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# FIXED POINT THEOREMS FOR CONVEX MINIMIZATION PROBLEMS IN COMPLEX VALUED $CAT(0)$ SPACES

**Godwin Amechi Okeke[1], Mujahid Abbas[2] and Manuel de la Sen[3]**

[1]Department of Mathematics, School of Physical Sciences
Federal University of Technology Owerri, P.M.B. 1526 Owerri, Imo State, Nigeria
Abdus Salam School of Mathematical Sciences, Government College University
Lahore 54600 Pakistan
e-mail: godwin.okeke@futo.edu.ng

[2]Department of Mathematics, Government College University, Lahore 54000 Pakistan
Department of Mathematics and Applied Mathematics, University of Pretoria
( Hatfield Campus), Lynnwood Road, Pretoria 0002, South Africa
e-mail: abbas.mujahid@gmail.com

[3]Institute of Research and Development of Processes
University of the Basque Country, Campus of Leioa (Bizkaia)
P.O. Box 644 - Bilbao, Barrio Sarriena, 48940- Leioa, Spain
e-mail: manuel.delasen@ehu.eus

**Abstract.** In this paper, we introduce the concept of a complex valued $CAT(0)$ space and propose a new proximal point algorithm for certain nonlinear operators satisfying rational expressions in the framework of complex valued $CAT(0)$ spaces. We prove existence of fixed point of these nonlinear mappings in such spaces. And we prove strong and $\Delta$-convergence of the iterative sequence generated through our proposed algorithm to the minimizer of a convex function and the fixed point of these mappings. We show that the new iterative algorithm converges faster than the modified Mann-type and Ishikawa-type proximal point algorithms.

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[0]Corresponding author: G. A. Okeke(godwin.okeke@futo.edu.ng).

## 1. Introduction

Suppose $(X, d)$ is a metric space. A geodesic path joining $x \in X$ to $y \in X$, or a geodesic from $x$ to $y$ is a map $c$ from a closed interval $[0, \ell] \subseteq \mathbb{R}$ to $X$ such that $c(0) = x$, $c(\ell) = y$, and $d(c(t), c(t')) = |t - t'|$ for each $t, t' \in [0, \ell]$. In particular, $c$ is an isometry and $d(x, y) = \ell$. The image $\alpha$ of $c$ is known as a geodesic (or metric) segment joining $x$ and $y$. When it is unique, the geodesic segment is denoted by $[x, y]$. The space $(X, d)$ is said to be a geodesic space if every two points of $X$ is joined by a geodesic segment, and $X$ is said to be uniquely geodesic if there is exactly one geodesic joining $x$ and $y$ for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if $Y$ includes every geodesic segment joining any two of its points.

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic space $(X, d)$ consists of three points $x_1, x_2$ and $x_3$ in $X$, the vertices of $\Delta$ and a geodesic segment between each pair of vertices (the edge of $\Delta$). A comparison triangle for geodesic triangle $\Delta(x_1, x_2, x_3)$ in $(X, d)$ is a triangle $\overline{\Delta}(x_1, x_2, x_3) := \Delta(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic space is said to be a $CAT(0)$ space if all geodesic triangles of appropriate size satisfy the following comparison axiom.

Let $\Delta$ be a geodesic triangle in $X$, and $\overline{\Delta}$ a comparison triangle for $\Delta$. Then, $\Delta$ is said to satisfy the $CAT(0)$ inequality if for each $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$, $d(x, y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y})$. Examples of $CAT(0)$ spaces includes: any complete simply connected Riemannian manifold having nonpositive sectional curvature, the complex Hilbert ball with a hyperbolic metric [21], Pre-Hilbert spaces [15], and $R$-trees [37]. The reader interested in detailed study of such spaces is reffered to ([2], [6], [33], [35], [36], [37], [40]).

The notion of complex valued metric spaces was introduced by Azam *et al.* [11] in 2011. They established some fixed point theorems for a pair of mappings satisfying rational inequality. Their results incorporated rational contractive operators which can not be studied in the setup of cone metric spaces as the definition of cone metric spaces rely on the underlying Banach space which is not a division ring. Several authors have obtained interesting and applicable results in complex valued metric spaces (see, e.g. [3], [4], [8], [7], [11], [18], [20], [27], [46], [47], [50], [51], [52], [53]).

The purpose of this paper is to define the concept of complex valued $CAT(0)$ space and propose a new proximal point algorithm for certain nonlinear mappings satisfying rational expressions in the framework of complex valued $CAT(0)$ spaces. We prove existence of fixed point results for these nonlinear mappings in complex valued $CAT(0)$ spaces. The strong and $\Delta$-convergence of the iterative sequence generated by these nonlinear mappings was proved. We propose a new modified proximal point algorithm for these nonlinear mappings in complex valued $CAT(0)$ spaces and prove that it converges faster to the minimizer of a convex function and the fixed point of these mappings than the modified Mann-type and Ishikawa-type proximal point algorithms. We obtain some numerical examples to validate our analytical results. Our results extend several known results to complex valued $CAT(0)$ spaces, including the results of Abbas *et al.* [2] and Khan and Abbas [33].

## 2. Preliminaries

The fixed point problem involves finding the fixed point of a mapping $T : X \to X$. The set of fixed point of the mapping $T$ is denoted by $F(T) := \{x \in X : x = Tx\}$. The fixed point problem $(FP)$ is then formulated as follows:

$$\text{find } x \in X \text{ such that } x = Tx.$$

Let $\mathbb{C}$ be the set of complex numbers. For the rest of this paper, we will adopt the partial order defined on $\mathbb{C}$ in [11].

**Definition 2.1.** ([11]) Let $X$ be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$, satisfies:

(1) $0 \precsim d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, y) \precsim d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then $d$ is called a complex valued metric on $X$, and $(X, d)$ is called a complex valued metric space.

Suppose $x, y_1, y_2$ are points in a $CAT(0)$ space, and $y_0$ is the midpoint of the segment $[y_1, y_2]$, then the $CAT(0)$ inequality implies

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2). \tag{2.1}$$

This is the (CN) inequality of Bruhat and Tits [16]. It is known (see [15]) that a geodesic space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality.

The following facts about $CAT(0)$ spaces can be seen in [24].

**Lemma 2.2.** ([24]) *Let $(X, d)$ be a $CAT(0)$ space. Then*

(i) *$(X, d)$ is uniquely geodesic.*
(ii) *Let $p, x, y$ be points of $X$, let $\alpha \in [0, 1]$, and let $m_1$ and $m_2$ be the points of $[p, x]$ and $[p, y]$, respectively, and satisfying $d(p, m_1) = \alpha d(p, x)$ and $d(p, m_2) = \alpha d(p, y)$. Then*

$$d(m_1, m_2) \leq \alpha d(x, y). \tag{2.2}$$

(iii) *Let* $x, y \in X$, $x \neq y$ *and* $z, w \in [x, y]$ *such that* $d(x, z) = d(x, w)$. *Then* $z = w$.

(iv) *Let* $x, y \in X$. *For each* $t \in [0, 1]$, *there exists a unique point* $z \in [x, y]$ *such that*

$$d(x, z) = td(x, y) \text{ and } d(y, z) = (1 - t)d(x, y). \tag{2.3}$$

We will use the following notations for convenience, from now on, $(1-t)x \oplus ty$ for the unique point $z$ satisfying (2.3).

Suppose $\{x_n\}$ is a bounded sequence in a $CAT(0)$ space $X$. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known (see, e.g. [21], Proposition 7) that in a $CAT(0)$ space, $A(\{x_n\})$ consists of exactly one point.

**Definition 2.3.** ([36], [39]) A sequence $\{x_n\}$ in $X$ is said to be $\Delta$-convergent to $x \in X$ if $x$ is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta\text{-}\lim_n x_n = x$ and call $x$ the $\Delta$-limit of $\{x_n\}$.

We use the following notations for the rest of this paper:

$$w_\Delta(x_n) := \bigcup \{A(\{u_n\})\},$$

where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$.

Motivated by the results above, we now define the concept of complex valued $CAT(0)$ space by assuming that the space $(X, d)$ is a complex valued metric space as follows:

**Definition 2.4.** A complex valued metric space $(X, d)$ is called a complex valued $CAT(0)$ space if it is geodesically connected, and if every geodesic triangle in $X$ is at least as "thin" as its comparison triangle in the Euclidean plane.

A geodesic space is said to be a complex valued $CAT(0)$ space if all geodesic triangles of appropriate size satisfy the following comparison axiom.

Let $\Delta$ be a geodesic triangle in $X$, and let $\overline{\Delta}$ be a comparison triangle for $\Delta$. Then, $\Delta$ is said to satisfy the complex valued $CAT(0)$ inequality if for each $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$, $d(x, y) \precsim d_{\mathbb{E}^2}(\overline{x}, \overline{y})$.

Suppose $x, y_1, y_2$ are points in a complex valued $CAT(0)$ space, and $y_0$ is the midpoint of the segment $[y_1, y_2]$. Then the complex valued $CAT(0)$ inequality implies

$$d^2(x, y_0) \precsim \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2). \tag{2.4}$$

This is the (CN) inequality of Bruhat and Tits [16]. Following the results of [15], we can easily show that a geodesic space is a complex valued $CAT(0)$ space if and only if it satisfies the (CN) inequality.

In 1975, Dass and Gupta [23] extended the Banach contraction mapping principle by using mappings satisfying contractive condition of the rational type in the framework of complete metric spaces. Folowing the results of [23], we define the following class of nonlinear mappings in complex valued $CAT(0)$ spaces.

$$\begin{aligned} \mathcal{M}_1 \quad = \quad & \{T : X \to X : T \text{ is continuous satisfying} \\ & d(Tx, Ty) \precsim \tfrac{\alpha d(y,Ty)(1+d(x,Tx))}{1+d(x,y)} + \beta d(x, y)\}, \end{aligned} \tag{2.5}$$

where $\alpha, \beta \in [0, 1)$ such that $\alpha + \beta < 1$ for each $x, y \in X$.

Similarly, we define the following class of mappings from the results of Jaggi [29].

$$\begin{aligned} \mathcal{M}_2 \quad = \quad & \{T : X \to X : T \text{ is continuous satisfying} \\ & d(Tx, Ty) \precsim \tfrac{\alpha d(x,Tx)d(y,Ty)}{d(x,y)} + \beta d(x, y)\}, \end{aligned} \tag{2.6}$$

where $\alpha, \beta \in [0, 1)$ such that $\alpha + \beta < 1$ for each $x, y \in X$, $x \neq y$.

Next, we define the following class of mappings from the results of Jaggi and Dass [30].

$$\begin{aligned} \mathcal{M}_3 \quad = \quad & \{T : X \to X : T \text{ is continuous satisfying} \\ & d(Tx, Ty) \precsim \tfrac{\alpha d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)} + \beta d(x, y)\}, \end{aligned} \tag{2.7}$$

where $\alpha, \beta \in [0, 1)$ such that $\alpha + \beta < 1$ for each $x, y \in X$.

Finally, we define the following class $\mathcal{F}$ of nonlinear mappings:

$$\mathcal{F} := \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3. \tag{2.8}$$

It is our purpose in this paper to prove some fixed point results for $\mathcal{F}$ class of nonlinear mappings in the framework of complex valued $CAT(0)$ spaces.

In 2013, Khan [34] introduced the Picard-Mann hybrid iterative process. The iterative process for one mapping case is given by the sequence $\{x_n\}_{n=1}^{\infty}$.

$$\begin{cases} x_1 = x \in D, \\ x_{n+1} = Tz_n, \\ z_n = (1-\alpha_n)x_n + \alpha_n Tx_n, \ n \in \mathbb{N}, \end{cases} \tag{2.9}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is in $(0,1)$. Khan [34] proved that this iterative process converges faster than all of Picard [48], Mann [41] and Ishikawa [28] iterative processes in the sense of Berinde [14] for contractive mappings.

In 2017, Okeke and Abbas [45] introduced the Picard-Krasnoselskii hybrid iterative process defined by the sequence $\{x_n\}_{n=1}^{\infty}$ as follows:

$$\begin{cases} x_1 = x \in D, \\ x_{n+1} = Ty_n, \\ y_n = (1-\ell)x_n + \ell Tx_n, \ n \in \mathbb{N}, \end{cases} \tag{2.10}$$

where $\ell \in (0,1)$. The authors proved that this new hybrid iteration process converges faster than all of Picard [48], Mann [41], Krasnoselskii [38] and Ishikawa [28] iterative processes in the sense of Berinde [14]. The authors used this iterative process to find the solution of delay differential equations. Moreover, Okeke [44] proved that the Picard-Mann hybrid iterative process [34] have the same rate of convergence as the Picard-Krasnoselskii hybrid iterative process [45].

Recently, Okeke [43] introduced the Picard-Ishikawa hybrid iterative process $\{x_n\}_{n=0}^{\infty}$ as follows: for any fixed $x_1$ in $D$, construct the sequence $\{x_n\}$ by

$$\begin{cases} x_1 = x \in D, \\ x_{n+1} = Tv_n, \\ v_n = (1-\alpha_n)x_n + \alpha_n Tu_n, \\ u_n = (1-\beta_n)x_n + \beta_n Tx_n, \ n \in \mathbb{N}, \end{cases} \tag{2.11}$$

where $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $(0,1)$. The author proved that this new hybrid iterative process converges faster than all of Picard, Krasnoselskii, Mann, Ishikawa, Noor, Picard-Mann and Picard-Krasnoselskii iterative processes.

Next, we modify iterative processes (2.9), (2.10) and (2.11) in complex valued $CAT(0)$ spaces as follows:

$$\begin{cases} x_1 = m \in D, \\ x_{n+1} = Tz_n, \\ z_n = (1-\alpha_n)x_n \oplus \alpha_n Tx_n, \ n \in \mathbb{N}, \end{cases} \tag{2.12}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is in $(0,1)$.

$$\begin{cases} x_1 = x \in D, \\ x_{n+1} = Ty_n, \\ y_n = (1-\ell)x_n \oplus \ell T x_n, \ n \in \mathbb{N}, \end{cases} \tag{2.13}$$

where $\ell \in (0,1)$.

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = Tv_n, \\ v_n = (1-\alpha_n)x_n \oplus \alpha_n T u_n, \\ u_n = (1-\beta_n)x_n \oplus \beta_n T x_n, \ n \in \mathbb{N}, \end{cases} \tag{2.14}$$

where $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $(0,1)$.

**Definition 2.5.** ([14]) Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ be two sequences of positive numbers that converge to $a$ and $b$, respectively. Assume there exists

$$l = \lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|}. \tag{2.15}$$

(1) If $l = 0$, then it is said that the sequence $\{a_n\}_{n=0}^{\infty}$ converges to $a$ faster than the sequence $\{b_n\}_{n=0}^{\infty}$ to $b$.
(2) If $0 < l < \infty$, then we say that the sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ have the same rate of convergence.

**Definition 2.6.** ([54]) Let $D$ be a nonempty, closed and convex subset of a complete $CAT(0)$ space. A function $f : D \to (-\infty, \infty]$ defined on the set $D$ is said to be convex, if for any geodesic $\gamma : [a,b] \to D$, the function $f \circ \gamma$ is convex. We say that a function $f$ defined on $D$ is lower semi-continuous at a point $x \in D$ if

$$f(x) \le \liminf_{n\to\infty} f(x_n) \tag{2.16}$$

for each sequence $x_n \to x$. A function $f$ is called lower semi-continuous on $D$ if it is lower semi-continuous at any point in $D$.

For each $\lambda > 0$, define the Moreau-Yosida resolvent of $f$ in $CAT(0)$ spaces as follows:

$$J_\lambda(x) = arg\min_{y\in X}[f(y) + \frac{1}{2\lambda} d^2(y,x)]$$

for each $x \in X$. The mapping $J_\lambda$ is well-defined (see [31]).

Suppose $f : X \to (-\infty, \infty]$ is a proper convex and lower semi-continuous function. Ariza-Ruiz *et al.* [10] proved that the set $F(J_\lambda)$ of fixed points of the resolvent associated with $f$ coincides with the set $arg\min_{y\in X} f(y)$ of minimizers of $f$.

The following lemmas will be useful in this study.

**Lemma 2.7.** ([24]) *Let $X$ be a $CAT(0)$ space. Then*

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z), \tag{2.17}$$

*for all $x, y, z \in X$ and $t \in [0,1]$.*

**Lemma 2.8.** ([24]) *Let $X$ be a $CAT(0)$ space. Then*

$$d^2((1-t)x \oplus ty, z) \leq (1-t)d^2(x,z) + td^2(y,z) - t(1-t)d^2(x,y), \tag{2.18}$$

*for all $x, y, z \in X$ and $t \in [0,1]$.*

**Lemma 2.9.** ([24]) *We have the following ststements.*

(i) *Every bounded sequence in $X$ has a $\Delta$-convergent subsequence.*
(ii) *If $C$ is a closed convex subset of $X$ and if $\{x_n\}$ is a bounded sequence in $C$, then the asymptotic center of $\{x_n\}$ is in $C$.*
(iii) *Let $C$ be a closed convex subset of $X$ and $f : C \to X$ be a nonexpansive mapping. If $\{x_n\}$ is $\Delta$-convergent to $x$ and $d(x_n, f(x_n)) \to 0$, then $x \in C$ and $f(x) = x$.*

**Lemma 2.10.** ([11]) *Let $(X,d)$ be a complex valued metric space and let $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$ converges to $x$ if and only if $|d(x_n, x)| \longrightarrow 0$ as $n \to \infty$.*

**Lemma 2.11.** ([11]) *Let $(X,d)$ be a complex valued metric space and let $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \longrightarrow 0$ as $n \to \infty$.*

**Lemma 2.12.** ([31]) *Let $(X,d)$ be a complete $CAT(0)$ space and $f : X \to (-\infty, \infty]$ be proper convex and lower semi-continuous. For any $\lambda > 0$, the resolvent $J_\lambda$ of $f$ is nonexpansive.*

**Lemma 2.13.** ([9]) *Let $(X,d)$ be a complete $CAT(0)$ space and $f : X \to (-\infty, \infty]$ be proper convex and lower semi-continuous. Then, for all $x, y \in X$ and $\lambda > 0$, we have*

$$\frac{1}{2\lambda}d^2(J_\lambda x, y) - \frac{1}{2\lambda}d^2(x,y) + \frac{1}{2\lambda}d^2(x, J_\lambda x) + f(J_\lambda x) \leq f(y).$$

**Proposition 2.14.** (The resolvent identity, [31]) *Let $(X,d)$ be a complete $CAT(0)$ space and $f : X \to (-\infty, \infty]$ be proper convex and lower semi-continuous. Then the following identity holds:*

$$J_\lambda x = J_\mu(\frac{\lambda-\mu}{\lambda}J_\lambda x \oplus \frac{\mu}{\lambda}x)$$

*for all $x \in X$ and $\lambda > \mu > 0$.*

**Lemma 2.15.** ([24]) *If $\{x_n\}$ is a bounded sequence in a complete $CAT(0)$ space with $A(\{x_n\}) = \{x\}$, $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

We now prove the following results which will be useful in the proofs our results in this paper.

**Lemma 2.16.** *Let $(X, d)$ be a complex valued $CAT(0)$ space and let $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$ is $\Delta$-convergent to $x$ if and only if $|d(x_n, x)| \longrightarrow 0$ as $n \to \infty$.*

*Proof.* Suppose $\{x_n\}$ is $\Delta$-convergent to $x$. This means that $x$ is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. This means that

$$\limsup_{n\to\infty} d(u_n, x) \precsim \limsup_{n\to\infty} d(x_n, x). \tag{2.19}$$

Therefore for some $0 \prec c \in \mathbb{C}$ we have

$$\limsup_{n\to\infty} |d(u_n, x)| \leq \limsup_{n\to\infty} |d(x_n, x)| < |c|. \tag{2.20}$$

Hence, we obtain $|d(x_n, x)| \longrightarrow 0$ as $n \to \infty$.

Conversely, suppose $|d(x_n, x)| \longrightarrow 0$ as $n \to \infty$. This means that $\{x_n\}$ is bounded. Then by Lemma 2.4 (i), there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{x\}$. This means that $x \in w_\Delta(x_n)$. By Lemma 2.4 (i), there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta$-$\lim_n v_n = v$ for some $v \in X$. We claim that $x = v$.

Assume for contradiction that $x \neq v$. Therefore, by the uniqueness of asymptotic centers, we have

$$\begin{aligned}\limsup_{n\to\infty} d(v_n, v) &\prec \limsup_{n\to\infty} d(v_n, x)\\ &\precsim \limsup_{n\to\infty} d(u_n, x)\\ &\prec \limsup_{n\to\infty} d(u_n, v)\\ &= \limsup_{n\to\infty} d(x_n, v)\\ &= \limsup_{n\to\infty} d(v_n, v),\end{aligned} \tag{2.21}$$

which is a contradiction. Hence, $x = v$. □

**Lemma 2.17.** *Let $(X, d)$ be a complex valued $CAT(0)$ space and let $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$ is a Cauchy sequence if and only if $|\Delta - \lim_n d(x_n, x_{n+m})| = 0$.*

*Proof.* Suppose $\{x_n\} \subset X$ is a Cauchy sequence. Then $\{x_n\}$ is convergent in $X$, hence $\{x_n\}$ is bounded. So that for some $0 \prec c \in \mathbb{C}$, we have $d(x_n, x_{n+m}) \prec c$, so that $|d(x_n, x_{n+m})| \prec |c|$. Clearly, $|\Delta - \lim_n d(x_n, x_{n+m})| = 0$.

Conversely, suppose $|\Delta - \lim_n d(x_n, x_{n+m})| = 0$. We see that $w_\Delta(x_n) = \{0\}$, and

$$|\limsup_{n\to\infty} d(x_n, x_{n+m})| = 0. \tag{2.22}$$

This means that $\{x_n\}$ is a Cauchy sequence. □

## 3. Existence of fixed points in complex valued $CAT(0)$ spaces

In this section we prove the existence of a unique fixed point of $\mathcal{F}$ class of nonlinear mappings satisfying (2.8) in complex valued $CAT(0)$ spaces. Our results generalize and extend several known results, including the famous Banach contraction principle [13] and Jaggi and Dass [30] among others.

**Theorem 3.1.** *Let $(X, d)$ be a complete complex valued $CAT(0)$ space and $T : X \to X$ be a mapping on $X$ such that*

$$T \in \mathcal{F} := \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3. \tag{3.1}$$

*Then $T$ has a unique fixed point.*

*Proof.* To prove this theorem, we consider the following three cases:

**Case I**: Suppose that $T \in \mathcal{M}_1$ and $x_0$ is an arbitrary point of $X$. Define a sequence of points $\{x_n\}$ generated recursively by $x_{n+1} = Tx_n$, that is,

$$x_1 = Tx_0,\ x_2 = Tx_1 = T(Tx_0) = T^2x_0, \cdots,\ x_{n+1} = T^{n+1}x_0. \tag{3.2}$$

Note that,

$$\begin{array}{rcl} d(x_n, x_{n+1}) & = & d(Tx_{n-1}, Tx_n) \\ & \precsim & \frac{\alpha d(x_n, Tx_n)[1+d(x_{n-1}, Tx_{n-1})]}{1+d(x_{n-1}, x_n)} + \beta d(x_{n-1}, x_n) \\ & = & \frac{\alpha d(x_n, x_{n+1})[1+d(x_{n-1}, x_n)]}{1+d(x_{n-1}, x_n)} + \beta d(x_{n-1}, x_n) \\ & = & \alpha d(x_n, x_{n+1}) + \beta d(x_{n-1}, x_n). \end{array} \tag{3.3}$$

This implies that

$$d(x_n, x_{n+1}) \precsim \frac{\beta}{1-\alpha} d(x_{n-1}, x_n). \tag{3.4}$$

Let $h = \frac{\beta}{1-\alpha} \in [0, 1)$. Then we have

$$\begin{array}{rcl} d(x_n, x_{n+1}) & \precsim & hd(x_{n-1}, x_n) \\ & \vdots & \\ & \precsim & h^{n+1} d(x_0, x_1). \end{array} \tag{3.5}$$

Hence, for each $m > n$, we have

$$\begin{array}{rcl} d(x_n, x_m) & \precsim & d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ & \precsim & [h^n + h^{n+1} + \cdots + h^{m-1}] d(x_0, x_1) \\ & \precsim & \frac{h^n}{1-h} d(x_0, x_1). \end{array} \tag{3.6}$$

This implies that

$$|d(x_n, x_m)| \leq \frac{h^n}{1-h}|d(x_0, x_1)| \longrightarrow 0 \text{ as } m, n \to \infty. \tag{3.7}$$

Therefore, the sequence $\{x_n\}$ is a Cauchy sequence by Lemma 2.6. Since $X$ is complete, there exists $x \in X$ such that $x_n \longrightarrow x$ as $n \to \infty$. We claim that $x$ is a unique fixed point of $T$.

Since $T$ is continuous, we have $x_{n+1} = Tx_n \longrightarrow Tx$. But $x_{n+1} \longrightarrow x$, which implies that $Tx = x$. That is, $x$ is a fixed point of $T$.

Next, we show that $x$ is a unique fixed point of $T$. Suppose there exists another fixed point $x^*$ of $T$. Then by (3.1), we have

$$\begin{aligned} d(x, x^*) &= d(Tx, Tx^*) \\ &\precsim \tfrac{\alpha d(x^*, Tx^*)(1+d(x,Tx))}{1+d(x,x^*)} + \beta d(x, x^*) \\ &= \tfrac{\alpha.0}{1+d(x,x^*)} + \beta d(x, x^*) \\ &= \beta d(x, x^*). \end{aligned} \tag{3.8}$$

This implies that $x = x^*$.

**Case II**: Suppose that $T \in \mathcal{M}_2$, this means that

$$\begin{aligned} T \in \mathcal{M}_2 &= \{T : X \to X : T \text{ is continuous satisfying} \\ &\quad d(Tx, Ty) \precsim \tfrac{\alpha d(x,Tx)d(y,Ty)}{d(x,y)} + \beta d(x, y)\}, \end{aligned} \tag{3.9}$$

where $\alpha, \beta \in [0, 1)$ such that $\alpha + \beta < 1$ for each $x, y \in X$, $x \neq y$.

Let $x_0$ be an arbitrary point of $X$. Define a sequence of points $\{x_n\}$ generated recursively by $x_{n+1} = Tx_n$, that is,

$$x_1 = Tx_0,\ x_2 = Tx_1 = T(Tx_0) = T^2x_0, \cdots,\ x_{n+1} = T^{n+1}x_0. \tag{3.10}$$

Note that

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\precsim \tfrac{\alpha d(x_{n-1},Tx_{n-1})d(x_n,Tx_n)}{d(x_{n-1},x_n)} + \beta d(x_{n-1}, x_n) \\ &= \tfrac{\alpha d(x_{n-1},x_n)d(x_n,x_{n+1})}{d(x_{n-1},x_n)} + \beta d(x_{n-1}, x_n) \\ &= \alpha d(x_n, x_{n+1}) + \beta d(x_{n-1}, x_n). \end{aligned} \tag{3.11}$$

It implies that

$$d(x_n, x_{n+1}) \precsim \frac{\beta}{1-\alpha} d(x_{n-1}, x_n). \tag{3.12}$$

If we put $h = \frac{\beta}{1-\alpha} \in [0, 1)$, then we have

$$\begin{aligned} d(x_n, x_{n+1}) &\precsim h d(x_{n-1}, x_n) \\ &\vdots \\ &\precsim h^{n+1} d(x_0, x_1). \end{aligned} \tag{3.13}$$

Now, for each $m > n$, we obtain that

$$\begin{array}{rcl} d(x_n, x_m) & \precsim & d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ & \precsim & [h^n + h^{n+1} + \cdots + h^{m-1}] d(x_0, x_1) \\ & \precsim & \frac{h^n}{1-h} d(x_0, x_1). \end{array} \tag{3.14}$$

This implies that

$$|d(x_n, x_m)| \leq \frac{h^n}{1-h} |d(x_0, x_1)| \longrightarrow 0 \text{ as } m, n \to \infty. \tag{3.15}$$

Therefore, the sequence $\{x_n\}$ is a Cauchy sequence by Lemma 2.11. Since $X$ is complete, there exists $x \in X$ such that $x_n \longrightarrow x$ as $n \to \infty$. We claim that $x$ is a unique fixed point of $T$.

Since $T$ is continuous, $x_{n+1} = Tx_n \longrightarrow Tx$. But $x_{n+1} \longrightarrow x$, which implies that $Tx = x$.

Next, we show that $x$ is a unique fixed point of $T$. Suppose there exists another fixed point $x^*$ of $T$. Then by (3.9), we have

$$\begin{array}{rcl} d(x, x^*) & = & d(Tx, Tx^*) \\ & \precsim & \frac{\alpha d(x, Tx) d(x^*, Tx^*)}{d(x, x^*)} + \beta d(x, x^*) \\ & = & \frac{\alpha . 0}{d(x, x^*)} + \beta d(x, x^*) \\ & = & \beta d(x, x^*). \end{array} \tag{3.16}$$

This implies that $x = x^*$.

**Case III**: Suppose that $T \in \mathcal{M}_3$, this means that

$$\begin{array}{rcl} T \in \mathcal{M}_3 & = & \{T : X \to X : T \text{ is continuous satisfying} \\ & & d(Tx, Ty) \precsim \frac{\alpha d(x, Tx) d(y, Ty)}{d(x,y) + d(x, Ty) + d(y, Tx)} + \beta d(x, y)\}, \end{array} \tag{3.17}$$

where $\alpha, \beta \in [0, 1)$ such that $\alpha + \beta < 1$ for each $x, y \in X$.

Let $x_0$ be an arbitrary point of $X$. Define a sequence of points $\{x_n\}$ generated recursively by $x_{n+1} = Tx_n$, that is,

$$x_1 = Tx_0, \ x_2 = Tx_1 = T(Tx_0) = T^2 x_0, \cdots, \ x_{n+1} = T^{n+1} x_0. \tag{3.18}$$

Since, $d(x_{n-1}, x_n) \precsim d(x_{n-1}, x_n) + d(x_{n-1}, x_{n+1})$, we have

$$\begin{array}{rcl} d(x_n, x_{n+1}) & = & d(Tx_{n-1}, Tx_n) \\ & \precsim & \frac{\alpha d(x_{n-1}, Tx_{n-1}) d(x_n, Tx_n)}{d(x_{n-1}, x_n) + d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})} + \beta d(x_{n-1}, x_n) \\ & = & \frac{\alpha d(x_{n-1}, x_n) d(x_n, x_{n+1})}{d(x_{n-1}, x_n) + d(x_{n-1}, x_{n+1}) + d(x_n, x_n)} + \beta d(x_{n-1}, x_n) \\ & = & \frac{\alpha d(x_{n-1}, x_n) d(x_n, x_{n+1})}{d(x_{n-1}, x_n) + d(x_{n-1}, x_{n+1})} + \beta d(x_{n-1}, x_n) \\ & \precsim & \alpha d(x_n, x_{n+1}) + \beta d(x_{n-1}, x_n). \end{array} \tag{3.19}$$

It implies that

$$d(x_n, x_{n+1}) \precsim \frac{\beta}{1-\alpha} d(x_{n-1}, x_n). \tag{3.20}$$

Let $h = \frac{\beta}{1-\alpha} \in [0, 1)$. Then we have

$$\begin{aligned} d(x_n, x_{n+1}) &\precsim hd(x_{n-1}, x_n) \\ &\vdots \\ &\precsim h^{n+1}d(x_0, x_1). \end{aligned} \tag{3.21}$$

Now, for each $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\precsim d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\precsim [h^n + h^{n+1} + \cdots + h^{m-1}]d(x_0, x_1) \\ &\precsim \tfrac{h^n}{1-h}d(x_0, x_1). \end{aligned} \tag{3.22}$$

This implies that

$$|d(x_n, x_m)| \leq \frac{h^n}{1-h}|d(x_0, x_1)| \longrightarrow 0 \text{ as } m, n \to \infty. \tag{3.23}$$

Therefore, the sequence $\{x_n\}$ is a Cauchy sequence by Lemma 2.11. Since $X$ is complete, there exists $x \in X$ such that $x_n \longrightarrow x$ as $n \to \infty$. We claim that $x$ is a unique fixed point of $T$.

Since $T$ is continuous, we have $x_{n+1} = Tx_n \longrightarrow Tx$. But $x_{n+1} \longrightarrow x$, this implies that $Tx = x$.

Next, we show that $x$ is a unique fixed point of $T$. Suppose there exists another fixed point $x^*$ of $T$, then by (3.17), we have

$$\begin{aligned} d(x, x^*) &= d(Tx, Tx^*) \\ &\precsim \tfrac{\alpha d(x,Tx)d(x^*,Tx^*)}{d(x,x^*)+d(x,Tx^*)+d(x^*,Tx)} + \beta d(x, x^*) \\ &= \tfrac{\alpha.0}{d(x,x^*)} + \beta d(x, x^*) \\ &= \beta d(x, x^*). \end{aligned} \tag{3.24}$$

This implies that $x = x^*$.

Therefore in all cases $T \in \mathcal{F} := \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$ has a unique fixed point. This completes the proof. □

## 4. Strong and Δ-convergence theorems in complex valued $CAT(0)$ spaces

In this section we prove the following convergence results for the $\mathcal{F}$ class of nonlinear mappings in the framework of complex valued $CAT(0)$ spaces.

**Lemma 4.1.** *Let $D$ be a nonempty closed convex subset of a complex valued $CAT(0)$ space $(X, d)$ and $T : D \to D$ be a mapping on $D$ such that*

$$T \in \mathcal{F} := \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3. \tag{4.1}$$

*If for an arbitrary chosen* $x_0 \in D$, *the sequence* $\{x_n\}$ *is generated by the Picard-Ishikawa hybrid iterative process* (2.14), *with real sequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *in* $(0,1)$ *satisfying* $0 < a \leq \alpha_n, \beta_n \leq b < 1$, *for some* $a, b \in \mathbb{R}$, *then*

(i) $\lim_{n\to\infty} |d(x_n, x)|$ *exists for* $x \in F(T)$.

(ii) $\lim_{n\to\infty} |d(x_n, Tx_n)| = 0$.

*Proof.* We consider the following cases:

**Case I**: Suppose $T \in \mathcal{M}_1$. Then by (2.14) and Lemma 2.2, we have

$$\begin{aligned} d(x_{n+1}, x) &= d(Tv_n, Tx) \\ &\precsim \tfrac{\lambda d(x,Tx)(1+d(v_n,Tv_n))}{1+d(v_n,x)} + \gamma d(v_n, x) \\ &= \tfrac{\lambda .0.(1+d(v_n,Tv_n))}{1+d(v_n,x)} + \gamma d(v_n, x) \\ &= \gamma d(v_n, x). \end{aligned} \tag{4.2}$$

Next, we have the following estimates:

$$\begin{aligned} d(v_n, p) &= d((1-\alpha_n)x_n \oplus \alpha_n Tu_n, x) \\ &\precsim (1-\alpha_n)d(x_n, x) + \alpha_n d(Tu_n, Tx) \\ &\precsim (1-\alpha_n)d(x_n, x) + \alpha_n \left[\tfrac{\lambda d(x,Tx)(1+d(u_n,Tu_n))}{1+d(u_n,x)} + \gamma d(u_n, x)\right] \\ &= (1-\alpha_n)d(x_n, x) + \alpha_n \gamma d(u_n, x). \end{aligned} \tag{4.3}$$

$$\begin{aligned} d(u_n, x) &= d((1-\beta_n)x_n \oplus \beta_n Tx_n, x) \\ &\precsim (1-\beta_n)d(x_n, x) + \beta_n d(Tx_n, Tx) \\ &\precsim (1-\beta_n)d(x_n, x) + \beta_n \left[\tfrac{\lambda d(x,Tx)(1+d(x_n,Tx_n))}{1+d(x_n,x)} + \gamma d(x_n, x)\right] \\ &= (1-\beta_n)d(x_n, x) + \beta_n \gamma d(x_n, x) \\ &= (1-\beta_n(1-\gamma))d(x_n, x). \end{aligned} \tag{4.4}$$

Using (4.4) in (4.3), we have

$$\begin{aligned} d(v_n, x) &\precsim (1-\alpha_n)d(x_n, x) + \alpha_n \gamma(1-\beta_n(1-\gamma))d(x_n, x) \\ &= (1-\alpha_n(1-\gamma(1-\beta_n(1-\gamma))))d(x_n, x). \end{aligned} \tag{4.5}$$

Using (4.5) in (4.2), we have

$$d(x_{n+1}, x) \precsim \gamma(1-\alpha_n(1-\gamma(1-\beta_n(1-\gamma))))d(x_n, x). \tag{4.6}$$

Since $\gamma(1-\alpha_n(1-\gamma(1-\beta_n(1-\gamma)))) < 1$, from (4.6) we obtain

$$d(x_{n+1}, x) \precsim d(x_n, x). \tag{4.7}$$

This means that $\{d(x_n, x)\}$ is decreasing, so $\lim_{n\to\infty} |d(x_n, x)|$ exists. This proves part (i). Suppose

$$\lim_{n\to\infty} |d(x_n, x)| = k. \tag{4.8}$$

Next, we prove part (ii). We first show that $\lim_{n\to\infty}|d(u_n,x)| = k$. From relation (4.4), we have

$$d(u_n,x) \precsim (1-\beta_n(1-\gamma))d(x_n,x). \tag{4.9}$$

Thus

$$\lim_{n\to\infty}|d(u_n,x)| \le k. \tag{4.10}$$

From (4.5), we obtain that

$$\lim_{n\to\infty}|d(v_n,x)| \le k. \tag{4.11}$$

Using (4.3), we get that

$$\begin{aligned} d(v_n,x) &\precsim (1-\alpha_n)d(x_n,x) + \alpha_n\gamma d(u_n,x) \\ &\precsim (1-\alpha_n)d(x_n,x) + \alpha_n d(u_n,x). \end{aligned} \tag{4.12}$$

This implies that

$$\alpha_n d(x_n,x) \precsim d(x_n,x) - d(v_n,x) + \alpha_n d(u_n,x). \tag{4.13}$$

Therefore,

$$d(x_n,x) \precsim \frac{1}{\alpha_n}[d(x_n,x) - d(v_n,x)] + d(u_n,x). \tag{4.14}$$

which implies that

$$\liminf_{n\to\infty}|d(x_n,x)| \le \frac{1}{\alpha}\lim_{n\to\infty}[|d(x_n,x) - d(v_n,x)|] + \liminf_{n\to\infty}|d(u_n,x)|. \tag{4.15}$$

Hence

$$k \le \liminf_{n\to\infty}|d(u_n,x)|. \tag{4.16}$$

By (4.10) and (4.16), we have

$$\lim_{n\to\infty}|d(u_n,x)| = k. \tag{4.17}$$

Next, by Lemma 2.8, we have

$$\begin{aligned} d^2(u_n,x) &= d^2((1-\beta_n)x_n \oplus \beta_n Tx_n, x) \\ &\precsim (1-\beta_n)d^2(x_n,x) + \beta_n d^2(Tx_n,x) - \beta_n(1-\beta_n)d^2(x_n,Tx_n) \\ &\precsim (1-\beta_n)d^2(x_n,x) + \beta_n d^2(x_n,x) - \beta_n(1-\beta_n)d^2(x_n,Tx_n) \\ &\precsim d^2(x_n,x) - \beta_n(1-\beta_n)d^2(x_n,Tx_n). \end{aligned} \tag{4.18}$$

This implies that

$$\beta_n(1-\beta_n)d^2(x_n,Tx_n) \precsim d^2(x_n,x) - d^2(u_n,x). \tag{4.19}$$

Hence, we have

$$\begin{aligned} d^2(x_n,Tx_n) &\precsim \tfrac{1}{\beta_n(1-\beta_n)}\left[d^2(x_n,x) - d^2(u_n,x)\right] \\ &\precsim \tfrac{1}{\beta(1-\beta)}\left[d^2(x_n,x) - d^2(u_n,x)\right]. \end{aligned} \tag{4.20}$$

This implies that

$$|d^2(x_n, Tx_n)| \leq \frac{1}{\beta(1-\beta)} | \left[ d^2(x_n, x) - d^2(u_n, x) \right] |. \tag{4.21}$$

It follows from (4.8) and (4.17) and (4.21) that

$$\lim_{n\to\infty} |d(x_n, Tx_n)| = 0. \tag{4.22}$$

**Case 2 and Case 3**: Similarly, we can prove (i) and (ii) for the cases $T \in \mathcal{M}_2$ and $T \in \mathcal{M}_3$, respectively. The proof of Lemma 4.1 is completed. □

Next, we have the following lemmas as a consequence of Lemma 4.1.

**Lemma 4.2.** *Let $D$ be a nonempty closed convex subset of a complex valued $CAT(0)$ space $(X, d)$ and $T : D \to D$ be a mapping on $D$ such that*

$$T \in \mathcal{F} := \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3. \tag{4.23}$$

*If for an arbitrary chosen $x_0 \in D$, the sequence $\{x_n\}$ is generated by the Picard-Mann hybrid iterative process* (2.12), *with a real sequence $\{\alpha_n\}$ in $(0,1)$ satisfying $0 < a \leq \alpha_n \leq b < 1$, for some $a, b \in \mathbb{R}$, then*

(i) $\lim_{n\to\infty} |d(x_n, x)|$ *exists for $x \in F(T)$.*
(ii) $\lim_{n\to\infty} |d(x_n, Tx_n)| = 0$.

*Proof.* Using arguments in the the proof of Lemma 4.1, the result follows. □

**Lemma 4.3.** *Let $D$ be a nonempty closed convex subset of a complex valued $CAT(0)$ space $(X, d)$ and $T : D \to D$ be a mapping on $D$ such that*

$$T \in \mathcal{F} := \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3. \tag{4.24}$$

*If for an arbitrary chosen $x_0 \in D$, the sequence $\{x_n\}$ is generated by the Picard-Krasnoselskii hybrid iterative process* (2.13), *with $\ell$ in $(0,1)$, then*

(i) $\lim_{n\to\infty} |d(x_n, x)|$ *exists for $x \in F(T)$.*
(ii) $\lim_{n\to\infty} |d(x_n, Tx_n)| = 0$.

*Proof.* It follows from the arguments similar to those in the proof of Lemma 4.1. □

**Theorem 4.4.** *Let $D$ be a nonempty closed convex subset of a complex valued $CAT(0)$ space $(X, d)$ and $T : D \to D$ be a mapping on $D$ such that*

$$T \in \mathcal{F} := \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3. \tag{4.25}$$

*If for any $x_0 \in D$, the sequence $\{x_n\}$ is generated by the Picard-Ishikawa hybrid iterative process* (2.14), *with real sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0,1)$ satisfying $0 < a \leq \alpha_n, \beta_n \leq b < 1$, for some $a, b \in \mathbb{R}$, then $\{x_n\}$ is $\Delta$-convergent to a fixed point $x$ of $T$.*

*Proof.* First, suppose that $T \in \mathcal{M}_1$. Then it follows from Lemma 4.1 that $\lim_{n\to\infty} |d(x_n, x)|$ exists for $x \in F(T)$ and also $\lim_{n\to\infty} |d(x_n, Tx_n)| = 0$. Therefore, $\{x_n\}$ is bounded.

We first show that $w_\Delta(x_n) \subseteq F(T)$. If $u \in w_\Delta(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. Using Lemma 2.4, it follows that there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_n v_n = v$ for some $v \in D$. Using Lemma 2.9, we see that $v \in F(T)$. By Lemma 4.1, $\lim_{n\to\infty} |d(x_n, v)|$ exists. We claim that $u = v$. Assume on contrary that $u \neq v$. Then by the uniqueness of asymptotic center, we obtain that

$$\begin{aligned} \limsup_{n\to\infty} d(v_n, v) &\prec \limsup_{n\to\infty} d(v_n, u) \\ &\precsim \limsup_{n\to\infty} d(u_n, u) \\ &\prec \limsup_{n\to\infty} d(u_n, v) \\ &= \limsup_{n\to\infty} d(x_n, v) \\ &= \limsup_{n\to\infty} d(v_n, v). \end{aligned} \tag{4.26}$$

This implies that

$$\limsup_{n\to\infty} |d(v_n, v)| < \limsup_{n\to\infty} |d(v_n, v)|, \tag{4.27}$$

which is a contradiction. Therefore, we have $u = v \in F(T)$. Hence, we obtain $w_\Delta(x_n) \subseteq F(T)$.

Next, we show that $\{x_n\}$ is $\Delta$-convergent to a fixed point of $T$. To do this, we show that $w_\Delta(x_n)$ consists of exactly one point. Suppose $\{u_n\}$ is a subsequence of $\{x_n\}$. Then by Lemma 2.9, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_n v_n = v$ for some $v \in D$. Suppose $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. We have previously established that $u = v$ and $v \in F(T)$.

Finally, we claim that $x = v$. If this is not the case, then the existence of $\lim_{n\to\infty} |d(x_n, v)|$ and uniqueness of asymptotic centers imply that

$$\begin{aligned} \limsup_{n\to\infty} d(v_n, v) &\prec \limsup_{n\to\infty} d(v_n, x) \\ &\precsim \limsup_{n\to\infty} d(x_n, x) \\ &\prec \limsup_{n\to\infty} d(x_n, v) \\ &= \limsup_{n\to\infty} d(v_n, v). \end{aligned} \tag{4.28}$$

This implies that

$$\limsup_{n\to\infty} |d(v_n, v)| < \limsup_{n\to\infty} |d(v_n, v)|, \tag{4.29}$$

which is a contradiction. Therefore, we have $x = v \in F(T)$. Hence,

$$w_\Delta(x_n) = \{x\}.$$

Similarly, we can prove that $\{x_n\}$ is $\Delta$-convergent to the fixed point $x$ of $T$ for the cases $T \in \mathcal{M}_2$ and $T \in \mathcal{M}_3$, respectively. The proof of Theorem 4.4 is completed. □

Next, we establish the following theorems as consequence of Theorem 4.4.

**Theorem 4.5.** *Let $D$ be a nonempty closed convex subset of a complex valued $CAT(0)$ space $(X, d)$ and $T : D \to D$ be a mapping on $D$ such that*

$$T \in \mathcal{F} := \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3. \tag{4.30}$$

*If for an arbitrary chosen $x_0 \in D$, the sequence $\{x_n\}$ is generated by the Picard-Mann hybrid iterative process* (2.12)*, with a real sequence $\{\alpha_n\}$ in $(0, 1)$ satisfying $0 < a \leq \alpha_n \leq b < 1$, for some $a, b \in \mathbb{R}$, then $\{x_n\}$ is $\Delta$-convergent to a fixed point $x$ of $T$.*

*Proof.* The proof of Theorem 4.5 follows on the similar lines as in the proof of Theorem 4.4. □

**Theorem 4.6.** *Let $D$ be a nonempty closed convex subset of a complex valued $CAT(0)$ space $(X, d)$ and $T : D \to D$ be a mapping on $D$ such that*

$$T \in \mathcal{F} := \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3. \tag{4.31}$$

*If for any $x_0 \in D$, the sequence $\{x_n\}$ is generated by the Picard-Krasnoselskii hybrid iterative process* (2.13)*, with $\ell$ in $(0, 1)$, then $\{x_n\}$ is $\Delta$-converges to ass fixed point $x$ of $T$.*

*Proof.* The proof of Theorem 4.6 follows on the similar lines as in the proof of Theorem 4.4. □

**Lemma 4.7.** *Let $D$ be a nonempty closed convex subset of a complex valued $CAT(0)$ space $(X, d)$ and $T : D \to D$ is a mapping on $D$ such that*

$$T \in \mathcal{F} := \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3. \tag{4.32}$$

*Then, $\{x_n\}$ is $\Delta$-convergent to a point $x$ and $|d(x_n, Tx_n)| \to 0$ as $n \to \infty$, imply $x \in D$ and $Tx = x$.*

*Proof.* First, suppose $T \in \mathcal{M}_1$. Then we have

$$\begin{aligned}
\limsup_{n\to\infty} d(Tx, x_n) &\precsim \limsup_{n\to\infty}[d(Tx, Tx_n) + d(Tx_n, x_n)] \\
&\precsim \limsup_{n\to\infty}\left[\frac{\lambda d(x, Tx)(1 + d(x_n, Tx_n))}{1 + d(x_n, x)}\right] \\
&\quad +\beta \limsup_{n\to\infty} d(x_n, x) \\
&\quad + \limsup_{n\to\infty} d(Tx_n, x_n) \\
&= \beta \limsup_{n\to\infty} d(x_n, x) + \limsup_{n\to\infty} d(Tx_n, x_n).
\end{aligned} \tag{4.33}$$

This implies that

$$\limsup_{n\to\infty} |d(Tx, x_n)| \leq \limsup_{n\to\infty} |d(x_n, x)| + \limsup_{n\to\infty} |d(Tx_n, x_n)|. \tag{4.34}$$

Since $\limsup_{n\to\infty} |d(Tx_n, x_n)| = 0$, we have

$$\limsup_{n\to\infty} |d(Tx, x_n)| \precsim \limsup_{n\to\infty} |d(x_n, x)| = r(x, x_n). \tag{4.35}$$

Similarly, we can prove the results for the cases $T \in \mathcal{M}_2$ and $T \in \mathcal{M}_3$, respectively. The proof of Lemma 4.7 is completed. □

## 5. Applications to convex minimization problem

Suppose that $(X, d)$ is a complex valued $CAT(0)$ space and $f$ is a proper and convex function from the set $X$ to $(-\infty, \infty]$. Then an optimization problem is to find $x \in X$ such that

$$f(x) = \min_{y \in X} f(y). \tag{5.1}$$

The set of minimizers of $f$ is denoted by $arg\min_{y \in X} f(y)$.

In 1970, Martinet [42] introduced an important tool for solving this problem. This method is called the proximal point algorithm (abbreviated, PPA). In 1976, Rockafellar [49] established that the PPA converges to the solution of the convex minimization problem in Hilbert spaces.

Suppose $f$ is a proper, convex and lower semi-continuous function on a Hilbert space $H$ which attains its minimum. The PPA is defined as follows: $x_1 \in H$ and

$$x_{n+1} = arg\min_{y \in H} \left[ f(y) + \frac{1}{2\lambda_n} \|y - x_n\|^2 \right] \tag{5.2}$$

for all $n \in \mathbb{N}$, where $\lambda_n > 0$ for all $n \in \mathbb{N}$. It was established that the sequence $\{x_n\}$ converges weakly to a minmizer of $f$ provided $\sum_{n=1}^{\infty} \lambda_n = \infty$. Moreover, Guler [22] proved that PPA does not necessarily converge strongly in general.

In 2000, Kamimura and Takahashi [32] combined the PPA with the Halpern algorithm [25] to obtain strong convergence results.

In 2013, Bačák [12] introduced the PPA in a $CAT(0)$ space $(X, d)$ as follows: let $x_1 \in X$ and

$$x_{n+1} = arg\min_{y \in X} \left[ f(y) + \frac{1}{2\lambda_n} d^2(y, x_n) \right] \tag{5.3}$$

for each $n \in \mathbb{N}$, where $\lambda_n > 0$ for all $n \in \mathbb{N}$. It was proved based on the concept of Fejér monotonicity that if $f$ has a minimizer and $\sum_{n=1}^{\infty} \lambda_n = \infty$, then the sequence $\{x_n\}$ is $\Delta$-convergent to its minmizer. Since then, several authors have constructed some PPA and proved interesting results in the framework of $CAT(0)$ spaces (see, e.g. [17], [54]).

The Mann and Ishikawa PPA are given as follows:

$$\begin{cases} z_n = arg\min_{y \in X} [f(y) + \frac{1}{2\lambda_n} d^2(y, x_n)], \\ x_{n+1} = (1 - \alpha_n) x_n \oplus \alpha_n T z_n, \end{cases} \tag{5.4}$$

for all $n \geq 1$, where $\{\alpha_n\}$ is a real sequence in $[0, 1]$.

$$\begin{cases} z_n = arg\min_{y\in X}[f(y) + \frac{1}{2\lambda_n}d^2(y, x_n)], \\ w_n = (1-\alpha_n)x_n \oplus \alpha_n T z_n, \\ x_{n+1} = (1-\beta_n)x_n \oplus \beta_n T w_n, \end{cases} \tag{5.5}$$

for all $n \geq 1$, where $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $[0, 1]$.

Motivated by the results above, we introduce the following modified Picard-Ishikawa hybrid PPA, which is constructed as follows:

$$\begin{cases} z_n = arg\min_{y\in X}[f(y) + \frac{1}{2\lambda_n}d^2(y, x_n)], \\ u_n = (1-\alpha_n)x_n \oplus \alpha_n T z_n, \\ v_n = (1-\beta_n)x_n \oplus \beta_n T u_n, \\ x_{n+1} = T v_n, \end{cases} \tag{5.6}$$

for all $n \geq 1$, where $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $[0, 1]$.

Next, we prove the following results.

**Lemma 5.1.** *Let $(X, d)$ be a complete complex valued $CAT(0)$ space and $f : X \to (\infty, \infty]$ be a proper, convex and lower semi-continuous function. Suppose that $T : X \to X$ is a mapping on $X$ satisfying*

$$T \in \mathcal{F} := \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3, \tag{5.7}$$

*such that $\omega = F(T) \cap arg\min_{y\in X} f(y)$ is nonempty for each $T \in \mathcal{F}$. Suppose that $\{x_n\}$ is a sequence generated by the PPA (5.6), where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences such that $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for all $n \in \mathbb{N}$, for some $a, b \in \mathbb{R}$ and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for some $\lambda \in \mathbb{R}$. Then*

(i) $\lim_{n\to\infty} |d(x_n, x)|$ *exists for all* $x \in \omega$,
(ii) $\lim_{n\to\infty} |d(x_n, z_n)| = 0$,
(iii) $\lim_{n\to\infty} |d(x_n, T x_n)| = 0$.

*Proof.* The proof of Lemma 5.1 follows on the similar lines as in the proof of Lemma 4.1. Therefore, the proof is omitted. □

**Theorem 5.2.** *Let $(X, d)$ be a complete complex valued $CAT(0)$ space and $f : X \to (\infty, \infty]$ be a proper, convex and lower semi-continuous function. Suppose that $T : X \to X$ is a mapping on $X$ satisfying*

$$T \in \mathcal{F} := \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3, \tag{5.8}$$

*such that $\omega = F(T) \cap arg\min_{y\in X} f(y)$ is nonempty for each $T \in \mathcal{F}$. Suppose that $\{x_n\}$ is a sequence generated by the PPA (5.6), where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences such that $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for all $n \in \mathbb{N}$, for some $a, b \in \mathbb{R}$ and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for some $\lambda \in \mathbb{R}$. Then the sequence $\{x_n\}$ is $\Delta$-convergent to a point of $\omega$.*

*Proof.* From, Proposition 2.14 and Lemma 5.1 (ii), for each $T \in \mathcal{F}$, we have

$$\begin{aligned} d(J_\lambda x_n, x_n) &\precsim d(J_\lambda x_n, z_n) + d(z_n, x_n) \\ &= d(J_\lambda x_n, J_{\lambda_n} x_n) + d(z_n, x_n) \\ &= d\left(J_\lambda x_n, J_\lambda\left(\tfrac{\lambda_n - \lambda}{\lambda_n} J_{\lambda_n} x_n \oplus \tfrac{\lambda}{\lambda_n} x_n\right)\right) + d(z_n, x_n) \\ &\precsim d\left(x_n, \left(1 - \tfrac{\lambda}{\lambda_n}\right) J_{\lambda_n} x_n \oplus \tfrac{\lambda}{\lambda_n} x_n\right) + d(z_n, x_n) \\ &\precsim \left(1 - \tfrac{\lambda}{\lambda_n}\right) d(x_n, J_{\lambda_n} x_n) + \tfrac{\lambda}{\lambda_n} d(x_n, x_n) + d(z_n, x_n) \\ &= \left(1 - \tfrac{\lambda}{\lambda_n}\right) d(x_n, z_n) + d(z_n, x_n). \end{aligned} \tag{5.9}$$

This implies that

$$|d(J_\lambda x_n, x_n)| \leq \left(1 - \frac{\lambda}{\lambda_n}\right) |d(x_n, z_n)| + |d(z_n, x_n)| \longrightarrow 0 \text{ as } n \to \infty. \tag{5.10}$$

We next show that

$$w_\Delta(x_n) := \bigcup_{\{u_n\} \subset \{x_n\}} A(\{u_n\}) \subset \omega. \tag{5.11}$$

Suppose that $u \in w_\Delta(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. Hence, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_{n\to\infty} v_n = v$ for some $v \in D$. By Lemma 4.4, we have $v \in \omega$. Hence, $u = v$ by Lemma 2.15. This implies that $w_\Delta(x_n) \subset \omega$.

Next, we show that the sequence $\{x_n\}$ is $\Delta$-convergent to a point of $\omega$. Hence, it suffices to show that $w_\Delta(x_n)$ consists of exactly one point. Suppose $\{u_n\}$ is a subsequence of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. Since $u \in w_\Delta(x_n) \subset \omega$ and $\{d(x_n, u)\}$ converges, it follows by Lemma 2.15 that $x = u$. Therefore, $w_\Delta(x_n) = \{x\}$. The proof of Theorem 5.2 is completed. □

## 6. Numerical examples

In this section, we give some numerical examples to validate our analytical results. We compare the speed of convergence of various proximal point algorithms discussed in section 5, viz: the Mann-type proximal point algorithm (5.4) denoted by PPAM, the Ishikawa-type proximal point algorithm (5.5) denoted by PPAI and the Picard-Ishikawa hybrid-type proximal point algorithm (5.6) denoted by PPAPI. All the codes were written in Matlab (R2010a) and run on PC with Intel(R) Core(TM) i3-4030U CPU @ 1.90 GHz.

**Example 6.1.** Suppose $X = [1, 2]$ with the partial order " $\precsim$ ". Define the metric "$d$" by

$$d(x, y) = |x - y|\sqrt{2}e^{i\frac{\pi}{4}} = |x - y|(1 + i),$$

for each $x, y \in X$. Let $T : X \to X$ be defined by $Tx = x + \frac{1}{x} - \frac{1}{2}$. Define $f : X \to (-\infty, \infty]$ by $f(x) = \frac{1}{2}d^2(x, 1)$. Take $\ell = \nu = 0.3$. Then, we can easily see that $(X, d)$ is a complete complex valued $CAT(0)$ space, $T \in \mathcal{F}$ and $f$ is a proper, convex and lower semi-continuous mapping with

$$\omega = F(T) \cap arg \min_{y \in X} f(y) = \{2\}.$$

By the proximity operator [19], we know that

$$arg \min_{y \in X} \left[ f(y) + \frac{1}{2} d^2(y, x) \right] = prox_f x = \frac{x+1}{2}.$$

Next, we compute the iterates of PPAM (5.4), PPAI (5.5) and PPAPI (5.6). The numerical experiments of all iterations for approximating the common element $\omega = \{2\}$ is given in the graphs below.

We consider the following three cases:

**Case I**: Take $x_1 = 1.9$, $\alpha_n = \frac{1}{5n+1}$, and $\beta_n = \frac{n}{10n+1}$.
**Case II**: Take $x_1 = 1.84$, $\alpha_n = \frac{1}{2n+1}$, and $\beta_n = \frac{1}{5n+1}$.
**Case III**: Take $x_1 = 1.2$, $\alpha_n = \frac{n}{9n+1}$, and $\beta_n = \frac{1}{4n+1}$.

Next, we present the following graphs of errors versus iteration numbers (n) for each case.

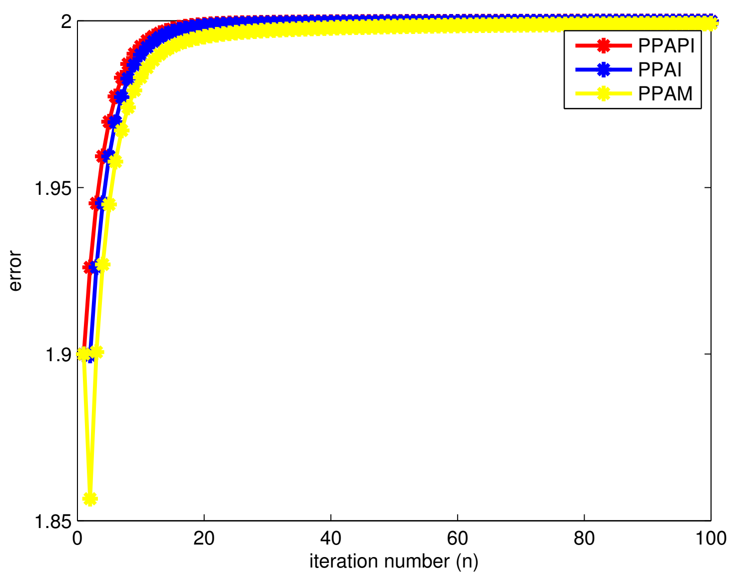


FIGURE 1. Error versus iteration number (n)

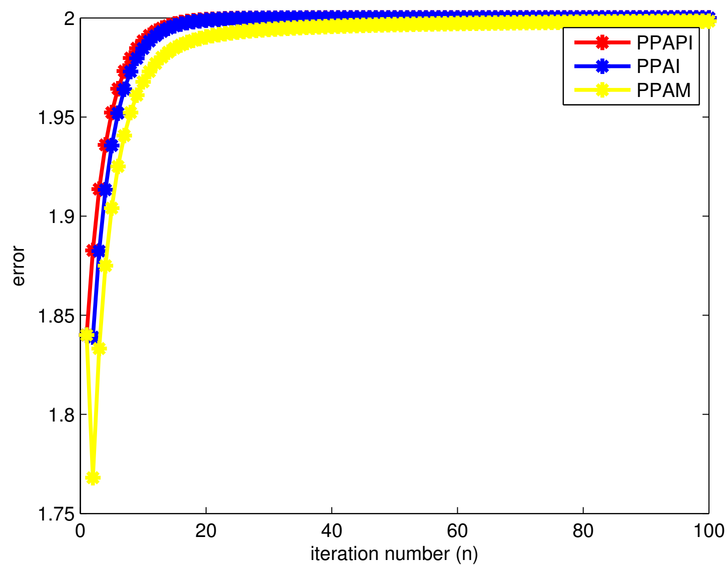


FIGURE 2. Error versus iteration number (n)

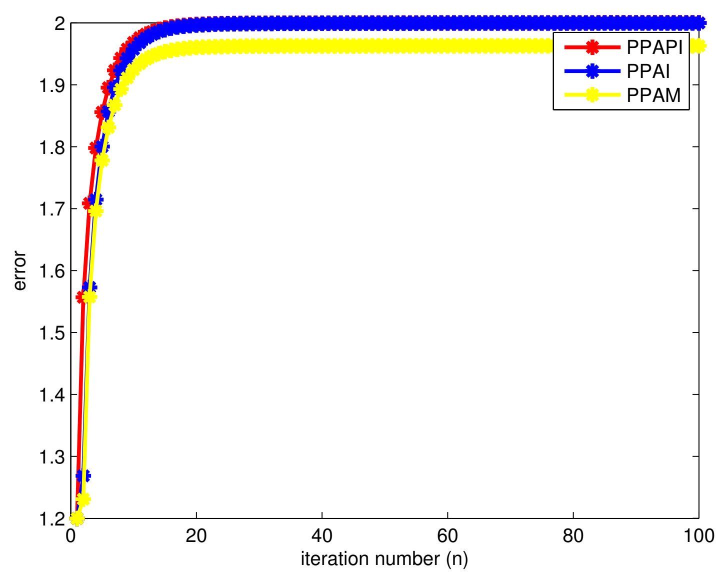


FIGURE 3. Error versus iteration number (n)

**Remark 6.2.** Clearly, from Figure 1, Figure 2 and Figure 3 of Case I, Case II and Case III respectively, we see that our proposed Picard-Ishikawa hybrid type proximal point algorithm PPAPI converges faster to the minimizer of a convex function and the fixed point of the mapping $T$ than the modified Mann-type proximal point algorithm PPAM and the modified Ishikawa-type proximal point algorithm PPAI.

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