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EXTREME POINTS OF A NEW CLASS OF HARMONIC MULTIVALENT FUNCTIONS DEFINED BY GENERALIZED DERIVATIVE OPERATOR INVOLVING METTAG-LEFFLER FUNCTION

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Abstract. In this paper, we have introduced and studied a new class of multivalent harmonic function associated with generalized derivative operator involving Mettag-Leffler function. Several results like, coefficient inequality, Hadamard product (convolution), extreme points, convex linear combination for functions belong to this class are obtained.

1. Introduction

Denote by $\mathfrak{A}(p)$ the function class represented by the following form

$$f(z) = z^p + \sum_{j=p+1}^{\infty} a_j z^j, \tag{1.1}$$

which are analytic and multivalent in open unit disc $\mathfrak{U} = \{z \in \mathbb{C} : |z| < 1\}$. Using the generalized hypergeometric function Dziok and Srivastava [8] defined the multivalent function $\mathcal{X}_p(a_1, \ldots, a_r, b_1, \ldots, b_s; z)$ as follows

$$\mathcal{X}_p(a_1, \ldots, a_r, b_1, \ldots, b_s; z) = z^p + \sum_{j=p+1}^{\infty} \frac{\prod_{i=1}^{r}(a_i)_{j-p}}{\prod_{n=1}^{s}(b_n)_{j-p}} \frac{z^j}{(j-p)!},\ p \in \mathbb{N} \tag{1.2}$$

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where $a_i \in \mathbb{C},\ b_s \in \mathbb{C}\setminus\{0,-1,-2,\dots\},(i=1,\dots,r,\ n=1,\dots,s)$ *and* $r \leq s+1$, $r,s\in\mathbb{N}_0=\mathbb{N}\bigcup\{0\}$, and the Pochhammer $(\upsilon)_j$ symbol is defined by

$$(\upsilon)_j=\frac{\Gamma(\upsilon+j)}{\Gamma(\upsilon)}=\begin{cases}\upsilon(\upsilon+1)\dots(\upsilon+j-1), & j=1,2,3,\dots\\ 1, & j=0,\end{cases}$$

where $\upsilon\in\mathbb{C}$ and $Re(\upsilon)>0$ and $\Gamma(\upsilon)$ denotes the Gamma function. The Mettag-Leffler function $E_\upsilon(z)$ (see [15],[16]) which described below. Also, Wiman [21] introduced generalization $E_{\upsilon,\varrho}(z)$ of the same function

$$E_\upsilon(z)=\sum_{j=0}^{\infty}\frac{z^j}{\Gamma(\upsilon j+1)}$$

and

$$E_{\upsilon,\varrho}(z)=\sum_{j=0}^{\infty}\frac{z^j}{\Gamma(\upsilon j+\varrho)}, \tag{1.3}$$

where $\upsilon,\varrho\in\mathbb{C}$, $Re(\upsilon)>0$ and $Re(\varrho)>0$.

In the recent years, interest in the Mettag-Leffler function has increased for use in application problems like, electric network, fluid flow, probability, statistical distribution theory, etc. For more information on this function and its applications (see [3], [4], [11], [12], [18] and [20]).

Geometric properties for the function $E_{\upsilon,\varrho}(z)$ including starlikeness, convexity and closed-to-convex were recently investigated by Bansal and Prajapat in [5]. Furthermore, certain result on partial sum of the Mettag-Leffler function were also obtained in [17]. Some special cases (see [17]) of Mettag-Leffler function $E_{\upsilon,\varrho}(z)$ are:

$$E_{2,1}(z)=\text{z}\ \cosh\sqrt{z}$$
$$E_{2,2}(z)=\sqrt{z}\sinh\left(\sqrt{z}\right)$$
$$E_{2,3}(z)=2\left[\cosh\left(\sqrt{z}\right)\ -1\right]$$
$$E_{2,4}(z)=6\left[\sinh\left(\sqrt{z}\right)-\sqrt{z}\right]/\sqrt{z}.$$

For $f(z)\in\mathfrak{A}(p)$, using Equations (1.2) and (1.3), Elhaddad and Darus [10] introduced the generalization differential operator

$$\widetilde{\mathfrak{D}}^m_{\lambda,p}(\upsilon,\varrho,a_1,b_1)f(z)=z^p+\sum_{j=p+1}^{\infty}\left(\frac{p+(j-p)\lambda}{p}\right)^m\frac{\Upsilon_{(j-p,\upsilon,\varrho)}(a_r,b_s)}{(j-p)!}a_jz^j, \tag{1.4}$$

where $m\in\mathbb{N}_o,\ \lambda\geq 0$ and $\Upsilon_{(j-p,\upsilon,\varrho)}(a_r,b_s)$ is given by

$$\Upsilon_{(j-p,\upsilon,\varrho)}(a_r,b_s)=\frac{\Gamma(\varrho)}{\Gamma(\upsilon(j-p)+\varrho)}\left(\frac{\prod_{i=1}^{r}(a_r)_{j-p}}{\prod_{n=1}^{s}(b_s)_{j-p}}\right),$$

which includes well-known operators like, the Elhaddad et al. operator (see [12],[13]), the Al-Oboudi operator [2], the Dozik and Srivastava operator [8], the Ruscheweyh operator [19], the Hohlov [14] and the Carlson and Shaffer operator [6].

A continuous function $f = u+iv$ is called complex-valued harmonic function in a domain $D \subseteq \mathbb{C}$ if both u and v are real harmonic functions in D. In any simply connected domain, denoted by S_H the class of all functions of the form $f = h+\overline{g}$ that are harmonic, univalent, normalized and sense preserving within the open unit disc $\mathfrak{U}$ where h and g are analytic in $\mathfrak{U}$. We call h the analytic part and g the co-analytic part of f. Clunie and Sheil-Small [7] initiated the study on these functions by introducing the class S_H, they gave necessary and sufficient conditions for f to be locally univalent and sense preserving in $\mathfrak{U}$, also, see Ref. [2]. Note that if the co-analytic part of f is zero, then the class S_H reduces to S, the class of normalized univalent analytic functions.

Multivalent harmonic functions in $\mathfrak{U}$ were introduced by Duren, Hengartner and Laugesen in [9] via the argument principle. Denoted by $H_p(n)$ $(p,n \in \mathbb{N})$, the class of all p-valent harmonic functions $f = h+\overline{g}$ that are sense-preserving in $\mathfrak{U}$ where h and g are of the form

$$h(z) = z^p + \sum_{j=2}^{\infty} a_{j+p-1} z^{j+p-1}, g(z) = \sum_{j=1}^{\infty} b_{j+p-1} z^{j+p-1}. \tag{1.5}$$

In this work we modified the operator $\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(\upsilon,\varrho,a_1,b_1)f(z)$ in Equation (1.4), of harmonic functions $f = h+\overline{g}$ as

$$\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(\upsilon,\varrho,a_1,b_1)f(z) = \widetilde{\mathfrak{D}}_{\lambda,p}^{m}(\upsilon,\varrho,a_1,b_1)h(z) + \overline{\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(\upsilon,\varrho,a_1,b_1)g(z)}, \tag{1.6}$$

where

$$\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(\upsilon,\varrho,a_1,b_1)h(z) = z^p + \sum_{j=2}^{\infty} (\frac{p+(j-1)\lambda}{p})^m \frac{\Upsilon_{(j-1,\upsilon,\varrho)}(a_r,b_s)}{(j-1)!} a_{j+p-1} z^{j+p-1}$$

and

$$\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(\upsilon,\varrho,a_1,b_1)g(z) = \sum_{j=1}^{\infty} \left(\frac{p+(j-1)\lambda}{p}\right)^m \frac{\Upsilon_{(j-1,\upsilon,\varrho)}(a_r,b_s)}{(j-1)!} b_{j+p-1} z^{j+p-1}.$$

Using the operator $\widetilde{\mathfrak{D}}_{\lambda,p}^{m}(\upsilon,\varrho,a_1,b_1)f(z)$, we introduce the class $H_{\lambda,p}^{m}(\gamma,\theta)$ of harmonic multivalent functions in the following definition:

Definition 1.1. For $0 \leq \theta < 1$, the function $f = h+\overline{g}$ is in the class $H^m_{\lambda,p}(\gamma,\theta)$ if satisfy the inequality

$$Re\left\{\frac{\widetilde{\mathfrak{D}}^m_{\lambda,p}(\upsilon,\varrho,a_1,b_1)\,h(z)+\overline{\widetilde{\mathfrak{D}}^m_{\lambda,p}(\upsilon,\varrho,a_1,b_1)\,g(z)}}{(1-\gamma)\,z^p+\gamma\widetilde{\mathfrak{D}}^m_{\lambda,p}(\upsilon,\varrho,a_1,b_1)\,h(z)+\gamma\overline{\widetilde{\mathfrak{D}}^m_{\lambda,p}(\upsilon,\varrho,a_1,b_1)\,g(z)}}\right\} \geq \theta,$$
$$|z| = r < 1. \tag{1.7}$$

Let $\overline{H}^m_{\lambda,p}(\gamma,\theta)$ denote the subclass of $H^m_{\lambda,p}(\gamma,\theta)$ consisting of harmonic multivalent functions $f = h+\overline{g}$ so that h and g are of the form:

$$h(z) = z^p - \sum_{j=2}^{\infty}|a_{j+p-1}|\,z^{j+p-1}, \quad g(z) = \sum_{j=1}^{\infty}|b_{j+p-1}|\,z^{j+p-1}. \tag{1.8}$$

2. Main results

In our first theorem, we give the sufficient coefficient condition for harmonic functions belonging to the class $H^m_{\lambda,p}(\gamma,\theta)$.

Theorem 2.1. *Let $f = h+\overline{g}$ be given by (1.5). If*

$$\sum_{j=2}^{\infty}\left(|a_{j+p-1}|+\frac{(1+\theta)}{(1-\theta)}|b_{j+p-1}|\right)\Omega^m_{\lambda,p}(\upsilon,\varrho) \leq \frac{1}{\gamma}-\frac{(1+\theta)}{(1-\theta)}|b_p|, \tag{2.1}$$

where $|b_p| < (1-\theta)/(1+\theta)$, $0 \leq \theta < 1$, $0 < \gamma < 1$ and $\Omega^m_{\lambda,p}(\upsilon,\varrho)$ is given by

$$\Omega^m_{\lambda,p}(\upsilon,\varrho) = \left(\frac{p+(j-1)\lambda}{p}\right)^m\frac{\Upsilon_{(j-1,\upsilon,\varrho)}(a_r,b_s)}{(j-1)!}, \tag{2.2}$$

then $f \in H^m_{\lambda,p}(\gamma,\theta)$.

Proof. Now, in order to prove that $f \in H^m_{\lambda,p}(\gamma,\theta)$. From Equation (1.7), we can write

$$Re\left\{\frac{\widetilde{\mathfrak{D}}^m_{\lambda,p}(\upsilon,\varrho,a_1,b_1)\,h(z)-\overline{\widetilde{\mathfrak{D}}^m_{\lambda,p}(\upsilon,\varrho,a_1,b_1)\,g(z)}}{(1-\gamma)\,z^p+\gamma\widetilde{\mathfrak{D}}^m_{\lambda,p}(\upsilon,\varrho,a_1,b_1)\,h(z)+\gamma\overline{\widetilde{\mathfrak{D}}^m_{\lambda,p}(\upsilon,\varrho,a_1,b_1)\,g(z)}}\right\}$$
$$= Re\left\{\frac{L(z)}{K(z)}\right\} \geq \theta,$$

where

$$L(z)=\widetilde{\mathfrak{D}}^m_{\lambda,p}(\upsilon,\varrho,a_1,b_1)\,h(z)-\overline{\widetilde{\mathfrak{D}}^m_{\lambda,p}(\upsilon,\varrho,a_1,b_1)\,g(z)}$$
$$= z^p+\sum_{j=2}^{\infty}\Omega^m_{\lambda,p}(\upsilon,\varrho)\,a_{j+p-1}z^{j+p-1}-\sum_{j=1}^{\infty}\Omega^m_{\lambda,p}(\upsilon,\varrho)\,a_{j+p-1}\overline{z}^{j+p-1}$$

and

$$K(z) = (1-\gamma) z^p + \gamma \widetilde{\mathfrak{D}}^m_{\lambda,p}(\upsilon, \varrho, a_1, b_1) h(z) + \gamma \overline{\widetilde{\mathfrak{D}}^m_{\lambda,p}(\upsilon, \varrho, a_1, b_1) g(z)}$$
$$= z^p + \gamma \sum_{j=2}^{\infty} \Omega^m_{\lambda,p}(\upsilon, \varrho) a_{j+p-1} z^{j+p-1} + \gamma \sum_{j=1}^{\infty} \Omega^m_{\lambda,p}(\upsilon, \varrho) a_{j+p-1} \overline{z}^{j+p-1}.$$

Using the fact that $Re(w) \geq \alpha$ if and only if $|1-\mu+w| \geq |1+\mu-w|$, it suffices to show that

$$|L(z) + (1-\mu) K(z)| - |L(z) - (1+\mu) K(z)| \geq 0 . \tag{2.3}$$

Substituting L(z) and K(z) in (2.3), we obtain

$$|L(z) + (1-\mu) K(z)| - |L(z) - (1+\mu) K(z)|$$
$$= \Big| (2-\theta) z^p + \sum_{j=2}^{\infty} (2-\theta) \Omega^m_{\lambda,p}(\upsilon, \varrho) a_{j+p-1} z^{j+p-1}$$
$$- \sum_{j=1}^{\infty} \Omega^m_{\lambda,p}(\upsilon, \varrho) a_{j+p-1} \overline{z}^{j+p-1} \Big|$$
$$- \Big| -\theta z^p + \sum_{j=2}^{\infty} [\gamma - (1+\theta)\gamma] \Omega^m_{\lambda,p}(\upsilon, \varrho) a_{j+p-1} z^{j+p-1}$$
$$- \sum_{j=1}^{\infty} (2+\theta) \gamma \Omega^m_{\lambda,p}(\upsilon, \varrho) a_{j+p-1} \overline{z}^{j+p-1} \Big|$$
$$\geq (2-\theta) |z|^{\mathrm{p}} - \sum_{j=2}^{\infty} (2-\theta) \gamma \Omega^m_{\lambda,p}(\upsilon, \varrho) |a_{j+p-1}| |z|^{j+p-1}$$
$$- \sum_{j=1}^{\infty} \Omega^m_{\lambda,p}(\upsilon, \varrho) |b_{j+p-1}| |z|^{j+p-1}$$
$$= -\theta |z|^p - \sum_{j=2}^{\infty} [\gamma - (1+\theta)\gamma] \Omega^m_{\lambda,p}(\upsilon, \varrho) |a_{j+p-1}| |z|^{j+p-1}$$
$$- \sum_{j=1}^{\infty} (2+\theta) \gamma \Omega^m_{\lambda,p}(\upsilon, \varrho) |b_{j+p-1}| |z|^{j+p-1}$$

$$= 2\,(1-\theta)\,|z|^{p}\Big\{1-\sum_{j=2}^{\infty}\gamma\Omega^{m}_{\lambda,p}\,(\upsilon,\varrho)\,|a_{j+p-1}|\,|z|^{j-1}$$

$$-\sum_{j=1}^{\infty}\frac{(1+\theta)}{(1-\theta)}\gamma\Omega^{m}_{\lambda,p}\,(\upsilon,\varrho)\,|b_{j+p-1}|\,|z|^{j-1}\Big\}$$

$$= 2(1-\theta)\Big\{1-\frac{(1+\theta)}{(1-\theta)}\gamma\,|b_p|-\left(\sum_{j=2}^{\infty}|a_{j+p-1}|+\frac{(1+\theta)}{(1-\theta)}\,|b_{j+p-1}|\right)\gamma\Omega^{m}_{\lambda,p}\,(\upsilon,\varrho)\Big\}.$$

Using the inequality (2.1), we note that the last equation is non-negative. This implies that $f\in H^{m}_{\lambda,p}\,(\gamma,\theta)$. This completes the proof. □

Now, we show that the condition (2.1) is also necessary for the function $f=h+\overline{g}$ to belong to $\overline{H}^{m}_{\lambda,p}\,(\gamma,\theta)$, where h and g are given by (1.8).

Theorem 2.2. *Let $f=h+\overline{g}$ be given by* (1.8). *Then $f\in\overline{H}^{m}_{\lambda,p}\,(\gamma,\theta)$ if and only if*

$$\sum_{j=2}^{\infty}\left(|a_{j+p-1}|+\frac{(1+\theta)}{(1-\theta)}\,|b_{j+p-1}|\right)\Omega^{m}_{\lambda,p}\,(\upsilon,\varrho)\le\frac{1}{\gamma}-\frac{(1+\theta)}{(1-\theta)}\,|b_p|\;, \tag{2.4}$$

where $|b_p|<(1-\theta)/(1+\theta)\,,\ 0\le\theta<1,\ 0<\gamma<1$ and $\Omega^{m}_{\lambda,p}\,(\upsilon,\varrho)$ is given by Equation (2.2).

Proof. Since $\overline{H}^{m}_{\lambda,p}\,(\gamma,\theta)\subset H^{m}_{\lambda,p}\,(\gamma,\theta)$, we need only to prove the "only if" part of the theorem. For functions $f\in\overline{H}^{m}_{\lambda,p}\,(\gamma,\theta)$, we notice that the condition (1.7) is equivalent to

$$Re\left\{\frac{\widetilde{\mathfrak{D}}^{m}_{\lambda,p}\,(\upsilon,\varrho,a_1,b_1)\,h\,(z)+\overline{\widetilde{\mathfrak{D}}^{m}_{\lambda,p}\,(\upsilon,\varrho,a_1,b_1)\,g\,(z)}}{(1-\gamma)\,z^{p}+\gamma\widetilde{\mathfrak{D}}^{m}_{\lambda,p}\,(\upsilon,\varrho,a_1,b_1)\,h\,(z)+\gamma\overline{\widetilde{\mathfrak{D}}^{m}_{\lambda,p}\,(\upsilon,\varrho,a_1,b_1)\,g\,(z)}}-\theta\right\}\ge 0,$$

or equivalent to

$$Re\left\{\frac{M(z)}{N(z)}\right\}\ge 0, \tag{2.5}$$

where

$$M(z)=-\gamma z^{p}+\sum_{j=2}^{\infty}(\theta\gamma-1)\Omega^{m}_{\lambda,p}(\upsilon,\varrho)\,|a_{j+p-1}|\,z^{j+p-1}$$

$$+\sum_{j=1}^{\infty}(1-\theta\gamma)\Omega^{m}_{\lambda,p}(\upsilon,\varrho)\,|b_{j+p-1}|\,z^{j+p-1}$$

and

$$N(z) = z^p - \sum_{j=2}^{\infty} \gamma\Omega_{\lambda,p}^{m}(\upsilon, \varrho)\, |a_{j+p-1}|\, z^{j+p-1} + \sum_{j=1}^{\infty} \gamma\Omega_{\lambda,p}^{m}(\upsilon, \varrho)\, |b_{j+p-1}|\, z^{j+p-1}.$$

Now, the last inequality (2.5) must hold for all values of z in U. Choosing the value of z on the positive real axis where $0 \le z = r < 1$, we should have

$$\frac{(1-\theta\gamma)|b_p| - \gamma + \sum_{j=2}^{\infty} \Omega_{\lambda,p}^{m}(\upsilon, \varrho)((\theta\gamma-1)|a_{j+p-1}| + (1-\theta\gamma)|b_{j+p-1}|) r^{j-1}}{1 + \gamma\,|b_p| + \sum_{j=2}^{\infty} \gamma\Omega_{\lambda,p}^{m}(\upsilon, \varrho)\,(|b_{j+p-1}| - |a_{j+p-1}|)\, r^{j-1}} \ge 0. \tag{2.6}$$

We notice that the expression in the Equation (2.6) is negative for r sufficiently close to 1, when the condition (2.4) does not hold, hence there exist $z_0 = r_0$ in $(0, 1)$ for which the quotient in (2.6) is negative, this contradicts the required condition for $f \in \overline{H}_{\lambda,p}^{m}(\gamma, \theta, \upsilon, \varrho, a_1, b_1)$ and the proof is completed. □

We consider the extreme points of closed convex hull of $\overline{H}_{\lambda,p}^{m}(\gamma, \theta)$, denoted by $clco\,\overline{H}_{\lambda,p}^{m}(\gamma, \theta)$.

Theorem 2.3. *Let f be given by Equation* (1.8)*. Then $f \in clco\overline{H}_{\lambda,p}^{m}(\gamma, \theta)$ if and only if f can be expressed in the form:*

$$f(z) = \sum_{j=1}^{\infty} \left[x_{j+p-1} h_{j+p-1}(z) + y_{j+p-1} g_{j+p-1}(z)\right], \tag{2.7}$$

where $h_p(z) = z^p$, $h_{j+p-1}(z) = z^p - \frac{1}{\gamma\Omega_{\lambda,p}^{m}(\upsilon,\varrho)} z^{j+p-1}$, $(j \ge 2, 3, \dots)$ *and*

$$g_{j+p-1}(z) = z^p + \frac{(1-\theta)}{\gamma(1+\theta)\,\Omega_{\lambda,p}^{m}(\upsilon, \varrho)} \overline{z}^{j+p-1}, (j \ge 1, 2, 3, \dots),$$

$\Omega_{\lambda,p}^{m}(\upsilon, \varrho)$ *is given by Equation* (2.2)*,* $x_{j+p-1} \ge 0$ *,* $y_{j+p-1} \ge 0$*, and*

$$x_p = 1 - \sum_{j=2}^{\infty} x_{j+p-1} - \sum_{1=1}^{\infty} y_{j+p-1}.$$

In particular, the extreme points of $\overline{H}_{\lambda,p}^{m}(\gamma, \theta)$ *are* $\{h_{j+p-1}\}$ *and* $\{g_{j+p-1}\}$.

Proof. Let f be of the form in Equation (2.7), then we have

$$\begin{aligned} f(z) &= x_p h_p(z) + \sum_{j=2}^{\infty} x_{j+p-1}\left(z^p - \frac{1}{\gamma \Omega_{\lambda,p}^m(\upsilon,\varrho)} z^{j+p-1}\right) \\ &\quad + \sum_{j=1}^{\infty} x_{j+p-1}\left(z^p + \frac{(1-\theta)}{\gamma(1+\theta)\Omega_{\lambda,p}^m(\upsilon,\varrho)} \overline{z}^{j+p-1}\right) \\ &= z^p - \sum_{j=2}^{\infty} \frac{1}{\gamma \mathit{\Omega}_{\lambda,p}^m(\upsilon,\varrho)} x_{j+p-1} z^{j+p-1} \\ &\quad + \sum_{j=1}^{\infty} \frac{(1-\theta)}{\gamma(1+\theta)\mathit{\Omega}_{\lambda,p}^m(\upsilon,\varrho)} y_{j+p-1} \overline{z}^{j+p-1}. \end{aligned} \tag{2.8}$$

On the other hand, let

$$|a_{j+p-1}| = \frac{1}{\gamma \mathit{\Omega}_{\lambda,p}^m(\upsilon,\varrho)} x_{j+p-1}$$

and

$$\begin{aligned} |b_{j+p-1}| &= \frac{(1-\theta)}{\gamma(1+\theta)\mathit{\Omega}_{\lambda,p}^m(\upsilon,\varrho)} y_{j+p-1} \\ &= \sum_{\mathrm{j}=2}^{\infty} \mathit{\Omega}_{\lambda,p}^m(\upsilon,\varrho)\left(\frac{1}{\mathit{\Omega}_{\lambda,p}^m(\upsilon,\varrho)\gamma} x_{j+p-1}\right) \\ &\quad + \sum_{\mathrm{j}=1}^{\infty} \frac{(1+\theta)\mathit{\Omega}_{\lambda,p}^m(\upsilon,\varrho)}{(1-\theta)}\left(\frac{(1-\theta)}{\gamma(1+\theta)\mathit{\Omega}_{\lambda,p}^m(\upsilon,\varrho)} y_{j+p-1}\right) \\ &= \sum_{j=2}^{\infty} x_{j+p-1} + \sum_{j=1}^{\infty} y_{j+p-1} \\ &= 1 - x_p \le 1. \end{aligned}$$

Therefore, $f \in clco\ \overline{H}_{\lambda,p}^m(\gamma,\theta)$.

Conversely, if $f \in clco\ \overline{H}_{\lambda,p}^m(\gamma,\theta)$. Set

$$x_{k+p-1} = \gamma \mathit{\Omega}_{\lambda,p}^m(\upsilon,\varrho)\, a_{j+p-1},\ j = 2, 3, \ldots,$$

$$y_{j+p-1} = \frac{\gamma(1+\theta)\mathit{\Omega}_{\lambda,p}^m(\upsilon,\varrho)}{(1-\theta)} b_{j+p-1}\ ,\ \ j = 1, 2, 3, \ldots,$$

and using

$$x_p = 1 - \sum_{j=2}^{\infty} x_{j+p-1} - \sum_{1=1}^{\infty} y_{j+p-1}.$$

Then, we have

$$\begin{aligned}
f(z) &= z^p - \sum_{j=2}^{\infty} |a_{j+p-1}| z^{j+p-1} + \sum_{j=1}^{\infty} |b_{j+p-1}| \overline{z}^{j+p-1} \\
&= z^p - \sum_{j=2}^{\infty} \frac{x_{j+p-1}}{\Omega^m_{\lambda,p}(\upsilon,\varrho)\gamma} z^{j+p-1} + \sum_{k=1}^{\infty} \frac{(1-\theta)}{\gamma(1+\theta)\,\Omega^m_{\lambda,p}(\upsilon,\varrho)} y_{k+p-1}\overline{z}^{j+p-1} \\
&= x_p z^p - \sum_{j=2}^{\infty} x_{j+p-1}\left(z^p - \frac{1}{\gamma\Omega^m_{\lambda,p}(\upsilon,\varrho)} z^{j+p-1}\right) \\
&\quad + \sum_{j=1}^{\infty} y_{j+p-1}\left(z^p + \frac{(1-\theta)}{\gamma(1+\theta)\,\Omega^m_{\lambda,p}(\upsilon,\varrho)}\overline{z}^{j+p-1}\right) \\
&= \sum_{j=1}^{\infty}\left[x_{k+p-1}h_{k+p-1}(z) + x_{k+p-1}h_{k+p-1}(z)\right].
\end{aligned}$$

This completes the proof. □

Next, we want to prove two theorems, the first theorem is about convolution for the class $\overline{H}^m_{\lambda,p}(\gamma,\theta)$ and secondly, we prove that the class $\overline{H}^m_{\lambda,p}(\gamma,\theta)$ is closed under convex combination. The convolution of two multivalent harmonic functions

$$f_1(z) = z^p - \sum_{j=2}^{\infty} a_{j+p-1} z^{j+p-1} + \sum_{k=1}^{\infty} b_{j+p-1}\overline{z}^{j+p-1} \tag{2.9}$$

and

$$f_2(z) = z^p - \sum_{j=2}^{\infty} c_{j+p-1} z^{j+p-1} + \sum_{k=1}^{\infty} d_{j+p-1}\overline{z}^{k+p-1} \tag{2.10}$$

is defined by

$$\begin{aligned}
(f_1 * f_2)(z) &= f_1(z) * f_2(z) \\
&= z^p - \sum_{j=2}^{\infty} a_{j+p-1}c_{j+p-1} z^{j+p-1} + \sum_{k=1}^{\infty} b_{j+p-1}d_{j+p-1}\overline{z}^{j+p-1}.
\end{aligned} \tag{2.11}$$

Using this definition, we show that the class $\overline{H}^m_{\lambda,p}(\gamma,\theta)$ is closed under convolution.

Theorem 2.4. *For $0 \le \theta_1 \le \theta < 1$, $0 < \gamma < 1$. Let $f_1 \in \overline{H}^m_{\lambda,p}(\gamma,\theta)$ and $f_2 \in \overline{H}^m_{\lambda,p}(\gamma,\theta_1)$. Then*

$$f_1 * f_2 \in \overline{H}^m_{\lambda,p}(\gamma,\theta) \subset \overline{H}^m_{\lambda,p}(\gamma,\theta_1).$$

Proof. Let

$$f_1(z) = z^p - \sum_{j=2}^{\infty} a_{j+p-1} z^{j+p-1} + \sum_{k=1}^{\infty} b_{j+p-1} \overline{z}^{j+p-1}$$

be in the class $\overline{H}^m_{\lambda,p}(\gamma, \theta)$ and

$$f_2(z) = z^p - \sum_{j=2}^{\infty} c_{j+p-1} z^{j+p-1} + \sum_{k=1}^{\infty} d_{j+p-1} \overline{z}^{k+p-1}$$

be in the class $\overline{H}^m_{\lambda,p}(\gamma, \theta_1)$. Then the convolution $f_1 * f_2$ is given by Equation (2.11), we want to show that the coefficients of $f_1 * f_2$ satisfies the required condition given in Theorem 2.1.

For $f_2 \in \overline{H}^m_{\lambda,p}(\gamma, \theta_1)$, we note that $c_{k+p-1} < 1$ and $d_{k+p-1} < 1$. Now consider convolution functions $f_1 * f_2$ as follows:

$$\begin{aligned}
&\sum_{j=2}^{\infty} \gamma \Omega^m_{\lambda,p}(\upsilon, \varrho)\, a_{j+p-1} c_{j+p-1} + \sum_{j=1}^{\infty} \frac{(1+\theta_1)}{(1-\theta_1)} \gamma \Omega^m_{\lambda,p}(\upsilon, \varrho)\, b_{j+p-1} d_{j+p-1} \\
&\leq \sum_{j=2}^{\infty} \gamma \Omega^m_{\lambda,p}(\upsilon, \varrho)\, a_{j+p-1} + \sum_{k=1}^{\infty} \frac{(1+\theta_1)}{(1-\theta_1)} \gamma \Omega^m_{\lambda,p}(\upsilon, \varrho)\, b_{j+p-1} \\
&\leq \sum_{j=2}^{\infty} \gamma \Omega^m_{\lambda,p}(\upsilon, \varrho)\, a_{j+p-1} + \sum_{k=1}^{\infty} \frac{(1+\theta)}{(1-\theta)} \gamma \Omega^m_{\lambda,p}(\upsilon, \varrho)\, b_{j+p-1} \\
&\leq 1.
\end{aligned}$$

□

Next, we introduce the last theorem in this paper of the class $\overline{H}^m_{\lambda,p}(\gamma, \theta)$.

Theorem 2.5. *The class $\overline{H}_p(n, \mu, \gamma, \delta)$ is closed under convex combinations.*

Proof. For s = 1, 2, ..., suppose that $f_{n,S} \in \overline{H}_p(n, \mu, \gamma, \delta)$, where $f_{n,S}$ is given by

$$f_{n,s}(z) = z^p - \sum_{j=2}^{\infty} |a_{s,j+p-1}|\, z^{j+p-1} + \sum_{k=1}^{\infty} |b_{s,j+p-1}|\, \overline{z}^{j+p-1}.$$

Then by Theorem 2.2, we have

$$\sum_{j=2}^{\infty} \gamma \Omega^m_{\lambda,p}(\upsilon, \varrho)\, |a_{s,j+p-1}| + \sum_{j=1}^{\infty} \frac{(1+\theta)}{(1-\theta)} \gamma \Omega^m_{\lambda,p}(\upsilon, \varrho)\, |b_{s,j+p-1}| \leq 1. \tag{2.12}$$

For $\sum_{\mathrm{s}=1}^{\infty}\mathrm{t_s}=1,\ 0\leq \mathrm{t_s}\leq 1,$ the convex combinations of $f_{n,s}$ written as

$$\sum_{s=1}^{\infty} t_s f_{n,s}(z)=z^p-\sum_{j=2}^{\infty}\left(\sum_{s=1}^{\infty} t_s\left|a_{s+p-1}\right|\right)z^{j+p-1}+\sum_{j=1}^{\infty}\left(\sum_{s=1}^{\infty} t_s\left|b_{s+p-1}\right|\right)\overline{z}^{+p-1}.$$

Now, by using the inequality (2.12), we obtain

$$\sum_{\mathrm{j}=2}^{\infty}\gamma\Omega_{\lambda,p}^{m}(\upsilon,\varrho)\left(\sum_{\mathrm{s}=1}^{\infty}\mathrm{t_s}\left|\mathrm{a_{s,j+p-1}}\right|\right)+\sum_{\mathrm{j}=1}^{\infty}\frac{(1+\theta_1)}{(1-\theta_1)}\gamma\Omega_{\lambda,p}^{m}(\upsilon,\varrho)\left(\sum_{\mathrm{s}=1}^{\infty}\mathrm{t_s}\left|\mathrm{b_{s,j+p-1}}\right|\right)$$

$$=\sum_{\mathrm{s}=1}^{\infty}\mathrm{t_s}\left(\sum_{\mathrm{k}=2}^{\infty}\gamma\Omega_{\lambda,p}^{m}(\upsilon,\varrho)\,\mathrm{a_{s,k+p-1}}+\sum_{\mathrm{k}=1}^{\infty}\frac{(1+\theta_1)}{(1-\theta_1)}\gamma\Omega_{\lambda,p}^{m}(\upsilon,\varrho)\,\mathrm{b_{s,k+p-1}}\right)$$

$$\leq\sum_{\mathrm{s}=1}^{\infty}\mathrm{t_s}=1,$$

which is the required coefficient condition. □

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