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A STUDY ON APPROXIMATION OF A CONJUGATE FUNCTION USING CESÀRO-MATRIX PRODUCT OPERATOR

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Abstract. In this paper, we establish a new theorem to study the error approximation of a function $\tilde{\zeta}$ conjugate of a function ζ (2π-periodic) in weighted Lipschitz class $W(L^p, p \geq 1,\ \xi(\omega))$, by Cesàro-Matrix ($C^\delta T$) product means of its CFS, where CFS denotes conjugate Fourier series. In fact, the results obtained in the paper provide the best approximation of the conjugate function $\tilde{\zeta}$ in $W(L^p, p \geq 1,\ \xi(\omega))$ class by $C^\delta T$ product means of its ate CFS for the cases $p > 1$ and $p = 1$. Our results generalize six previously known results. Thus, the results of [4], [11], [12], [13], [14], [15] become the particular cases of our results. Our theorems provide some important corollaries.

1. Introduction

The studies of estimations of conjugate of functions in different Lipschitz classes and Hölder classes using single summability operators, have been made by the researchers like [2, 5, 7, 8] etc., in past few decades. The studies of estimation of error of cojugate of functions in different Lipschitz classes

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and Hölder classes using different product operator, have been made by the researchers like [4], [6], [11], [12], [13], [14], [15] etc., in recent past.

In this work, we attempt to consider more advance class of function in order to arrive at the best approximation of function $\tilde{\zeta}$ conjugate of a function ζ (2π-periodic) by trigonometric polynomial of degree more than λ. It can be noted that the results obtained so far in the direction of present work could not provide the best approximation of the function, also we have used Cesàro-Matrix ($C^{\delta}T$) of product operators which is deploped here in order to work using most generalized operator. It is important to mention here that $C^{\delta}T$ is the most generalized product operator and no further generalization of such operator is possible. Therefore, we establish two theorems so obtain best error estimate of a function $\tilde{\zeta}$, conjugate to a 2π-periodic function ζ in weighted $W(L^p, \xi(\omega))$ classes of its CFS. Here we shall consider the two cases (i) $p > 1$ and (ii) $p = 1$ in order to get the Hölder's inequality satisfied.

Our theorems generalizes six previously known results. Thus, the results of [4], [6], [11], [12], [13], [14], [15] become the special cases of our theorem. Some inportant corollaries are also obtained from our theorems.

Note 1. The CFS is not necessarily a Fourier series (FS).

Example 1.1. The series

$$\sum_{\lambda=2}^{\infty} \left(\frac{\sin(\lambda x)}{\log \lambda} \right)$$

conjugate to the FS

$$\sum_{\lambda=2}^{\infty} \left(\frac{\cos(\lambda x)}{\log \lambda} \right)$$

is not a FS (Zygmund [1], p.186).

From above example, we conclude that, a separate study of conjugate series in the present direction of work is quite essential.

Let $C_{2\pi}$ is a Banach space of all periodic functions with period 2π and continuous on the interval $0 \le x \le 2\pi$ under the supremum norm.

The best λ-order error approximation of a function $\tilde{\zeta} \in C_{2\pi}$ is defined by

$$E_{\lambda}(\tilde{\zeta}) = \inf_{t_\lambda} \|\tilde{\zeta} - t_{\lambda}\|,$$

where t_{λ} is a trigonometric polynomial of degree λ (Bernstein [10]).

Let us define the L^p space of all 2π-periodic and integrable functions as

$$L^p[0, 2\pi] := \left\{ \tilde{\zeta} : [0, 2\pi] \to \mathbb{R} : \int_0^{2\pi} |\tilde{\zeta}(x)|^p \, dx < \infty \right\}, p \ge 1.$$

Now, $\|.\|_p$ is defined as

$$\|\tilde{\zeta}\|_p = \begin{cases} \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\tilde{\zeta}(x)\right|^p dx\right\}^{\frac{1}{p}} & \text{for } 1 \leq q < \infty, \\ \operatorname{ess\,sup}_{x\in(0,2\pi)}|\tilde{\zeta}(x)| & \text{for } p = \infty. \end{cases}$$

We consider the following classes of functions:

$$Lip\alpha := \{\zeta : [0, 2\pi] \to \mathbb{R} : |\zeta(x+\omega) - \zeta(x)| = O(\omega^\alpha)\}$$

for $0 < \alpha \leq 1$;

$$Lip(\alpha, p) := \{\zeta \in L^p[0, 2\pi] : \|\zeta(x+\omega) - \zeta(x)\|_p = O(\omega^\alpha)\}$$

for $p \geq 1, 0 < \alpha \leq 1$;

$$Lip(\xi(\omega), p) := \{\zeta \in L^p[0, 2\pi] : \|\zeta(x+\omega) - \zeta(x)\|_p = O(\xi(\omega))\}$$

for $p \geq 1, 0 < \alpha \leq 1$ & $\beta \geq 0$;

$$W(L^p, \xi(\omega)) := \left\{\zeta \in L^p[0, 2\pi] : \|(\zeta(x+\omega) - \zeta(x))\sin^\beta(\frac{\omega}{2})\|_p = O(\xi(\omega))\right\},$$

where $\xi(\omega)$ be a positive and increasing with $\omega > 0$ and L^p space of all 2π-periodic and integrable functions. Under above assumptions for $\alpha \in (0, 1], p \geq 1, \omega > 0$, we find that

$$W(L^p, \xi(\omega)) \xrightarrow{\beta=0} Lip(\xi(\omega), p) \xrightarrow{\xi(\omega)=\omega^\alpha} Lip(\alpha, p) \xrightarrow{p\to\infty} Lip\alpha.$$

Let $\sum_{\lambda=0}^{\infty} v_\lambda$ be an infinite series witht $s_k = \sum_{m=0}^{k} v_m$. The λ^{th} partial sums of the CFS is denoted by $s_\lambda(\tilde{\zeta}; x)$, and is given by ([1])

$$s_\lambda(\tilde{\zeta}; x) - \tilde{\zeta}(x) = \frac{1}{2\pi}\int_0^\pi \psi(x,\omega)\frac{\cos(\lambda + \frac{1}{2})\omega}{\sin\frac{\omega}{2}}\,d\omega.$$

We can consult for detailed works on FS and CFS [1]. Let $T = (l_{\lambda,k})$ be an infinite triangular matrix satisfying the conditions of regularity ([9]), that is,

$$\begin{cases} \sum_{k=0}^{\lambda} l_{\lambda,k} = 1 \quad \text{as} \quad \lambda \to \infty, \\ l_{\lambda,k} = 0 \quad \text{for} \quad k > \lambda, \\ \sum_{k=0}^{\lambda} |l_{\lambda,k}| \leq M, \text{ a finite constant.} \end{cases} \tag{1.1}$$

The sequence-to-sequence transformation:

$$t_\lambda^T(\tilde{\zeta}; x) := \sum_{k=0}^{\lambda} l_{\lambda,k}s_k = \sum_{k=0}^{\lambda} l_{\lambda,\lambda-k}s_{\lambda-k}$$

defines the sequence $t_\lambda^T(\zeta;x)$ of triangular matrix means of the sequence $\{s_\lambda\}$ generated by the sequence of coefficients $(l_{\lambda,k})$.

If $t_\lambda^T(\tilde{\zeta};x) \to s$ as $\lambda \to \infty$, then the infinite series $\sum_{\lambda=0}^{\infty} v_\lambda$ or the sequence $\{s_\lambda\}$ is summable to s by triangular matrix (T-method)([1]).

Following [3], Let us write $S_\lambda^0 = s_\lambda$, $S_\lambda^\delta = S_0^1 + S_1^{\delta-1} + ... + S_\lambda^{\delta-1}$ and E_λ^δ for the value of S_λ^δ when $v_0 = 1$ and $v_\lambda = 0$ for $\lambda > 0$, that is, when $s_\lambda = 1$ for all λ.

If $C_\lambda^\delta = \frac{S_\lambda^\delta}{E_\lambda^\delta} \to s$, when $\lambda \to \infty$, then we say that $\sum v_j$ is summable C^δ (Cesàro means of order $\delta > -1$) to sum s, where $S_j^\delta = \sum \binom{\lambda-\nu+\delta-1}{\delta-1} s_\nu$ and $E_\lambda^\delta = \binom{\lambda+\delta}{\delta}$.

Now, we define $C^\delta T$ means as

$$t_\lambda^{C^\delta T}(\tilde{\zeta};x) := \sum_{r=0}^{\lambda} \frac{\binom{\lambda-r+\delta-1}{\delta-1}}{\binom{\delta+\lambda}{\delta}} \sum_{k=0}^{r} l_{r,k} s_k(\tilde{\zeta};x). \tag{1.2}$$

If $t_\lambda^{C^\delta T}(\tilde{\zeta};x) \to s$ as $\lambda \to \infty$, then $\sum_{\lambda=0}^{\infty} v_\lambda$ is summable to s by $C^\delta T$ method.

Note 2. Since C^δ and T both are regular, $C^\delta T$ method is also regular.

Note 3. It can be observed that $C^\delta T$ operator can not be generalized furthermore.

Remark 1.2. The special cases of $C^\delta T$ means: $C^\delta T$ transform reduces to

(i) $C^\delta H$ transform if $l_{\lambda,k} = \frac{1}{(\lambda-k+1)\log(\lambda+1)}$;
(ii) $C^\delta N_p$ transform if $l_{\lambda,k} = \frac{p_{\lambda-k}}{P_\lambda}$ where $P_\lambda = \sum_{k=0}^{\lambda} p_k \neq 0$;
(iii) $C^\delta \bar{N}_p$ transform if $l_{\lambda,k} = \frac{p_k}{P_\lambda}$;
(iv) $C^\delta E^q$ transform when $a_{\lambda,k} = \frac{1}{(1+q)^\lambda}\binom{\lambda}{k} q^{\lambda-k}$;
(v) $C^\delta E^1$ when $l_{\lambda,k} = \frac{1}{2^\lambda}\binom{\lambda}{k}$;
(vi) $C^\delta N_{pq}$ transfoorm if $l_{\lambda,k} = \frac{p_{\lambda-k}q_k}{R_\lambda}$ where $R_\lambda = \sum_{k=0}^{\lambda} p_k q_{\lambda-k}$.

In above special cases (ii), (iii) and (vi), p_λ and q_λ are two non-negative monotonic non-increasing sequences of real constants.

Remark 1.3. C^1H, C^1N_p, C^1N_{pq}, C^1E^q and C^1E^1 transforms are also the special cases of $C^\delta T$ for $\delta = 1$.

Example 1.4. we consider

$$1 - 1574 \sum_{\lambda=1}^{\infty} (-1573)^{\lambda-1}. \tag{1.3}$$

The λ^{th} partial sum of the series (1.3) is given by

$$s_\lambda = (-1573)^\lambda,\ \forall \lambda \in \mathbb{N}_0.$$

If we take $l_{\lambda,k} = \frac{1}{(787)^\lambda}\binom{\lambda}{k}(786)^{\lambda-k}$, then

$$\begin{aligned} t_\lambda^T &= l_{\lambda,0}s_0 + l_{\lambda,1}s_1 + \ldots + \lambda_{\lambda,\lambda}s_\lambda \\ &= \frac{1}{(787)^\lambda}\left[\binom{\lambda}{0}(786)^\lambda - \binom{\lambda}{1}(786)^{\lambda-1}.1573 + \ldots + \binom{\lambda}{\lambda}(-1573)^\lambda\right] \\ &= \frac{1}{(787)^\lambda}(-787)^\lambda \\ &= \begin{cases} 1, & \lambda \text{ is even,} \\ -1, & \lambda \text{ is odd.} \end{cases} \end{aligned} \tag{1.4}$$

In this example, we know that the series is summable neither by Cesàro means nor Matrix means, but summable by Cesàro-Matrix. Thus, $C^\delta T$ means is more powerfull and effective than single C^δ and T means.

We write

$$\tilde{K}_\lambda^{C^\delta T} = \frac{1}{2\pi}\sum_{r=0}^{\lambda}\frac{\binom{r+\delta-1}{\delta-1}}{\binom{\delta+\lambda}{\delta}}\sum_{k=0}^{r} l_{r,r-k}\frac{\cos(r-k+\frac{1}{2})\omega}{\sin\frac{\omega}{2}},$$

$$\varrho = \text{ integral part of } \left(\frac{1}{\omega}\right),$$

$$\psi(x,\omega) = \zeta(x+\omega) - \zeta(x-\omega).$$

We use the following in our work ([1]).

$$\frac{1}{\sin(\frac{\omega}{2})} \leq \frac{\pi}{\omega},\ 0 < \omega \leq \pi, \tag{1.5}$$

$$\sin\omega \leq \omega,\ \omega \geq 0, \tag{1.6}$$

$$|\cos\lambda\omega| \leq 1,\ \forall \omega \in \mathbb{R}. \tag{1.7}$$

Note 4. Following conditions are used in the proof of the main results:

$$\begin{cases} l_{\lambda,\lambda-k} - l_{\lambda+1,\lambda+1-k} \geq 0 \ for\ 0 \leq k \leq \lambda, \\ L_{\lambda,k} = \sum_{r=k}^{\lambda} l_{\lambda,\lambda-r} \ and\ L_{\lambda,0} = 1, \forall \lambda \in \mathbb{N}_0. \end{cases} \tag{1.8}$$

Remark 1.5. Considering the matrix $T = (l_{\lambda,k})$ with

$$l_{\lambda,k} = \begin{cases} \frac{2018\times(2019)^k}{(2019)^{\lambda+1}-1}, & 0 \leq k \leq \lambda \\ 0, & k > \lambda \end{cases},$$

we can observe that (1.1) and (1.8) satisfied.

Note 5. Function $\bar{\zeta}$ denotes a conjugate to a 2π-period and Lebesgue integrable function and this notation is used throughout the paper.

2. Lemmas

We need the following lemmas for the proof of the main theorems:

Lemma 2.1. *If conditions* (1.1) *and* (1.8) *hold for* $\{l_{\lambda,k}\}$*, then*

$$|\tilde{K}_{\lambda}^{C^{\delta}T}(\omega)| = O\left(\frac{1}{\omega}\right) \ \forall\delta \geq 1,\ 0 < \omega \leq \frac{\pi}{\lambda+1}.$$

Proof. For $0 < \omega \leq \frac{\pi}{\lambda+1}$, using (1.5), (1.6) and (1.7), then for all $\delta \geq 1$,

$$\begin{aligned}
|\tilde{K}_{\lambda}^{C^{\delta}T}(\omega)| &= \left|\frac{1}{2\pi}\sum_{r=0}^{\lambda}\frac{\binom{r+\delta-1}{\delta-1}}{\binom{\delta+\lambda}{\delta}}\sum_{k=0}^{r} l_{r,r-k}\frac{\cos(r-k+\frac{1}{2})\omega}{\sin\frac{\omega}{2}}\right| \\
&\leq \frac{1}{2\pi}\sum_{r=0}^{\lambda}\frac{\binom{r+\delta-1}{\delta-1}}{\binom{\delta+\lambda}{\delta}}\sum_{k=0}^{r} l_{r,r-k}\frac{|\cos(r-k+\frac{1}{2})\omega|}{|\sin\frac{\omega}{2}|} \\
&\leq \frac{1}{2\omega}\sum_{r=0}^{\lambda}\frac{\binom{r+\delta-1}{\delta-1}}{\binom{\delta+\lambda}{\delta}}\sum_{k=0}^{r} l_{r,r-k} \\
&= \frac{1}{2\omega}\sum_{r=0}^{\omega}\frac{(r+\delta-1)!}{(\delta-1)!\ r!} \times \frac{\delta!\ \lambda!}{(\delta+\lambda)!}L_{r,0} \\
&= \frac{\lambda!\ \delta}{2\omega\ (\delta+\lambda)!}\sum_{r=0}^{\lambda}\frac{(r+\delta-1)!}{r!} \text{ since } L_{r,0} = 1 \\
&= \frac{\lambda!\ \delta}{2\omega\ \delta!\ (\delta+1)\cdots(\delta+\lambda)}\left[\frac{(\delta-1)!}{0!} + \frac{\delta!}{1!}\cdots + \frac{(\lambda+\delta-1)!}{\lambda!}\right] \\
&\leq \frac{\lambda!\ \delta}{2\omega\ \delta!\ (\delta+1)\cdots(\delta+\lambda)} \times (\lambda+1)\ \frac{\delta!\ (\delta+1)\cdots(\delta+\lambda-1)}{\lambda!} \\
&= \frac{\delta}{2\omega} \times \frac{\lambda+1}{\delta+\lambda} \\
&\leq \frac{\delta}{2\omega} \\
&= O\left(\frac{1}{\omega}\right).
\end{aligned}$$

□

Lemma 2.2. *If conditions* (1.1) *and* (1.8) *hold for* $\{l_{\lambda,k}\}$, *then*

$$\left|\tilde{K}_{\lambda}^{C^{\delta}T}(\omega)\right| = O\left(\frac{1}{(\lambda+1)\omega^2}\right) \ \forall\delta \geq 1, \ \frac{\pi}{\lambda+1} \leq \omega \leq \pi.$$

Proof. For $\frac{\pi}{\lambda+1} \leq \omega \leq \pi$, using (1.5), $l_{r,r-k} \geq l_{r+1,r+1-k} \geq l_{r+1,r-k}$ and $L_{\varrho+1,0} = 1$, then

$$\begin{aligned}\left|\tilde{K}_{\lambda}^{C^{\delta}T}(\omega)\right| &= \left|(2\pi)^{-1}\sum_{r=0}^{\lambda}\frac{\binom{r+\delta-1}{\delta-1}}{\binom{\delta+\lambda}{\delta}}\sum_{k=0}^{r} l_{r,r-k}\frac{\cos(r-k+\frac{1}{2})\omega}{\sin\frac{\omega}{2}}\right| \\ &= O\left(\frac{1}{\omega}\right)\left|\mathrm{Re}\sum_{r=0}^{\lambda}\frac{\binom{r+\delta-1}{\delta-1}}{\binom{\delta+\lambda}{\delta}}\sum_{k=0}^{r} l_{r,r-k}e^{i(r-k+\frac{1}{2})\omega}\right|. \qquad (2.1)\end{aligned}$$

Now, we consider

$$\begin{aligned}\left|\sum_{r=0}^{\lambda}\frac{\binom{r+\delta-1}{\delta-1}}{\binom{\delta+\lambda}{\delta}}\sum_{k=0}^{r} l_{r,r-k}e^{i(r-k+\frac{1}{2})\omega}\right| &\leq \left|\sum_{r=0}^{\varrho}\frac{\binom{r+\delta-1}{\delta-1}}{\binom{\delta+\lambda}{\delta}}\sum_{k=0}^{r} l_{r,r-k}e^{\iota(r-k)\omega}\right| \\ &\quad + \left|\sum_{r=\varrho+1}^{\lambda}\frac{\binom{r+\delta-1}{\delta-1}}{\binom{\delta+\lambda}{\delta}}\sum_{k=0}^{\varrho} l_{r,r-k}e^{i(r-k)\omega}\right| \\ &\quad + \left|\sum_{r=\varrho+1}^{\lambda}\frac{\binom{r+\delta-1}{\delta-1}}{\binom{\delta+\lambda}{\delta}}\sum_{k=\varrho+1}^{r} l_{r,r-k}e^{i(r-k)\omega}\right| \\ &= \Lambda_1 + \Lambda_2 + \Lambda_3. \qquad (2.2)\end{aligned}$$

Now, since $L_{r,0} = 1$, for all $\delta \geq 1$,

$$\begin{aligned}\Lambda_1 &\leq \sum_{r=0}^{\varrho}\frac{\binom{r+\delta-1}{\delta-1}}{\binom{\delta+\lambda}{\delta}}\sum_{k=0}^{r} l_{r,r-k}\left|e^{i(r-k)\omega}\right| \\ &\leq \sum_{r=0}^{\varrho}\frac{\binom{r+\delta-1}{\delta-1}}{\binom{\delta+\lambda}{\delta}}L_{r,0} \\ &= \sum_{r=0}^{\varrho}\frac{(r+\delta-1)!}{(\delta-1)!\ r!}\times\frac{\delta!\ \lambda!}{(\delta+\lambda)!} \\ &= \frac{\lambda!}{(\delta+1)...(\delta+\lambda)\ \delta!}\sum_{r=0}^{\varrho}\frac{(r+\delta-1)!\ \delta}{r!}\end{aligned}$$

$$
\begin{aligned}
&= \frac{\lambda!}{(\delta+1)...(\delta+\varrho)...(\delta+\lambda)}\left[1+\delta+\frac{\delta(\delta+1)}{2!}+...+\frac{(\varrho+\delta-1)!}{\varrho!(\delta-1)!}\right]\\
&\leq \frac{\lambda!}{(\delta+1)...(\delta+\varrho)...(\delta+\lambda)}\left[(\varrho+1)\frac{\delta(\delta+1)...(\delta+\varrho-1)}{\varrho!}\right]\\
&= \frac{\varrho!(\varrho+1)...\lambda}{(\delta+\varrho)...(\delta+\lambda-1)}\times\frac{\delta}{\delta+\lambda}\times\frac{\varrho+1}{\varrho!}\\
&\leq \frac{\delta}{\delta+\lambda}\times(\varrho+1)\\
&\leq \frac{\delta}{\delta+\lambda}\times\left(\frac{1}{\omega}+1\right)\\
&= O\left(\frac{1}{\omega(\lambda+1)}(1+\omega)\right).
\end{aligned}
$$

Changing the order of summation and applying Abel's transformation in Λ_2, we have

$$
\begin{aligned}
\Lambda_2 &= \left|\sum_{k=0}^{\varrho}\sum_{r=\varrho+1}^{\lambda}\frac{\binom{r+\delta-1}{\delta-1}}{\binom{\delta+\lambda}{\delta}}l_{r,r-k}e^{i(r-k)\omega}\right|\\
&= \frac{1}{\binom{\delta+\lambda}{\delta}}\left|\sum_{k=0}^{\varrho}\left[\left\{\sum_{r=\varrho+1}^{\lambda-1}\left(\binom{r+\delta-1}{\delta-1}l_{r,r-k}-\binom{r+\delta}{\delta-1}l_{r+1,r+1-k}\right)\right.\right.\right.\\
&\quad\left.\left.\left.\times\sum_{\nu=0}^{r}e^{i(\nu-k)\omega}\right\}+\binom{\lambda+\delta-1}{\delta-1}l_{\lambda,\lambda-k}\sum_{\nu=0}^{\lambda}e^{i(\nu-k)\omega}-\binom{\delta+\varrho}{\delta-1}l_{\varrho+1,\varrho+1-k}\right]\right|\\
&= O(\omega^{-1})\frac{1}{\binom{\eta+\lambda}{\eta}}\sum_{k=0}^{\varrho}\left[\left|\sum_{r=\varrho+1}^{\lambda-1}\left(\binom{r+\delta-1}{\delta-1}l_{r,r-k}-\binom{r+\delta}{\delta-1}l_{r+1,r+1-k}\right)\right|\right.\\
&\quad\left.+\left|\binom{\lambda+\delta-1}{\delta-1}l_{\lambda,\lambda-k}\right|+\left|\binom{\delta+\varrho}{\delta-1}l_{\varrho+1,\varrho+1-k}\right|\right]\\
&= O(\omega^{-1})\frac{1}{\binom{\delta+\lambda}{\delta}}\sum_{k=0}^{\varrho}\left[\binom{\delta+\varrho}{\delta-1}l_{\varrho+1,\varrho+1-k}+\binom{\delta+\lambda-1}{\delta-1}l_{\lambda,\lambda-k}\right.\\
&\quad\left.+\binom{\delta+\lambda-1}{\delta-1}l_{\lambda,\lambda-k}+\binom{\delta+\varrho}{\delta-1}l_{\varrho+1,\varrho+1-k}\right]\\
&= O(\omega^{-1})\frac{1}{\binom{\delta+\lambda}{\delta}}\sum_{k=0}^{\varrho}\left[\binom{\delta+\varrho}{\delta-1}l_{\varrho+1,\varrho+1-k}+\binom{\delta+\lambda-1}{\delta-1}l_{\lambda,\lambda-k}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= O(\omega^{-1})\frac{1}{\binom{\delta+\lambda}{\delta}}\sum_{k=0}^{\varrho}\left[\binom{\delta+\varrho}{\delta-1}l_{\varrho+1,\varrho+1-k}+\binom{\delta+\lambda-1}{\delta-1}l_{\lambda,\lambda-k}\right]\\
&= O(\omega^{-1})\frac{\lambda!}{(\delta+1)...(\delta+\lambda)}\sum_{k=0}^{\varrho}\left[\frac{\delta(\delta+1)...(\delta+\varrho)...(\delta+\lambda)}{(\delta+\varrho+1)...(\delta+\lambda)(\varrho+1)!}l_{\varrho+1,\varrho+1-k}\right.\\
&\quad\left.+\frac{\delta(\delta+1)...(\delta+\lambda-1)(\delta+\lambda)}{(\delta+\lambda)\ \lambda!}l_{\lambda,\lambda-k}\right]\\
&= O(\omega^{-1})\frac{\lambda!}{(\delta+1)...(\delta+\lambda)}\times\frac{\delta(\delta+1)...(\delta+\lambda)}{(\lambda+1)!}\sum_{k=0}^{\varrho}(l_{\varrho,\varrho-k}+l_{\lambda,\lambda-k})\\
&= O\left(\frac{1}{\omega(\lambda+1)}(L_{\varrho,0}+L_{\lambda,0})\right)\\
&= O\left(\frac{1}{\omega(\lambda+1)}\right).
\end{aligned}
$$

Applying Abel's transformation in Λ_3, we have

$$
\begin{aligned}
\Lambda_3 &= \Big|\sum_{r=\varrho+1}^{\lambda}\frac{\binom{r+\delta-1}{\delta-1}}{\binom{\delta+\lambda}{\delta}}\Big[\sum_{k=\varrho+1}^{r-1}(l_{r,r-k}-l_{r,r-k+1})\sum_{\nu=0}^{k}e^{i(r-\nu)\omega}\\
&\quad+l_{r,0}\sum_{\nu=0}^{r}e^{\iota(r-\nu)\omega}-l_{r,r-\tau-1}\sum_{\nu=0}^{\varrho}e^{i(r-\nu)\omega}\Big]\Big|\\
&= O(\omega^{-1})\sum_{r=\varrho+1}^{\lambda}\frac{\binom{r+\delta-1}{\delta-1}}{\binom{\delta+\lambda}{\delta}}\Big[\Big|\sum_{k=\varrho+1}^{r-1}(l_{r,r-k}-l_{r,r-k+1})\Big|+l_{r,0}+l_{r,r-\varrho-1}\Big]\\
&= O(\omega^{-1})\sum_{r=\varrho+1}^{\lambda}\frac{\binom{r+\delta-1}{\delta-1}}{\binom{\delta+\lambda}{\delta}}\left[\ |-l_{r,r-\varrho}+l_{r,1}|+l_{r,0}+l_{r,r-\varrho-1}\right]\\
&= O(\omega^{-1})\frac{1}{\binom{\delta+\lambda}{\delta}}\sum_{r=\varrho+1}^{\lambda}\binom{r+\delta-1}{\delta-1}l_{r,r-\varrho}\\
&= O(\omega^{-1})\frac{\lambda!}{(\delta+1)...(\delta+\lambda)}\times\frac{\delta(\delta+1)...(\delta+\lambda)}{\lambda!(\delta+\lambda)}\\
&\quad\times\left[\frac{\delta...(\delta+\varrho)}{(\varrho+1)!}l_{\varrho+1,1}+...+\frac{\delta...(\delta+\lambda-1)}{\lambda!}l_{\lambda,\lambda-\varrho}\right]
\end{aligned}
$$

$$= O(\omega^{-1}) \frac{\lambda!}{(\delta+1)...(\delta+\lambda)} \times \frac{\delta(\delta+1)...(\delta+\lambda)}{\lambda!(\delta+\lambda)} [l_{\varrho+1,1} + l_{\varrho+2,2} + ... l_{\lambda,\lambda-\varrho}]$$
$$= O(\omega^{-1}) \frac{\delta}{\delta+\lambda} [l_{\varrho+1,1} + l_{\varrho+1,2} + ... + l_{\varrho+1,\lambda-\varrho}]$$
$$= O\left(\frac{1}{\omega(\lambda+1)}\right) L_{\varrho+1,1}$$
$$= O\left(\frac{1}{\omega(\lambda+1)}\right).$$

Note that If $1+\frac{3}{\omega} \leq \frac{k}{\omega}$ for ω fixed, then $\min k = 3+\pi$. So, combining Λ_1, Λ_2 and Λ_3, we have

$$\begin{aligned} \Lambda_1 + \Lambda_2 + \Lambda_3 &= O\left[\frac{1}{\omega(\lambda+1)} \times (1+\omega)\right] + O\left[\frac{1}{\omega(\lambda+1)}\right] + O\left[\frac{1}{\omega(\lambda+1)}\right] \\ &= O\left[\frac{1}{(\lambda+1)}\left(1+\frac{3}{\omega}\right)\right] \\ &= O\left(\frac{1}{\lambda+1} \times \frac{3+\pi}{\omega}\right). \end{aligned} \tag{2.3}$$

Now, from (2.1), (2.2) and (2.3) we get

$$\left|\tilde{K}_\lambda^{C^\delta T}(\omega)\right| = O\left(\frac{1}{(\lambda+1)\omega^2}\right).$$

□

3. Main results

Theorem 3.1. *The error approximation of $\tilde{\zeta}$ in $W(L^p, \xi(\omega))$, $(p > 1)$, by $C^\delta T$ means of its CFS is given by*

$$\|t_\lambda^{C^\delta T}(\tilde{\zeta}; x) - \tilde{\zeta}(x)\|_p = O\left[(\lambda+1)^\beta \xi\left(\frac{1}{\lambda+1}\right)\right],$$

where $0 \leq \beta < \frac{1}{p}$ and condition (1.8) holds and positive increasing function $\xi(\omega)$ satisfies the following conditions:

$$\frac{\xi(\omega)}{\omega^{\beta+1-\sigma}} \text{ is nondecreasing;} \tag{3.1}$$

$$\left\{\int_0^{\frac{\pi}{\lambda+1}} \left(\frac{\lambda^{-\sigma}|\psi(x,\omega)| \sin^\beta(\frac{\omega}{2})}{\xi(\omega)}\right)^p d\omega\right\}^{\frac{1}{p}} = O((\lambda+1)^{\sigma-\frac{1}{p}}), \text{ for } \beta < \sigma < \frac{1}{p}; \tag{3.2}$$

$$\frac{\xi(\omega)}{\omega} \text{ is nonincreasing;} \tag{3.3}$$

$$\text{and } \left\{\int_{\frac{\pi}{\lambda+1}}^{\pi} \left(\frac{\omega^{-\eta}|\psi(x,\omega)|\sin^{\beta}(\frac{\omega}{2})}{\xi(\omega)}\right)^{p} d\omega\right\}^{\frac{1}{p}} = O((\lambda+1)^{\eta-\frac{1}{p}}), \tag{3.4}$$

where $\frac{1}{p} < \eta < \beta + \frac{1}{p}$ *for* η *being an arbitrary number and* $p+q = pq$. *Conditions* (3.2) *and* (3.4) *hold uniformly in* x.

Proof. The integral representation of $s_\lambda(\tilde{\zeta};x)$ is given by ([1])

$$s_\lambda(\tilde{\zeta};x) - \tilde{\zeta}(x) = \frac{1}{2\pi}\int_0^{\pi} \psi(x,\omega)\frac{\cos(\lambda+\frac{1}{2})\omega}{\sin(\frac{\omega}{2})}\, d\omega.$$

Denoting $C^\delta T$ means of $\{s_\lambda(\tilde{\zeta}:x)\}$ by $t_\lambda^{C^\delta T}(\tilde{\zeta}:x)$, we get

$$\begin{aligned}
t_\lambda^{C^\delta T}(\tilde{\zeta};x) - \tilde{\zeta}(x) &= \sum_{r=0}^{\lambda} \frac{\binom{\lambda-r+\delta-1}{\delta-1}}{\binom{\delta+\lambda}{\delta}} \sum_{k=0}^{r} l_{r,k}[s_k(\tilde{\zeta};x) - \tilde{\zeta}(x)] \\
&= \frac{1}{2\pi}\int_0^{\pi} \psi(x,\omega) \sum_{r=0}^{\lambda} \frac{\binom{r+\delta-1}{\delta-1}}{\binom{\delta+\lambda}{\delta}} \sum_{k=0}^{r} l_{r,r-k} \frac{\cos(r-k+\frac{1}{2})\omega}{\sin(\frac{\omega}{2})}\, d\omega \\
&= \int_0^{\pi} \psi(x,\omega)\tilde{K}_\lambda^{C^\delta T}(\omega)\, d\omega \\
&= \int_0^{\frac{\pi}{\lambda+1}} \psi(x,\omega)\tilde{K}_\lambda^{C_\delta T}(\xi)\, d\omega + \int_{\frac{\pi}{\lambda+1}}^{\pi} \psi(x,\omega)\tilde{K}_\lambda^{C_\delta T}(\omega)\, d\omega \\
&= I_1 + I_2.
\end{aligned} \tag{3.5}$$

Applying (1.5), Lemma 2.1, Hölder's inequality and second mean value theorem for integral, we have

$$\begin{aligned}
I_1 &= O(1)\left\{\int_0^{\frac{\pi}{\lambda+1}} \left(\frac{\omega^{-\sigma}|\psi(x,\omega)|\sin^{\beta}(\frac{\omega}{2})}{\xi(\omega)}\right)^{p} d\omega\right\}^{\frac{1}{p}} \\
&\quad \times \left\{\int_0^{\frac{\pi}{\lambda+1}} \left(\frac{\xi(\omega)}{\omega^{-\sigma+1}\sin^{\beta}(\frac{\omega}{2})}\right)^{q} d\omega\right\}^{\frac{1}{q}} \\
&= O\left[(\lambda+1)^{\sigma-\frac{1}{p}} \times \left\{\int_0^{\frac{\pi}{\lambda+1}} \left(\frac{\xi(\omega)}{\omega^{\beta+1-\sigma}}\right)^{q} d\omega\right\}^{\frac{1}{q}}\right]
\end{aligned}$$

$$= O\left[(\lambda+1)^{\sigma-\frac{1}{p}}(\lambda+1)^{\beta+\frac{1}{p}-\sigma}\xi\left(\frac{\pi}{\lambda+1}\right)\right]$$
$$= O\left[(\lambda+1)^{\beta}\xi\left(\frac{1}{\lambda+1}\right)\right]. \tag{3.6}$$

We note that from condition (3.1), we have $\frac{\xi(\frac{\pi}{\lambda+1})}{\frac{\pi}{\lambda+1}} \leq \frac{\xi(\frac{\pi}{\lambda+1})}{\frac{1}{\lambda+1}}$, it implies

$$\xi\left(\frac{\pi}{\lambda+1}\right) \leq \pi\xi\left(\frac{1}{\lambda+1}\right). \tag{3.7}$$

In view of condition (3.2), $p^{-1}+q^{-1}=1$ and (3.7), again using Lemma 2.2, Hölder's inequality and (1.5), we have

$$I_2 = O\left(\frac{1}{\lambda+1}\right)\int_{\frac{\pi}{\lambda+1}}^{\pi}\frac{|\psi(x,\omega)|}{\omega^2}\,d\omega$$
$$= O\left\{\frac{1}{\lambda+1}\int_{\frac{\pi}{\lambda+1}}^{\pi}\left(\frac{\omega-\eta|\psi(x,\omega)|\sin^{\beta}(\frac{\omega}{2})}{\xi(\omega)}\right)^{p}d\omega\right\}^{\frac{1}{p}}$$
$$\times\left\{\int_{\frac{\pi}{\lambda+1}}^{\pi}\left(\frac{\omega^{-1}\xi(\omega)}{\omega^{-\eta+1+\beta}}\right)^{q}d\omega\right\}^{\frac{1}{q}}$$
$$= O\left[(\lambda+1)^{-1+\eta-\frac{1}{p}}\xi\left(\frac{\pi}{\lambda+1}\right)\left(\frac{\lambda+1}{\pi}\right)\left(\int_{\frac{\pi}{\lambda+1}}^{\pi}\omega^{-(\beta+1-\eta)q}\,d\omega\right)^{\frac{1}{q}}\right]$$
$$= O\left[(\lambda+1)^{\eta-\frac{1}{p}}\xi\left(\frac{\pi}{\lambda+1}\right)(\lambda+1)^{\beta+1-\eta-\frac{1}{q}}\right]$$
$$= O\left[(\lambda+1)^{\beta}\xi\left(\frac{1}{\lambda+1}\right)\right]. \tag{3.8}$$

In view of (3.3), (3.4), the second mean value theorem for integrals, $0 < \eta < \beta + \frac{1}{p}$, $p+q=pq$ and (3.7), from (3.5), (3.6), (3.8), we get

$$\left|t_{\lambda}^{C^{\delta}T}(\tilde{\zeta},x)-\tilde{\zeta}(x)\right| = O\left[(\lambda+1)^{\beta}\xi\left(\frac{1}{\lambda+1}\right)\right].$$

Now, using L_p-norm of a function, we get

$$\left\|t_{\lambda}^{C^{\delta}T}(\tilde{\zeta},x)-\tilde{\zeta}(x)\right\|_{p} = O\left[(\lambda+1)^{\beta}\xi\left(\frac{1}{\lambda+1}\right)\right].$$

This completes the proof. □

Now, we establish the following theorem for the case $p=1$.

Theorem 3.2. *The error approximation of $\tilde{\zeta}$ in $W(L^1,\xi(\omega))$, by $C^\delta T$ means of its CFS is given by*

$$\|t_\lambda^{C^\delta T}(\tilde{\zeta};x)-\tilde{\zeta}(x)\|_1 = O\left[(\lambda+1)^\beta \xi\left(\frac{1}{\lambda+1}\right)\right],$$

where $0 \le \beta < 1$, provided (1.8) holds and a positive increasing function $\xi(\omega)$ satisfies conditions (3.1) to (3.4) of Theorem 3.1 for $p=1$, $\beta < \sigma < 1$ and $1 < \eta < \beta + 1$.

Proof. Following the proof of Theorem 3.1, for $p=1$, i.e., $q=\infty$, we have

$$\begin{aligned}
I_1 &= O\left\{\int_0^{\frac{\pi}{\lambda+1}} \left(\frac{\omega^{-\sigma}|\psi(x,\omega)|\sin^\beta\left(\frac{\omega}{2}\right)}{\xi(\omega)}\right) d\omega \times \operatorname{ess}\sup_{0<\omega\le\frac{\pi}{\lambda+1}} \left|\frac{\xi(\omega)}{\omega^{-\sigma+1}\sin^\beta\left(\frac{\omega}{2}\right)}\right|\right\}\\
&= O\left((\lambda+1)^{\sigma-1}\right) \operatorname{ess}\sup_{0<\omega\le\frac{\pi}{\lambda+1}} \left|\frac{\xi(\omega)}{\omega^{\beta-\sigma+1}}\right|\\
&= O\left((\lambda+1)^{\sigma-1}\right)\left\{\frac{\xi\left(\frac{\pi}{\lambda+1}\right)}{\left(\frac{\pi}{\lambda+1}\right)^{\beta-\sigma+1}}\right\}\\
&= O\left((\lambda+1)^\beta \xi\left(\frac{1}{\lambda+1}\right)\right).
\end{aligned}\tag{3.9}$$

In view of conditions (3.1) and (3.2) for $p=1$,

$$\begin{aligned}
I_2 &= O\left(\frac{1}{\lambda+1}\right)\int_{\frac{\pi}{\lambda+1}}^{\pi} \frac{|\psi(x,\omega)|}{\omega^2}\, d\omega\\
&= O\left\{\frac{1}{\lambda+1}\int_{\frac{\pi}{\lambda+1}}^{\pi} \frac{\omega^{-\eta}|\psi(x,\omega)|\sin^\beta(\frac{\omega}{2})}{\xi(\omega)}\, d\omega\right\} \times \operatorname{ess}\sup_{\frac{\pi}{\lambda+1}\le\omega\le\pi} \left|\frac{\xi(\omega)}{\omega^{-\eta+\beta+2}}\right|\\
&= O\left[(\lambda+1)^{\eta-2}\xi\left(\frac{\pi}{\lambda+1}\right)\left(\frac{(\lambda+1)^{2+\beta-\eta}}{\pi^{2+\beta-\eta}}\right)\right]\\
&= O\left[(\lambda+1)^\beta \xi\left(\frac{\pi}{\lambda+1}\right)\right].
\end{aligned}\tag{3.10}$$

In view of (3.1) and (3.2). Collecting (3.9) and (3.10), we get

$$|t_\lambda^{C^\delta T}(\tilde{\zeta};x)-\tilde{\zeta}(x)| = O\left[(\lambda+1)^\beta \xi\left(\frac{1}{\lambda+1}\right)\right]. \tag{3.11}$$

Finally from (3.11),

$$\|t_\lambda^{C^\delta T}(\tilde{\zeta};x)-\tilde{\zeta}(x)\|_1 = O\left[(\lambda+1)^\beta \xi\left(\frac{1}{\lambda+1}\right)\right].$$

This completes the proof. $\square$

4. Corollaries

Now, we give some corollaries.

Corollary 4.1. *The error estimate of $\tilde{\zeta}$ in Lip$(\xi(\omega), p)$ class by $C^\delta T$ means of its CFS is given by*

$$\left\| t_\lambda^{C^\delta T}(\tilde{\zeta}, x) - \tilde{\zeta}(x) \right\|_p = O\left[(\lambda+1)^{\frac{1}{p}} \xi\left(\frac{1}{\lambda+1} \right) \right],$$

where $C^\delta T$ is defined as (1.2).

Proof. Taking $\beta = 0$ in Theorem 3.1, we can get the proof. □

Corollary 4.2. *The error estimate of $\tilde{\zeta}$ in Lip(α, p) class by $C^\delta T$ product means of its CFS is given by*

$$\left\| t_\lambda^{C^\delta T}(\tilde{\zeta}, x) - \tilde{\zeta}(x) \right\|_p = O\left[(\lambda+1)^{-\alpha+\frac{1}{p}} \right],$$

where $C^\delta T$ is defined as (1.2).

Proof. Taking $\beta = 0$ and $\xi(\omega) = \omega^\alpha$ in Theorem 3.1, we can get the proof. □

Corollary 4.3. *The error estimate of $\tilde{\zeta}$ in Lipα class by $C^\delta T$ product means of its CFS is given by*

$$\left\| t_\lambda^{C^\delta T}(\tilde{\zeta}, x) - \tilde{\zeta}(x) \right\|_p = O\left[(\lambda+1)^{-\alpha} \right]$$

where, $C^\delta T$ is defined as (1.2).

Proof. Taking $\beta = 0$, $\xi(\omega) = \omega^\alpha$ and $p \to \infty$ in Theorem 3.1, we can get the proof. □

Corollary 4.4. *The error estimate of $\tilde{\zeta} \in W(L^p, \xi(\omega))$ class by $C^\delta H$ means*

$$t_\lambda^{C^\delta H} = \sum_{r=0}^{\lambda} \frac{\binom{\lambda-r+\delta-1}{\delta-1}}{\binom{\delta+\lambda}{\delta}} (\log(r+1))^{-1} \sum_{k=0}^{r} \frac{1}{(r-k+1)} s_k,$$

of the CFS is given by

$$\| t_\lambda^{C^\delta H}(\tilde{\zeta}; x) - \tilde{\zeta}(x) \|_p = O\left[(\lambda+1)^{\beta+\frac{1}{p}} \xi\left(\frac{1}{\lambda+1} \right) \right],$$

provided $C^\delta T$ defined in (1.2) *and $\xi(\omega)$ satisfies the conditions* (3.1) *to* (3.4).

Corollary 4.5. *The error estimate of* $\tilde{\zeta} \in W(L^p, \xi(\omega))$ *class by* $C^\delta N_p$ *means*

$$t_\lambda^{C^\delta N_p} = \sum_{r=0}^{\lambda} \frac{\binom{\lambda-r+\delta-1}{\delta-1}}{\binom{\delta+\lambda}{\delta}} \frac{1}{P_r} \sum_{k=0}^{r} p_{r-k} s_k,$$

of the CFS is given by

$$\|t_\lambda^{C^\delta N_p}(\tilde{\zeta}; x) - \tilde{\zeta}(x)\|_p = O\left[(\lambda+1)^{\beta+\frac{1}{p}} \xi\left(\frac{1}{\lambda+1}\right)\right]$$

provided $C^\delta T$ *defined in* (1.2) *and* $\xi(\omega)$ *satisfies the conditions* (3.1) *to* (3.4)

Corollary 4.6. *The error estimate of* $\tilde{\zeta} \in W(L^p, \xi(\omega))$ *class by* $C^\delta N_{pq}$ *means*

$$t_\lambda^{C^\delta N_{pq}} = \sum_{r=0}^{\lambda} \frac{\binom{\lambda-r+\delta-1}{\delta-1}}{\binom{\delta+\lambda}{\delta}} \frac{1}{R_r} \sum_{k=0}^{r} p_{r-k} q_k s_k,$$

of the CFS is given by

$$\|\omega_\lambda^{C^\delta N_{pq}}(\tilde{\zeta}; x) - \tilde{\zeta}(x)\|_p = O\left[(\lambda+1)^{\beta+\frac{1}{p}} \xi\left(\frac{1}{\lambda+1}\right)\right]$$

provided $C^\delta T$ *defined in* (1.2) *and* $\xi(\omega)$ *satisfies the conditions* (3.1) *to* (3.4).

Corollary 4.7. *The error approximation of* $\tilde{\zeta} \in W(L^p, \xi(\omega))$ *class by* $C^\delta \bar{N}_p$ *means*

$$t_\lambda^{C^\delta \bar{N}_p} = \sum_{r=0}^{\lambda} \frac{\binom{\lambda-r+\delta-1}{\delta-1}}{\binom{\delta+\lambda}{\delta}} \frac{1}{P_r} \sum_{k=0}^{r} p_k s_k,$$

of the CFS is given by

$$\|t_\lambda^{C^\delta \bar{N}_p}(\tilde{\zeta}; x) - \tilde{\zeta}(x)\|_p = O\left[(\lambda+1)^{\beta+\frac{1}{p}} \xi\left(\frac{1}{\lambda+1}\right)\right]$$

provided $C^\delta T$ *defined in* (1.2) *and* $\xi(\omega)$ *satisfies the conditions* (3.1) *to* (3.4).

Corollary 4.8. *The error estimate of* $\tilde{\zeta} \in W(L^p, \xi(\omega))$ *class by* $C^\delta E^q$ *means*

$$t_\lambda^{C^\delta E^q} = \sum_{r=0}^{\delta} \frac{\binom{\delta-r+\delta-1}{\delta-1}}{\binom{\delta+\delta}{\delta}} \frac{1}{(1+q)^r} \sum_{k=0}^{r} \binom{r}{k} q^{r-k} s_k,$$

of the CFS is given by

$$\|t_\lambda^{C^\delta E^q}(\tilde{\zeta}; x) - \tilde{\zeta}(x)\|_p = O\left[(\lambda+1)^{\beta+\frac{1}{p}} \xi\left(\frac{1}{\lambda+1}\right)\right]$$

provided $C^\delta T$ *defined in* (1.2) *and* $\xi(\omega)$ *satisfies the conditions* (3.1) *to* (3.4).

Corollary 4.9. *The error estimate of* $\zeta \in W(L^p, \xi(\omega))$ *class by* $C^\delta E^1$ *means*

$$t_\lambda^{C^\delta E^1} = \sum_{r=0}^{\lambda} \frac{\binom{\lambda-r+\delta-1}{\delta-1}}{\binom{\delta+\lambda}{\delta}} \frac{1}{2^r} \sum_{k=0}^{r} \binom{r}{k} s_k,$$

of the FS is given by

$$\|t_\lambda^{C^\delta E^1}(\tilde{\zeta}; x) - \tilde{h}(x)\|_p = O\left[(\lambda+1)^{\beta+\frac{1}{p}} \xi\left(\frac{1}{\lambda+1}\right)\right]$$

provided $C^\delta T$ *defined in* (1.2) *and* $\xi(\omega)$ *satisfies the conditions* (3.1) *to* (3.4).

Remark 4.10. The Corollaries 4.1 to 4.9 can also be obtained for the special cases $C^1 H, C^1 N_p$, $C^1 N_{pq}$, $C^1 \tilde{N}_p, C^1 E^q$ and $C^1 E^1$ in view of Remark 1.3.

5. Particular cases

Some special cases of our main results are:

(i) Let us consider Remark 1.3(iv) for $\delta = 1$ and if $\beta = 0,\ \xi(\omega) = \omega^\alpha,\ 0 < \alpha \leq 1$ in our theorem, then the Theorem 2 of [11] become a special case of our result.
(ii) Let us take $\delta = 1$ and if, $\beta = 0,\ \xi(\omega) = \omega^\alpha,\ 0 < \alpha \leq 1$ & $p \to \infty$ in our theorem, then the Theorem 3.3 of [12] become a special case of our result.
(iii) If we consider Remark 1.3(iv) for $\delta = 1$ in our result then the main Theorem 2.2. of [4] become a special case of our result.
(iv) If we take $\delta = 1$ in our result then the Theorem 2 of [13] become a special case of our result.
(v) If we consider remark 1.3(iv) for $\delta = 1$, then the Theorem 3.1 of [14] become a special case of our result.
(vi) If we consider remark 1.3(ii) for $\delta = 1$, then the main Theorem 1 of [15] become a special case of our result.

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