**Nonlinear Functional Analysis and Applications**
**Vol. 15, No. 4 (2010), pp. 569-579**

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm


# ON THE STABILITY PROBLEMS IN QUASI-BANACH SPACES

**G. Zamani Eskandani[1] and P. Găvruta[2]**

[1]Faculty of Mathematical Sciences, University of Tabriz,
Tabriz, Iran
e-mail: zamani@tabrizu.ac.ir

[2]Department of Mathematics, University Politehnica of Timisoara,
Piata Victoriei, No. 2, 300006 Timisoara, Romania
e-mail: pgavruta@yahoo.com

**Abstract.** In this paper, we give a generalization of the results from [7] and pose an open problem in quasi-Banach spaces.

## 1. Introduction

In 1940, S.M. Ulam [34] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

*Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality*

$$d(h(x * y), h(x) \diamond h(y)) < \delta$$

*for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with*

$$d(h(x), H(x)) < \epsilon$$

*for all $x \in G_1$?* If this problem has a solution, we say that the homomorphisms from $G_1$ to $G_2$ are stable or the functional equation $h(x * y) = h(x) \diamond h(y)$ is stable.

[0]Received June 2, 2009. Revised September 2, 2009.
[0]2000 Mathematics Subject Classification: Primary 39B72, 46B03, 47Jxx.
[0]Keywords: Hyers–Ulam–Rassias stability, additive mapping, quasi-Banach space, $p$-Banach space.

In 1941, D. H. Hyers [12] considered the case of approximately additive mappings $f : E \to E'$, where $E$ and $E'$ are Banach spaces and $f$ satisfies *Hyers inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in E$. It was shown that the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \leq \epsilon.$$

In 1950, Hyers' Theorem was generalized by Aoki [1] for additive mappings and and independently, in 1978, by Rassias [27] for linear mappings by considering an *unbounded Cauchy difference*.

**Theorem 1.1.** (Th.M. Rassias)*: Let $f : E \longrightarrow E'$ be a mapping from a normed vector space $E$ into a Banach space $E'$ subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

*for all $x, y \in E$, where $\epsilon$ and $p$ are constants with $\epsilon > 0$ and $0 \leq p < 1$. Then the limit $L(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$ exists for all $x \in E$ and $L : E \longrightarrow E'$ is the unique additive mapping which satisfies*

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p$$

*for all $x \in E$. Also, if for each $x \in E$ the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then $L$ is linear.*

In 1990, Th.M. Rassias [29] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Z. Gajda [6] gave an affirmative solution to this question for $p > 1$. It was shown by Z. Gajda [6], as well as by Th.M. Rassias and P. Šemrl [32] that one cannot prove a Th.M. Rassias type theorem when $p = 1$. P. Găvruta [8] proved that the function $f(x) = xln|x|$, if $x \neq 0$ and $f(0) = 0$ satisfies (1.1) with $\epsilon = p = 1$ but

$$\sup_{x\neq 0} \frac{|f(x) - A(x)|}{|x|} \geq \sup_{n\in\mathbb{N}} \frac{|n\ln n - A(n)|}{n} = \sup_{n\in\mathbb{N}} |\ln n - A(1)| = \infty$$

for any additive function $A : \mathbb{R} \to \mathbb{R}$. The paper of Th.M. Rassias has provided a lot of influence in the development of what we now call Hyers–Ulam–Rassias stability of functional equations (cf. the books of P. Czerwik [3] and D.H. Hyers, G. Isac and Th.M. Rassias [13]). J.M. Rassias [25] followed the innovative approach of Th.M Rassias theorem in which he replaced the factor

$\|x\|^p + \|y\|^p$ by $\|x\|^p\|y\|^p$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$ (see also [24, 26]). In 1994, a further generalization of Th.M Rassias' Theorem was obtained by P. Găvruta [7], in which he replaced the bound $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. G. Isac and Th. M. Rassias [16] replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^{p_1} + \|y\|^{p_2}$ in Theorem 1.1 and solved stability problem when $p_2 \leq p_1 < 1$ or $1 < p_2 \leq p_1$, also they asked the question whether such a theorem can be proved for $p_2 < 1 < p_1$. P. Găvruta [8] gave a negative answer to this question. G. Isac and Th.M. Rassias [14] applied the Hyers-Ulam-Rassias stability theory to prove fixed point theorems and study some new applications in Nonlinear Analysis. During the last two decades, a number of papers and research monographs have been published on various generalizations and applications of the Hyers-Ulam–Rassias stability a number of functional equations and mappings (see [5, 10, 17, 18, 19, 21, 23, 30, 31]).

Th.M. Rassias [28] has obtained the following theorem and posed a problem:

**Theorem 1.2.** *Let $E_1$ and $E_2$ to be two Banach spaces, and let $f : E_1 \to E_2$ be a mapping such that $f(tx)$ is continuous in $t$ for each fixed $x$. Assume that there exists $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\|f(x + y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{1.2}$$

*for all $x, y \in X$. Then exists a unique liner mapping $T : E_1 \to E_2$ such that*

$$\|f(x) - T(x)\| \leq \frac{k\theta}{k - k^p}\|x\|^p s(k, p).$$

*for all $x \in X$, where*

$$s(k, p) = 1 + \frac{1}{k}\sum_{m=2}^{k-1} m^p$$

*and any given positive integer $k > 2$.*

**Rassias Problem**: What is the best possible value of $k$ in Theorem 1.2?

P. Găvruta et al. have given a generalization of [7] and have answered to Th. M Rassias problem [11].

In this paper, we give a generalization of the results from [7] and pose an open problem in quasi-Banach spaces. We recall some basic facts concerning quasi-Banach spaces and some preliminary results.

**Definition 1.3.** [2, 33] *Let $X$ be a real linear space. A quasi-norm is a real-valued function on $X$ satisfying the following:*

(*i*) $\|x\| \geq 0$ *for all* $x \in X$ *and* $\|x\| = 0$ *if and only if* $x = 0$.
(*ii*) $\|\lambda x\| = |\lambda|\|x\|$ *for all* $\lambda \in \mathbb{R}$ *and all* $x \in X$.
(*iii*) *There is a constant* $K \geq 1$ *such that* $\|x + y\| \leq K(\|x\| + \|y\|)$ *for all* $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a *quasi-normed space* if $\|\cdot\|$ is a quasi-norm on X. The smallest possible $K$ is called the *modulus of concavity* of $\|\cdot\|$. A *quasi-Banach space* is a complete quasi-normed space. The most significant class of quasi-Banach spaces which are not Banach spaces are the $L_p$ spaces for $0 < p < 1$ with the $L_p$-norm $\|\cdot\|_p$.

A quasi-norm $\|\cdot\|$ is called a *p-norm* $(0 < p \leq 1)$ if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a *p-Banach space.*

By the Aoki–Rolewicz theorem [33] (see also [2]), each quasi-norm is equivalent to some $p$-norm. Since it is much easier to work with $p$-norms than quasi-norms, henceforth we restrict our attention mainly to $p$-norms.

G.Z. Eskandani [4] has investigated the Hyers–Ulam–Rassias stability of the following functional equation

$$\sum_{i=1}^{m} f\big(mx_i + \sum_{j=1, j\neq i}^{m} x_j\big) + f\big(\sum_{i=1}^{m} x_i\big) = 2f\big(\sum_{i=1}^{m} mx_i\big).$$

in quasi-Banach spaces. C. Park [22] has proved the Hyers–Ulam–Rassias stability of homomorphisms in quasi-Banach algebras. M.S. Moslehian and Gh. Sadeghi [20] have proved the Hyers–Ulam–Rassias stability of linear mappings in quasi-Banach modules associated to the Cauchy functional equation and a generalized Jensen functional equation.

Throughout this paper, assume that $k$ is a fixed integer greater than 1.

## 2. Stability of Cauchy functional equation

Assume that $X$ is a quasi-normed space with quasi-norm $\|\cdot\|_X$ and that $Y$ is a $p$-Banach space with $p$-norm $\|.\|_Y$. Let $K$ be the modulus of concavity of $\|\cdot\|_Y$.

**Theorem 2.1.** *Let $\varphi : X \times X \to [0, \infty)$ be a mapping such that*

$$\sum_{n=0}^{\infty} \frac{1}{k^{np}} \varphi^p(k^n x, k^n y) < \infty \tag{2.1}$$

*for all $x, y \in X$. Suppose that a mapping $f : X \to Y$ satisfies the inequality*

$$\|f(x + y) - f(x) - f(y)\|_Y \leq \varphi(x, y) \tag{2.2}$$

*for all $x, y \in X$. Then the limit*

$$A_k(x) = \lim_{n\to\infty} \frac{1}{k^n} f(k^n x)$$

*exists for all* $x \in X$ *and the mapping* $A_k : X \to Y$ *is a unique additive mapping satisfying*

$$\|f(x) - A_k(x)\|_Y \leq \frac{1}{k}\Big[\sum_{n=0}^{\infty} \frac{\widetilde{\varphi}_k^p(k^n x)}{k^{np}}\Big]^{\frac{1}{p}} \tag{2.3}$$

*for all* $x \in X$, *where*

$$\widetilde{\varphi}_k(x) := K^{k-2}\varphi(x,x) + \sum_{i=2}^{k-1} K^{k-i}\varphi(x, ix).$$

*Proof.* By induction on $k$, we show that

$$\|f(kx) - kf(x)\|_Y \leq \widetilde{\varphi}_k(x) \tag{2.4}$$

for all $x \in X$. Letting $y = x$ in (2.2), we get

$$\|f(2x) - 2f(x)\|_Y \leq \varphi(x,x) \tag{2.5}$$

for all $x \in X$. So we get (2.4) for $k = 2$.

Assume that (2.4) holds for $k$. Letting $y = kx$ in (2.2), we get

$$\|f\big((k+1)x\big) - f(x) - f(kx)\|_Y \leq \varphi(x, kx) \tag{2.6}$$

for all $x \in X$. It follows from (2.4) and (2.6) that

$$\begin{aligned}\|f\big((k+1)x\big) - (k+1)f(x)\|_Y &\leq K\|f\big((k+1)x\big) - f(x) - f(kx)\|_Y \\ &\quad + K\|f(kx) - kf(x)\|_Y \\ &\leq K\varphi(x,kx) + K\widetilde{\varphi}_k(x) \\ &= \widetilde{\varphi}_{k+1}(x)\end{aligned}$$

This completes the induction argument. Replacing $x$ by $k^n x$ in (2.4) and dividing both sides of (2.4) by $k^{n+1}$, we get

$$\Big\|\frac{1}{k^{n+1}}f(k^{n+1}x) - \frac{1}{k^n}f(k^n x)\Big\|_Y \leq \frac{1}{k^{n+1}}\widetilde{\varphi}_k(k^n x) \tag{2.7}$$

for all $x \in X$ and all non-negative integers $n$. Since $Y$ is a $p$-Banach space, we have

$$\begin{aligned}\Big\|\frac{1}{k^{n+1}}f(k^{n+1}x) - \frac{1}{k^m}f(k^m x)\Big\|_Y^p &\leq \sum_{i=m}^{n}\Big\|\frac{1}{k^{i+1}}f(k^{i+1}x) - \frac{1}{k^i}f(k^i x)\Big\|_Y^p \\ &\leq \frac{1}{k^p}\sum_{i=m}^{n}\frac{\widetilde{\varphi}_k^p(k^i x)}{k^{ip}}\end{aligned} \tag{2.8}$$

for all $x \in X$ and all non-negative integers $n$ and $m$ with $n \geq m$. Therefore, we conclude from (2.1) and (2.8) that the sequence $\{\frac{1}{k^n}f(k^n x)\}$ is a Cauchy sequence in $Y$ for all $x \in X$. Since $Y$ is complete, the sequence $\{\frac{1}{k^n} f(k^n x)\}$ converges in $Y$ for all $x \in X$. So one can define the mapping $A_k : X \to Y$ by

$$A_k(x) := \lim_{n\to\infty} \frac{1}{k^n} f(k^n x) \tag{2.9}$$

for all $x \in X$. Letting $m = 0$ and passing the limit $n \to \infty$ in (2.8), we get (2.3). Now, we show that $A$ is an additive mapping. It follows from (2.1), (2.2) and (2.9) that

$$\begin{aligned}\|A_k(x+y) - A_k(x) - A_k(y)\|_Y &= \lim_{n\to\infty} \frac{1}{k^n} \big\|f(k^n x + k^n y) - f(k^n x) - f(k^n y)\big\|_Y \\ &\leq \lim_{n\to\infty} \frac{1}{k^n} \varphi(k^n x, k^n y) = 0\end{aligned}$$

for all $x, y \in X$. Hence the mapping $A_k$ is additive. To prove the uniqueness of $A_k$, let $T : X \to Y$ be another additive mapping satisfying (2.3). It follows from (2.3) and (2.9) that

$$\begin{aligned}\|A_k(x) - T(x)\|_Y^p &= \lim_{n\to\infty} \frac{1}{k^{np}} \big\|f(k^n x) - T(k^n x)\big\|_Y^p \\ &\leq \frac{1}{k^p} \lim_{n\to\infty} \sum_{i=0}^{\infty} \frac{\widetilde{\varphi}_k^p(k^{n+i} x)}{k^{(n+i)p}} \\ &\leq \frac{1}{k^p} \lim_{n\to\infty} \sum_{i=n}^{\infty} \frac{\widetilde{\varphi}_k^p(k^i x)}{k^{ip}} = 0\end{aligned}$$

for all $x \in X$. So $A_k = T$. □

**Theorem 2.2.** *Let $\Phi : X \times X \to [0, \infty)$ be a mapping such that*

$$\sum_{n=1}^{\infty} k^{np} \Phi^p(\frac{x}{k^n}, \frac{y}{k^n}) < \infty \tag{2.10}$$

*for all $x, y \in X$. Suppose that a mapping $f : X \to Y$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\|_Y \leq \Phi(x, y)$$

*for all $x, y \in X$. Then the limit*

$$A_k(x) := \lim_{n\to\infty} k^n f(\frac{x}{k^n})$$

*exists for all $x \in X$ and the mapping $A_k : X \to Y$ is a unique additive mapping satisfying*

$$\|f(x) - A_k(x)\|_Y \leq \frac{1}{k} \big[\sum_{n=1}^{\infty} k^{np} \widetilde{\Phi}_k^p(\frac{x}{k^n})\big]^{\frac{1}{p}} \tag{2.11}$$

*for all* $x \in X$, *where*

$$\widetilde{\Phi}_k(x) := K^{k-2}\Phi(x,x) + \sum_{i=2}^{k-1} K^{k-i}\Phi(x,ix).$$

*Proof.* Similar to the proof of Theorem 2.1, we have

$$\|f(kx) - kf(x)\|_Y \le \widetilde{\Phi}_k(x) \tag{2.12}$$

for all $x \in X$. Replacing $x$ by $\frac{x}{k^{n+1}}$ in (2.12) and multiplying both sides of (2.12) to $k^n$, we get

$$\Big\|k^{n+1}f\big(\frac{x}{k^{n+1}}\big) - k^n f\big(\frac{x}{k^n}\big)\Big\|_Y \le k^n \widetilde{\Phi}_k\big(\frac{x}{k^{n+1}}\big)$$

for all $x \in X$ and all non-negative integers $n$. Since $Y$ is a $p$-Banach space, we have

$$\begin{aligned}\Big\|k^{n+1}f\big(\frac{x}{k^{n+1}}\big) - k^m f\big(\frac{x}{k^m}\big)\Big\|_Y^p &\le \sum_{i=m}^{n} \Big\|k^{i+1}f\big(\frac{x}{k^{i+1}}\big) - k^i f\big(\frac{x}{k^i}\big)\Big\|_Y^p \\ &\le \sum_{i=m}^{n} k^{ip}\widetilde{\Phi}_k^p\big(\frac{x}{k^{i+1}}\big) \qquad (2.13) \\ &\le \frac{1}{k^p}\sum_{i=m+1}^{n} k^{ip}\widetilde{\Phi}_k^p\big(\frac{x}{k^i}\big)\end{aligned}$$

for all $x \in X$ and all non-negative integers $n$ and $m$ with $n \ge m$. Therefore, we conclude from (2.10) and (2.13) that the sequence $\{k^n f(x/k^n)\}$ is a Cauchy sequence in $Y$ for all $x \in X$. Since $Y$ is complete, the sequence $\{k^n f(x/k^n)\}$ converges in $Y$ for all $x \in X$. So one can define the mapping $A_k : X \to Y$ by

$$A_k(x) := \lim_{n\to\infty} k^n f\big(\frac{x}{k^n}\big)$$

for all $x \in X$. Letting $m = 0$ and passing the limit $n \to \infty$ in (2.13), we get (2.11). The rest of the proof is similar to the proof of Theorem 2.1. □

The following result is related to the result of [15].

**Theorem 2.3.** *Let* $\psi : [0,\infty) \to [0,\infty)$ *be a function such that*

(1) $\lim_{t\to\infty} \frac{\psi(t)}{t} = 0$,

(2) $\psi(ts) \le \psi(t)\psi(s)$,

(3) $\psi(t) < t$ *for all* $t > 1$.

*Suppose that a mapping $f : X \to Y$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\|_Y \leq \theta\big(\psi(\|x\|_X) + \psi(\|y\|_X)\big) \tag{2.14}$$

*for all $x, y \in X$. Then there exists a unique additive mapping $A_k : X \to Y$ satisfying*

$$\|f(x) - A_k(x)\|_Y \leq \frac{\sigma_k(\psi)\theta}{(k^p - \psi(k)^p)^{\frac{1}{p}}}\psi(\|x\|_X) \tag{2.15}$$

*for all $x \in X$, where*

$$\sigma_k(\psi) := 2K^{k-2} + \sum_{i=2}^{k-1} K^{k-i}(1 + \psi(i)).$$

*Moreover, $A_k = A_2$ for all $k \geq 2$.*

*Proof.* Let

$$\varphi(x, y) = \theta\big(\psi(\|x\|_X) + \psi(\|y\|_X)\big)$$

for all $x, y \in X$. It follows from (2) that $\psi(k^n) \leq \big(\psi(k)\big)^n$ and

$$\varphi(k^n x, k^n y) \leq \theta\big(\psi(k)\big)^n\big(\psi(\|x\|_X) + \psi(\|y\|_X)\big).$$

By using Theorem 2.1, we can get (2.15). Now, we show that $A_k = A_2$. Replacing $x$ by $2^n x$ in (2.15) and dividing both sides of (2.15) by $2^n$, we get

$$\begin{aligned}\|\frac{f(2^n x)}{2^n} - A_k(x)\|_Y &\leq \frac{\sigma_k\theta}{2^n(k^p - \psi(k)^p)^{\frac{1}{p}}}\psi(\|2^n x\|_X) \\ &\leq \frac{\sigma_k\theta}{2^n(k^p - \psi(k)^p)^{\frac{1}{p}}}\psi(\|x\|_X)\psi(2^n)\end{aligned} \tag{2.16}$$

for all $x \in X$. Using (1) and passing the limit $n \longrightarrow \infty$ in (2.16), we get $A_k = A_2$. □

**Theorem 2.4.** *Let $q$ be a non-negative real number such that $q \neq 1$ and $H : [0, \infty) \times [0, \infty) \to [0, \infty)$ be a homogeneous function of degree $q$. Suppose that a mapping $f : X \to Y$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\|_Y \leq H(\|x\|_X, \|y\|_X) \tag{2.17}$$

*for all $x, y \in X$. Then there exists a unique additive mapping $A_k : X \to Y$ satisfying*

$$\|f(x) - A_k(x)\|_Y \leq \frac{\sigma_k(H)}{|k^p - k^{pq}|^{\frac{1}{p}}}\|x\|_X^q \tag{2.18}$$

*for all $x \in X$, where*

$$\sigma_k(H) := K^{k-2}H(1, 1) + \sum_{i=2}^{k-1} K^{k-i}H(1, i).$$

*Moreover,* $A_k = A_2$ *for all* $k \geq 2$.

*Proof.* The proof follows from Theorems 2.1 and 2.2. □

For the particular cases $H(x,y) = \theta(x^q + y^q), H(x,y) = \theta x^r.y^s$ $(r+s=q)$ and $H(x,y) = \min\{x^q, y^q\}$, we have the following corollaries:

**Corollary 2.5.** *Let* $\theta$ *and* $q$ *be non-negative real numbers such that* $q \neq 1$. *Suppose that a mapping* $f : X \to Y$ *satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\|_Y \leq \theta(\|x\|_X^q + \|y\|_X^q)$$

*for all* $x, y \in X$. *Then there exists a unique additive mapping* $A_k : X \to Y$ *satisfying*

$$\|f(x) - A_k(x)\|_Y \leq \frac{\theta\sigma_k}{|k^p - k^{pq}|^{\frac{1}{p}}}\|x\|_X^q$$

*for all* $x \in X$, *where*

$$\sigma_k := 2K^{k-2} + \sum_{i=2}^{k-1} K^{k-i}(1+i^q),$$

*Moreover,* $A_k = A_2$ *for all* $k \geq 2$.

**Corollary 2.6.** *Let* $\theta, r, s$ *be non-negative real numbers such that* $q := r+s \neq 1$. *Suppose that a mapping* $f : X \to Y$ *satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\|_Y \leq \theta\|x\|_X^r\|y\|_X^s$$

*for all* $x, y \in X$. *Then there exists a unique additive mapping* $A_k : X \to Y$ *satisfying*

$$\|f(x) - A_k(x)\|_Y \leq \frac{\theta\sigma_k}{|k^p - k^{pq}|^{\frac{1}{p}}}\|x\|_X^q$$

*for all* $x \in X$, *where*

$$\sigma_k := K^{k-2} + \sum_{i=2}^{k-1} K^{k-i}i^s,$$

*Moreover,* $A_k = A_2$ *for all* $k \geq 2$.

The following result is related to the result of [9].

**Corollary 2.7.** *Let* $q$ *be a non-negative real number such that* $q \neq 1$. *Suppose that a mapping* $f : X \to Y$ *satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\|_Y \leq \min\{\|x\|_X^q, \|y\|_X^q\}$$

*for all* $x, y \in X$. *Then there exists a unique additive mapping* $A_k : X \to Y$ *satisfying*

$$\|f(x) - A_k(x)\|_Y \leq \frac{\theta\sigma_k}{|k^p - k^{pq}|^{\frac{1}{p}}}\|x\|_X^q$$

*for all* $x \in X$, *where*

$$\sigma_k := K^{k-2} + \sum_{i=2}^{k-1} K^{k-i},$$

*Moreover,* $A_k = A_2$ *for all* $k \geq 2$.

**Open Problem**: What is the best possible value of $k$ in Corollaries 2.5, 2.6 and 2.7.

## References


[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, **2** (1950), 64-66.

[2] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis,* vol. **1**, Colloq. Publ., vol. **48**, Amer. Math. Soc., Providence, RI, 2000.

[3] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, New Jersey, London, Singapore, Hong Kong, 2002.

[4] G.Z. Eskandani, *On the Hyers-Ulam-Rassias stability of an additive functional equation in quasi-Banach spaces*, J. Math. Anal. Appl., **345** (2008), 405-409.

[5] G.Z. Eskandani, H. Vaezi, F. Moradlou, *On the Hyers-Ulam-Rassias stability of functional equations in quasi-Banach spaces*, Int. J. Appl. Math. Stat., **15** (2009), 1-15.

[6] Z. Gajda, *On stability of additive mappings*, Int. J. Math. Math. Sci. **14** (1991) 431-434.

[7] P. Găvruta, *A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings,* J. Math. Anal. Appl., **184** (1994), 431–436.

[8] P. Găvruta, *On a problem of G. Isac and Th. M. Rassias concerning the stability of mappings,* J. Math. Anal. Appl., **261** (2001), 543–553.

[9] P. Găvruta,*On the Hyers-Ulam-Rassias stability of mappings,* in: G.V. Milovanovic(ed.), Recent Progress in Inequalities, Kluwer, 1998, pp.465–469.

[10] P. Găvruta and L. Cadariu, *General stability of the cubic functional equation,* Bul. St. Univ. Politehnica Timisoara, Ser. Mat. Fiz., **47**(2002), 57–70.

[11] P. Găvruta, M. Hossu, D. Popescu and C. Căprău, *On the stability of mappings and an anwser to a problem of Th. M Rassias* , Annales mathe′matiques Blaise Pascal, tome2, $n^0$2(1995), 55–60.

[12] D. H. Hyers, *On the stability of the linear functional equation,* Proc. Nat. Acad. Sci., **27** (1941), 222–224.

[13] D. H. Hyers, G. Isac and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.

[14] G. Isac and Th.M. Rassias, *Stability of $\psi$-additive mappings: Applications to nonlinear analysis*, Int. J. Math. Math. Sci., **19** (1996), 219-228.

[15] G. Isac and Th.M. Rassias, *On the Hyers-Ulam stability of psi-additive mappings,* J. Approx. Theory **72**(1993), 131–137.

[16] G. Isac and Th. M. Rassias, it Functional inequalities for approximately additive mappings. In: Stability of Mappings of Hyers–Ulam Type, 117–125, Hadronic Presscollect. orig. Artic., Hadronic Press, Palm Harbour, FL, 1994.

[17] K. Jun and Y. Lee, *On the Hyers–Ulam–Rassias stability of a Pexiderized quadratic inequality,* Math. Inequal. Appl., **4** (2001), 93–118.

[18] S. M. Jung, *Hyers–Ulam–Rassias Stability of Functional Equations in Mathimatical Analysis*, Hadronic Press, Palm Harbor, 2001.

[19] Pl. Kannappan, *Quadratic functional equation and inner product spaces,* Results Math., **27** (1995), 368–372.
[20] M.S. Moslehian and Gh. Sadeghi, *Stability of linear mappings in quasi-Banach modules*, Math. Inequal. Appl., **11** (2008), 549-557.
[21] A. Najati and G.Z. Eskandani, *Stability of a mixed additive and cubic functional equation in quasi-Banach spaces*, J. Math. Anal. Appl., **342** (2008), 1318-1331.
[22] C. Park, *Hyers–Ulam–Rassias stability of homomorphisms in quasi-Banach algebras*, Bull. Sci. Math., **132** (2008), 87–96.
[23] C. Park and F. Moradlou,*Stability of Homomorphisms and Derivations in C*-ternary Rings*, Taiwanese J. Math., (To appear)
[24] J.M. Rassias, *On approximation of approximately linear mappings by linear mappings*, J. Funct. Anal., **46** (1982) 126-130.
[25] J.M. Rassias, *On approximation of approximately linear mappings by linear mappings*, Bull. Sci. Math., **108** (1984), 445-446.
[26] J.M. Rassias, *Solution of a problem of Ulam*, J. Approx. Theory, **57**(1989), 268–273.
[27] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces,* Proc. Amer. Math. Soc., **72** (1978), 297–300.
[28] Th.M. Rassias, *On a modified Hyers–Ulam sequence,* J. Math. Anal. Appl., **158** (1991), 106–113.
[29] Th.M. Rassias, *Problem 16; 2*, Report of the $27^{\text{th}}$ International Symp. on Functional Equations, Aequationes Math., **39** (1990), 292–293.
[30] Th.M. Rassias, *On the stability of the quadratic functional equation and its applications,* Studia Univ. Babes-Bolyai Math., **43**(1998), 89-124.
[31] Th.M. Rassias (ed.), *Functional Equations and Inequalities,* Kluwer Academic Publishers, Dordrecht, Boston, London, 2000.
[32] Th.M. Rassias and P. Šemrl, *On the behaviour of mappings which do not satisfy Hyers-Ulam stability*, Proc. Amer. Math. Soc., **114** (1992) 989-993.
[33] S. Rolewicz, *Metric Linear Spaces,* PWN-Polish Sci. Publ., Warszawa, Reidel, Dordrecht, 1984.
[34] S.M. Ulam, *A Collection of the Mathematical Problems,* Interscience Publ. New York, 1960.