27)-(4.29), we get (4.26). The estimates (4.10)(ii) and (4.26) give

$$\begin{aligned} \mathcal{L}'(t) \leq & -\left(\lambda - \bar{\lambda} - \frac{\varepsilon_1}{2} - \delta\right) \left\|u'(t)\right\|^2 - \left(\bar{k} - \frac{\delta d_2}{p}\right)(g * u')(t) \\ & - \left[b_{2*} - \delta\left(\frac{\sigma_3}{2} + b_3^*\right)\right] \left\|u_x'(t)\right\|^2 \\ & -\delta\left(\chi_{1*} - \frac{d_2}{p} - \frac{\varepsilon_2}{2b_{1*}}\right) \int_0^{\|u_x(t)\|^2} B_1(y)dy \\ & -\delta\left(\frac{d_2(1-\delta_2)}{p}\eta^* - \frac{\sigma_3}{2}\right) \|u_x(t)\|^2 \\ & -\frac{\delta d_2 \delta_2}{p} I(t) - \delta d_2 \tilde{d}_1 \|u_x(t)\|_{L^p}^p + \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\right) \|f(t)\|^2, \end{aligned} \tag{4.30}$$

for all δ, ε_1, $\varepsilon_2 > 0$, $\delta_2 \in (0,1)$.

By $\dfrac{\sigma_3}{2\eta^*} < \dfrac{d_2}{p} < \chi_{1*}$, we can choose δ, ε_1, $\varepsilon_2 > 0$, $\delta_2 \in (0,1)$ such that

$$\begin{aligned} \theta_1 &= \lambda - \frac{\varepsilon_1}{2} - \delta > 0, \\ \theta_2 &= b_{2*} - \delta\left(\frac{\sigma_3}{2} + b_3^*\right) > 0, \\ \theta_3 &= \chi_{1*} - \frac{d_2}{p} - \frac{\varepsilon_2}{2b_{1*}} > 0, \\ \theta_4 &= \frac{d_2(1-\delta_2)}{p}\eta^* - \frac{\sigma_3}{2} > 0, \\ \theta_5 &= \bar{k} - \frac{\delta d_2}{p} > 0. \end{aligned} \tag{4.31}$$

By (4.30)-(4.31), we get

$$\mathcal{L}'(t) \leq -\gamma_1 E_1(t) + \tilde{C}_1 e^{-\bar{\eta}_1 t} \leq -\frac{\gamma_1}{\bar{\beta}_2}\mathcal{L}(t) + \tilde{C}_1 e^{-\bar{\eta}_1 t} \leq -\bar{\gamma}\mathcal{L}(t) + \tilde{C}_1 e^{-\bar{\eta}_1 t}, \tag{4.32}$$

where $\gamma_1 = \min\{\theta_1,\ \theta_2,\ \delta\theta_3,\ \dfrac{\delta d_2 \delta_2}{p},\ \delta d_2 \tilde{d}_1\}$, $0 < \bar{\gamma} < \min\{\frac{\gamma_1}{\bar{\beta}_2},\ \bar{\eta}_1\}$, $\tilde{C}_1 = \dfrac{1}{2}\left(\dfrac{1}{\varepsilon_1} + \dfrac{\delta}{\varepsilon_2}\right)\bar{C}_1$.

On the other hand, we also have

$$\begin{aligned} \mathcal{L}(t) &\geq \bar{\beta}_1 E_1(t) \\ &\geq \bar{\beta}_1 \min\{1, b_{1*}\} \left[\left\|u'(t)\right\|^2 + \|u_x(t)\|^2 + \left\|u_x'(t)\right\|^2 + \|u_x(t)\|_{L^p}^p\right]. \end{aligned} \tag{4.33}$$

Therefore, Theorem 4.1 is proved completely. □

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