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FINDING SOLUTIONS OF SECOND ORDER LINEAR HOMOGENEOUS QUATERNIONIC DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we propose a corresponding Wronskian method that can be used in quaternionic differential equations. By considering $M(\mathbb{C}, 2)$ consisting of 2×2 matrixes that correspond to the basis of quaternions, we define the Wronskian applicable to the quaternionic differential equations. In addition, we propose some steps for finding another solution using the reduction of the order of quaternionic homogeneous differential equations, and show how the steps can be used in examples.

1. Introduction

The representation of rotation and transformation using quaternions is useful for modeling three-dimensional motion over time, so quaternionic differential equations are used for kinematic modeling, attitude dynamics, fluid mechanics, quantum mechanics, etc.(see [10]). Since Hamilton introduced the quaternions, many studies have been conducted on the holomorphicity and regularity of the quaternionic functions by the algebraic properties of quaternions(see [1, 11, 12]). In addition to studying quaternion functions of the real variables, we examined the properties of functions for special quaternion variables, such as dual quaternions and split quaternions(see [5, 6, 7, 8, 9]).

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Let $\mathbb{H}$ be a quaternionic field denoted by

$$\mathbb{H} = \{q \mid q = x_0 + ix_1 + jx_2 + kx_3, \quad x_r \in \mathbb{R}\ (r = 0, 1, 2, 3)\},$$

where $\{i, j, k\}$ satisfying

$$i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j, \tag{1.1}$$

consists of the elements of $\mathbb{H}$. Let $\Psi : \mathbb{R} \to \mathbb{H}$ be a quaternionic function of a real variable such that

$$\Psi(t) = u_0(t) + iu_1(t) + ju_2(t) + ku_3(t),$$

where $u_r : \mathbb{R} \to \mathbb{R}$ $(r = 0, 1, 2, 3)$ are real-valued functions of the real variable t. Let $\mathcal{F}$ be the set of real-valued functions of a real variable. Then, we can denote $\Psi(t)$ as $\Psi(t) \in \mathbb{H} \otimes \mathcal{F}$. These notations are referred to by [3]. The first and second derivative of quaternionic functions with respect to the real variable t are denoted by

$$\Psi'(t) = \frac{d}{dt}\Psi(t) \quad \text{and} \quad \Psi''(t) = \frac{d^2}{dt^2}\Psi(t),$$

respectively.

Put the 2×2 matrices

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ e_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ e_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

satisfying the following relations computed by the multiplication of matrix algebra:

$$e_0^2 = e_0,\ e_1^2 = e_2^2 = e_3^2 = -e_0,$$

$$e_1e_2 = -e_2e_1 = e_3,\ e_2e_3 = -e_3e_2 = e_1,\ e_3e_1 = -e_1e_3 = e_2.$$

Then, we can express the quaternion as

$$q = e_0x_0 + e_1x_1 + e_2x_2 + e_3x_3 = \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix}.$$

and if z_1 and z_2 are denoted by $z_1 = x_0 + ix_1$ and $z_2 = x_2 + ix_3$ with the conjugation of z_1 and z_2 are $\overline{z_1} = x_0 - ix_1$ and $\overline{z_2} = x_2 - ix_3$, we can write q as

$$q = \begin{pmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{pmatrix}. \tag{1.2}$$

The set $M(2, \mathbb{C})$ denoted by

$$M(2, \mathbb{C}) = \{q \mid q = \begin{pmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{pmatrix}, \quad z_1,\ z_2 \in \mathbb{C}\}$$

is isomorphic to $\mathbb{H}$ whose elements consist of $\{1, i, j, k\}$ satisfying (1.1).

Consider the existence and uniqueness for the quaternionic initial value problems

$$\Psi'' = \alpha\Psi' + \beta\Psi + \gamma,\ \Psi(t_0) = f,\ \Psi'(t_0) = g, \tag{1.3}$$

where

$$\alpha(t) = a_0(t) + ia_1(t) + ja_2(t) + ka_3(t),\ \beta(t) = b_0(t) + ib_1(t) + jb_2(t) + kb_3(t),$$
$$\gamma(t) = c_0(t) + ic_1(t) + jc_2(t) + kc_3(t)$$

in $\mathbb{H} \otimes \mathcal{F}$ for $t_0 \in I$, being an open interval I of $\mathbb{R}$. Also,

$$f = f_0 + if_1 + jf_2 + kf_3 \quad \text{and} \quad g = g_0 + ig_1 + jg_2 + kg_3$$

in $\mathbb{H}$.

Theorem 1.1. *Let I be an open interval containing the point $t = t_0$. If α, β, γ in Equation* (1.3) *are continuous functions of t on I, then Equation* (1.3) *has a unique solution Ψ on I.*

Referring to [3] and [10], if α, β, γ in Equation (1.3) are continuous functions of t on an open interval I containing the point $t = t_0$, then the initial value problem (1.3) has a solution Ψ on this interval and this solution is unique. By using the matrix representation (1.2), we can write the quaternionic initial value problem (1.3) as follows:

$$\begin{aligned}\begin{pmatrix} \Phi_1'' & \Phi_2'' \\ -\overline{\Phi_2}'' & \overline{\Phi_1}'' \end{pmatrix} &= \begin{pmatrix} \mu_1 & \mu_2 \\ -\overline{\mu_2} & \overline{\mu_1} \end{pmatrix}\begin{pmatrix} \Phi_1' & \Phi_2' \\ -\overline{\Phi_2}' & \overline{\Phi_1}' \end{pmatrix} \\ &\quad + \begin{pmatrix} \nu_1 & \nu_2 \\ -\overline{\nu_2} & \overline{\nu_1} \end{pmatrix}\begin{pmatrix} \Phi_1 & \Phi_2 \\ -\overline{\Phi_2} & \overline{\Phi_1} \end{pmatrix} + \begin{pmatrix} \lambda_1 & \lambda_2 \\ -\overline{\lambda_2} & \overline{\lambda_1} \end{pmatrix},\end{aligned} \tag{1.4}$$

$$\begin{pmatrix} \Phi_1 & \Phi_2 \\ -\overline{\Phi_2} & \overline{\Phi_1} \end{pmatrix} = \begin{pmatrix} F_1 & F_2 \\ -\overline{F_2} & \overline{F_1} \end{pmatrix}$$

and

$$\begin{pmatrix} \Phi_1' & \Phi_2' \\ -\overline{\Phi_2}' & \overline{\Phi_1}' \end{pmatrix} = \begin{pmatrix} G_1 & G_2 \\ -\overline{G_2} & \overline{G_1} \end{pmatrix},$$

where

$$\Phi_1'' = u_0'' + iu_1'',\ \Phi_2'' = u_2'' + iu_3'',\ \Phi_1' = u_0' + iu_1',\ \Phi_2' = u_2' + iu_3',$$
$$\Phi_1 = u_0 + iu_1,\ \Phi_2 = u_2 + iu_3,\ \mu_1 = a_0 + ia_1,\ \mu_2 = a_2 + ia_3,$$
$$\nu_1 = b_0 + ib_1,\ \nu_2 = b_2 + ib_3,\ \lambda_1 = c_0 + ic_1,\ \lambda_2 = c_2 + ic_3,$$
$$F_1 = f_0 + if_1,\ F_2 = f_2 + if_3,\ G_1 = g_0 + ig_1,\ G_2 = g_2 + ig_3.$$

The functions $a_r,\ b_r,\ c_r$ $(r = 0, 1, 2, 3)$ are continuous real functions of t on an open interval I. Using the proof method in [2], the linear system (1.4) has a unique solution

$$\begin{pmatrix} \Phi_1 & \Phi_2 \\ -\overline{\Phi_2} & \overline{\Phi_1} \end{pmatrix}$$

on I satisfying the initial conditions in (1.3).

2. Linear independence and dependence of solutions

By referring to the Wronskian method discussed in Kou and Xia [10] and Leo and Ducati [3, 4], we propose the Wronskian method that can be used in quaternionic differential equations. Without loss of generality, our discussion is presented on quaternionic second-order differential equations for simplicity.

Consider the linear independence and dependence of the solutions of second order homogeneous differential equations such that

$$\Psi'' = \alpha\Psi' + \beta\Psi, \tag{2.1}$$

where each component of α and β is a quaternionic continuous function of t on an open interval I. The general solution of equation (2.1) is given in the form $\Psi = \varphi q + \xi p$ by two linearly independent solutions φ and ξ, where

$$\varphi = \varphi_0 + i\varphi_1 + j\varphi_2 + k\varphi_3$$

and

$$\xi = \xi_0 + i\xi_1 + j\xi_2 + k\xi_3$$

in $\mathbb{H} \otimes \mathcal{F}$ for $t_0 \in I$ and $q,\ p \in \mathbb{H}$.

In complex analysis, the concept of the Wronskian for second order homogeneous differential equations is a useful criterion to distinguish linear independence and dependence of two solutions of a homogeneous second order differential equation, defined by

$$W = \begin{vmatrix} \varphi & \xi \\ \varphi' & \xi' \end{vmatrix} = \varphi\xi' - \xi\varphi', \quad \text{when } \varphi,\ \xi \in \mathbb{C} \otimes \mathcal{F}, \tag{2.2}$$

and $W \in \mathbb{C} \otimes \mathcal{F}$. However, in the form of Equation (2.2), it is not possible to determine whether the quaternionic solutions φ and ξ of Equation (2.1) are independent or dependent on each other. Since the commutative rule does not hold for the product, W may not be zero even if φ and ξ are linearly dependent on each other. In other words, if $\xi = \varphi q$,

$$W = \varphi\xi' - \xi\varphi' = \varphi\varphi' q - \varphi q\varphi'.$$

When q is a real constant, the property of Wronskian holds with $W = 0$. When q is a non-real constant in quaternions, since $\varphi' q \neq q\varphi'$, it may be that $W \neq 0$. Therefore, let us consider the following expression to determine the linear independence of two solutions of homogeneous second order differential equation such that Equation (2.1) is true for the quaternionic functions.

Let us consider two linearly dependent solutions of Equation (2.1), that is, $\xi = \varphi q$, where $\varphi,\ \xi \in \mathbb{H} \otimes \mathcal{F}$ and q is a constant in quaternions. Without loss of generality, suppose the quaternionic constant multiple is calculated on the right. We can also write as

$$\varphi = \begin{pmatrix} \phi_1 & \phi_2 \\ -\overline{\phi_2} & \overline{\phi_1} \end{pmatrix} \quad \text{and} \quad \xi = \begin{pmatrix} \eta_1 & \eta_2 \\ -\overline{\eta_2} & \overline{\eta_1} \end{pmatrix},$$

where $\phi_1 = \varphi_0 + i\varphi_1$, $\phi_2 = \varphi_2 + i\varphi_3$, $\eta_1 = \xi_0 + i\xi_1$ and $\eta_2 = \xi_2 + i\xi_3$.

We define the quaternionic Wronskian such that

$$W = \begin{vmatrix} \phi_1 & \phi_2 & \eta_1 & \eta_2 \\ -\overline{\phi_2} & \overline{\phi_1} & -\overline{\eta_2} & \overline{\eta_1} \\ \phi_1' & \phi_2' & \eta_1' & \eta_2' \\ -\overline{\phi_2}' & \overline{\phi_1}' & -\overline{\eta_2}' & \overline{\eta_1}' \end{vmatrix}. \tag{2.3}$$

In detail,

$$\begin{aligned} W &= (\eta_1' \, \overline{\eta_1}' + \eta_2' \, \overline{\eta_2}')(\phi_1 \, \overline{\phi_1} + \phi_2 \, \overline{\phi_2}) + (-\eta_1' \, \overline{\eta_1} - \eta_2' \, \overline{\eta_2})(\phi_1 \, \overline{\phi_1}' + \phi_2 \, \overline{\phi_2}') \\ &\quad + (-\overline{\eta_1} \, \overline{\eta_2}' + \overline{\eta_2} \, \overline{\eta_1}')(\phi_1 \, \phi_2' - \phi_1' \, \phi_2) + (\eta_1 \, \overline{\eta_1}' + \eta_2 \, \overline{\eta_2}')(-\overline{\phi_1} \, \phi_1' - \overline{\phi_2} \, \phi_2') \\ &\quad + (\eta_1 \, \eta_2' - \eta_2 \, \eta_1')(-\overline{\phi_1} \, \overline{\phi_2}' + \overline{\phi_2} \, \overline{\phi_1}') + (\eta_1 \, \overline{\eta_1} + \eta_2 \, \overline{\eta_2})(\phi_1' \, \overline{\phi_1}' + \phi_2' \, \overline{\phi_2}') \\ &= (|\eta_1'|^2 + |\eta_2'|^2)(|\phi_1|^2 + |\phi_2|^2) + (-\eta_1' \, \overline{\eta_1} - \eta_2' \, \overline{\eta_2})(\phi_1 \, \overline{\phi_1}' + \phi_2 \, \overline{\phi_2}') \\ &\quad + (-\overline{\eta_1} \, \overline{\eta_2}' + \overline{\eta_2} \, \overline{\eta_1}')(\phi_1 \, \phi_2' - \phi_1' \, \phi_2) + (\eta_1 \, \overline{\eta_1}' + \eta_2 \, \overline{\eta_2}')(-\overline{\phi_1} \, \phi_1' - \overline{\phi_2} \, \phi_2') \\ &\quad + (\eta_1 \, \eta_2' - \eta_2 \, \eta_1')(-\overline{\phi_1} \, \overline{\phi_2}' + \overline{\phi_2} \, \overline{\phi_1}') + (|\eta_1|^2 + |\eta_2|^2)(|\phi_1'|^2 + |\phi_2'|^2). \end{aligned}$$

Theorem 2.1. *Let α and β in Equation* (2.1) *be quaternionic continuous functions of t on an open interval I. If two solutions φ and ξ of Equation* (2.1) *on I are linearly dependent on I, then the quaternionic Wronskian is zero for any t in I.*

Proof. Suppose two solutions φ and ξ of Equation (2.1) on I are linearly dependent on I. The quaternionic Wronskian is

$$\begin{aligned} W &= (|\eta_1'|^2 + |\eta_2'|^2)(|\phi_1|^2 + |\phi_2|^2) + (-\eta_1' \, \overline{\eta_1} - \eta_2' \, \overline{\eta_2})(\phi_1 \, \overline{\phi_1}' + \phi_2 \, \overline{\phi_2}') \\ &\quad + (-\overline{\eta_1} \, \overline{\eta_2}' + \overline{\eta_2} \, \overline{\eta_1}')(\phi_1 \, \phi_2' - \phi_1' \, \phi_2) + (\eta_1 \, \overline{\eta_1}' + \eta_2 \, \overline{\eta_2}')(-\overline{\phi_1} \, \phi_1' - \overline{\phi_2} \, \phi_2') \\ &\quad + (\eta_1 \, \eta_2' - \eta_2 \, \eta_1')(-\overline{\phi_1} \, \overline{\phi_2}' + \overline{\phi_2} \, \overline{\phi_1}') + (|\eta_1|^2 + |\eta_2|^2)(|\phi_1'|^2 + |\phi_2'|^2). \end{aligned}$$

Since two solutions φ and ξ of Equation (2.1) on I are linearly dependent on I, we can express ξ as φq for the quaternionic constant $q = q_0 + iq_1 + jq_2 + kq_3$. By using the matrix expression, we can write

$$\xi = \begin{pmatrix} \eta_1 & \eta_2 \\ -\overline{\eta_2} & \overline{\eta_1} \end{pmatrix}, \; \varphi = \begin{pmatrix} \phi_1 & \phi_2 \\ -\overline{\phi_2} & \overline{\phi_1} \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} Q_1 & Q_2 \\ -\overline{Q_2} & \overline{Q_1} \end{pmatrix}$$

and then, we have

$$\eta_1 = Q_1\ \phi_1 - \overline{Q_2}\ \phi_2,\ \eta_2 = Q_2\ \phi_1 + \overline{Q_1}\ \phi_2,$$
$$\overline{\eta_1} = \overline{Q_1}\ \overline{\phi_1} - Q_2\ \overline{\phi_2},\ \overline{\eta_2} = Q_1\ \overline{\phi_2} + \overline{Q_2}\ \overline{\phi_1}.$$

Thus, W has the following components

$$|\eta_1'|^2 + |\eta_2'|^2 = (|Q_1|^2 + |Q_2|^2)(|\phi_1'|^2 + |\phi_2'|^2),$$
$$\eta_1'\ \overline{\eta_1} + \eta_2'\ \overline{\eta_2} = (|Q_1|^2 + |Q_2|^2)(\overline{\phi_1}\ \phi_1' + \overline{\phi_2}\ \phi_2'),$$
$$-\overline{\eta_1}\ \overline{\eta_2}' + \overline{\eta_2}\ \overline{\eta_1}' = (|Q_1|^2 + |Q_2|^2)(\overline{\phi_2}\ \overline{\phi_1}' - \overline{\phi_1}\ \overline{\phi_2}'),$$
$$\eta_1\ \overline{\eta_1}' + \eta_2\ \overline{\eta_2}' = (|Q_1|^2 + |Q_2|^2)(\phi_1\ \overline{\phi_1}' + \phi_2\ \overline{\phi_2}'),$$
$$\eta_1\ \eta_2' - \eta_2\ \eta_1' = (|Q_1|^2 + |Q_2|^2)(\phi_1\ \phi_2' - \phi_1'\ \phi_2),$$
$$|\eta_1|^2 + |\eta_2|^2 = (|Q_1|^2 + |Q_2|^2)(|\phi_1|^2 + |\phi_2|^2).$$

By calculating the above equations, W is obtained as zero. □

Example 2.2. For the following homogeneous second order differential equation

$$\Psi'' + j\Psi' + (1-k)\Psi = 0, \tag{2.4}$$

the quaternionic functions $\varphi = \exp(-it)$ and $\xi = \exp((i-j)t)$ form a basis of solutions of Equation (2.4) on any interval. Indeed, by the expression of the elementary functions (see [1]), we can write as $\varphi = \cos t - i \sin t$ and $\xi = \cos\sqrt{2}t + \frac{i-j}{\sqrt{2}}\sin\sqrt{2}t$. Then, we have

$$W = -3 - \sin^2 2t - 2\cos^2 t\cos 2t - i(\cos 2t\sin 2t + 2\sin 2t\sin^2 t) \neq 0 \tag{2.5}$$

for any t_0 in I. For this example, the form of the Wronskian presented above is represented in Appendix 1. Thus, φ and ξ are linearly independent on I.

3. Reduction of order for homogeneous differential equations

Let φ be a solution of Equation (2.1) on some interval I. Looking for a solution in the form $\xi = \varphi\tau$ and substituting φ and its derivatives

$$\xi' = \varphi'\tau + \varphi\tau' \quad \text{and} \quad \xi'' = \varphi''\tau + 2\varphi'\tau' + \varphi\tau''$$

into Equation (2.1),

$$\varphi''\tau + 2\varphi'\tau' + \varphi\tau'' = \alpha\varphi'\tau + \alpha\varphi\tau' + \beta\varphi\tau.$$

Since ξ is a solution of Equation (2.1),

$$\varphi''\tau = \alpha\varphi'\tau + \beta\varphi\tau$$

is satisfied, so we obtain

$$2\varphi'\tau' + \varphi\tau'' = \alpha\varphi\tau'$$

and then,

$$\tau'' = (\varphi^{-1}\alpha\varphi - 2\varphi^{-1}\varphi')\tau'. \tag{3.1}$$

For homogeneous second order equations with quaternion constant coefficients, $\alpha(t)$ and $\beta(t)$ replace to α and β, respectively, at least one solution is in the form of a quaternionic exponential, $\varphi = \exp(qt)$, and since $\frac{d}{dt}\exp(qt) = q\exp(qt)$ (see Appendix 2), Equation (3.1) reduces to

$$\varphi\tau'' = (\alpha - 2q)\varphi\tau'. \tag{3.2}$$

Let us introduce the quaternionic function $\sigma = \varphi\tau'$. Observing that

$$\sigma' = \varphi'\tau' + \varphi\tau'' = q\varphi\tau' + \varphi\tau''.$$

Equation (3.2) can be rewritten as follows:

$$\sigma' = q\varphi\tau' + (\alpha - 2q)\varphi\tau' = q\sigma + (\alpha - 2q)\sigma = (\alpha - q)\sigma.$$

This equation can be immediately integrated; its solution reads

$$\sigma = \exp((\alpha - q)t).$$

Thus, the second solution of the homogeneous second order differential equation with constant coefficients is given by

$$\xi = \exp(qt)\int \exp(-qt)\exp((\alpha - q)t)\ dt. \tag{3.3}$$

The solution of this integral will give interesting information about the second solution of quaternionic differential equations with constant coefficients when the associated characteristic quadratic equation has a unique solution. To solve the integral in Equation (3.3), we first consider by observing that

$$q = q_0 + X,\ p = p_0 + Y,$$

$$X = iq_1 + jq_2 + kq_3,\ Y = ip_1 + jp_2 + kp_3,$$

$$|X| = \sqrt{q_1^2 + q_2^2 + q_3^2},\ |Y| = \sqrt{p_1^2 + p_2^2 + p_3^2},$$

$$e^{qt} = e^{q_0 t}\cos(|X|t) + \frac{X}{|X|}\sin(|X|t),\ e^{pt} = e^{p_0 t}\cos(|Y|t) + \frac{Y}{|Y|}\sin(|Y|t).$$

Using these notations, we obtain the following formulas

$$\begin{aligned}
&\int e^{qt}e^{pt}\ dt \\
&= \frac{1}{2}\int e^{(q_0+p_0)t}\{\cos((|X|+|Y|)t)+\cos((|X|-|Y|)t)\}\ dt \\
&\quad + \frac{-X\cdot Y+X\times Y}{|X||Y|}\frac{1}{2}\int e^{(q_0+p_0)t}\{\cos((|X|-|Y|)t)-\cos((|X|+|Y|)t)\}\ dt \\
&\quad + \frac{Y}{|Y|}\frac{1}{2}\int e^{(q_0+p_0)t}\{\sin((|X|+|Y|)t)-\sin((|X|-|Y|)t)\}\ dt \\
&\quad + \frac{X}{|X|}\frac{1}{2}\int e^{(q_0+p_0)t}\{\sin((|X|+|Y|)t)+\sin((|X|-|Y|)t)\}\ dt,
\end{aligned}$$

and the specific calculation process can be checked in Appendix 3. So, we express the formula we need by using the following,

$$X \leftrightarrow -iq_1-jq_2-kq_3,\ Y \leftrightarrow i(a_1-q_1)+j(a_2-q_2)+k(a_3-q_3),$$

$$|X|=\sqrt{q_1^2+q_2^2+q_3^2},\ |Y|=\sqrt{(a_1-q_1)^2+(a_2-q_2)^2+(a_3-q_3)^2}$$

we have the solution ξ of Equation (2.1). The formula of ξ and the specific calculation process can be seen in Appendix 4.

Example 3.1. For the homogeneous second order equation

$$\Psi''+j\Psi'+(1-k)\Psi=0,\ \varphi=\exp(-it),$$

find ξ as a solution:

We have $\alpha=-j$, $q=-i$, $X=i$, $Y=i-j$, $|X|=1$, $|Y|=\sqrt{2}$, $X\cdot Y=1$ and $X\times Y=-k$. So, we obtain

$$\begin{aligned}
&\int \exp(-qt)\exp((\alpha-q)t)\ dt \\
&= \frac{1}{2}\int e^{(q_0+p_0)t}\{\cos((|X|+|Y|)t)+\cos((|X|-|Y|)t)\}\ dt \\
&\quad + \frac{-X\cdot Y+X\times Y}{|X||Y|}\frac{1}{2}\int e^{(q_0+p_0)t}\{\cos((|X|-|Y|)t)-\cos((|X|+|Y|)t)\}\ dt \\
&\quad + \frac{Y}{|Y|}\frac{1}{2}\int e^{(q_0+p_0)t}\{\sin((|X|+|Y|)t)-\sin((|X|-|Y|)t)\}\ dt \\
&\quad + \frac{X}{|X|}\frac{1}{2}\int e^{(q_0+p_0)t}\{\sin((|X|+|Y|)t)+\sin((|X|-|Y|)t)\}\ dt
\end{aligned}$$

$$= \frac{1}{2}\left\{\frac{1}{1+\sqrt{2}}\sin((1+\sqrt{2})t) + \frac{1}{1-\sqrt{2}}\sin((1-\sqrt{2})t)\right\}$$
$$+ \frac{-1-k}{\sqrt{2}}\frac{1}{2}\left\{\frac{1}{1-\sqrt{2}}\sin((1-\sqrt{2})t) - \frac{1}{1+\sqrt{2}}\sin((1+\sqrt{2})t)\right\}$$
$$+ \frac{i-j}{\sqrt{2}}\frac{1}{2}\left\{-\frac{1}{1+\sqrt{2}}\cos((1+\sqrt{2})t) + \frac{1}{1-\sqrt{2}}\cos((1-\sqrt{2})t)\right\}$$
$$+ i\frac{1}{2}\left\{-\frac{1}{1+\sqrt{2}}\cos((1+\sqrt{2})t) - \frac{1}{1-\sqrt{2}}\cos((1-\sqrt{2})t)\right\}$$
$$= \frac{1}{2\sqrt{2}}\Big\{(\sin((1+\sqrt{2})t) - \sin((1-\sqrt{2})t))$$
$$- i(\cos((1+\sqrt{2})t) - \cos((1-\sqrt{2})t))$$
$$+ j((\sqrt{2}-1)\cos((1+\sqrt{2})t) + (\sqrt{2}+1)\cos((1-\sqrt{2})t))$$
$$+ k((\sqrt{2}-1)\sin((1+\sqrt{2})t) + (\sqrt{2}+1)\sin((1-\sqrt{2})t))\Big\}$$

and

$$\xi = \exp(qt)\int \exp(-qt)\exp((\alpha-q)t)\ dt$$
$$= (\cos t - i\sin t)\int \exp(-qt)\exp((\alpha-q)t)\ dt$$
$$= \frac{1}{\sqrt{2}}(\sin(\sqrt{2}t) + j\cos(\sqrt{2}t) - k\sin(\sqrt{2}t))$$
$$= j(\cos(\sqrt{2}t) + \frac{i-j}{\sqrt{2}}\sin(\sqrt{2}t)) = j\exp((i-j)t).$$

Example 3.2. Find another solutions for the following equation

$$\Psi'' + i\Psi' + \frac{k}{2} = 0$$

having $\varphi = \exp(-\frac{i+j}{2}t)$ as a solution.

We have $\alpha = -i$, $q = -\frac{i+j}{2}$, $X = \frac{1}{2}i + \frac{1}{2}j$, $Y = -\frac{1}{2}i + \frac{1}{2}j$, $|X| = \frac{1}{\sqrt{2}}$, $|Y| = \frac{1}{\sqrt{2}}$, $X \cdot Y = 0$ and $X \times Y = \frac{1}{2}k$. Hence, we can write

$$\int \exp(-qt)\exp((\alpha-q)t)\ dt$$
$$= \frac{1}{2}\int e^{(q_0+p_0)t}\{\cos((|X|+|Y|)t) + \cos((|X|-|Y|)t)\}\ dt$$
$$+ \frac{-X\cdot Y + X\times Y}{|X||Y|}\frac{1}{2}\int e^{(q_0+p_0)t}\{\cos((|X|-|Y|)t) - \cos((|X|+|Y|)t)\}\ dt$$

$$
\begin{aligned}
&+ \frac{Y}{|Y|}\frac{1}{2}\int e^{(q_0+p_0)t}\{\sin((|X|+|Y|)t) - \sin((|X|-|Y|)t)\}\ dt \\
&+ \frac{X}{|X|}\frac{1}{2}\int e^{(q_0+p_0)t}\{\sin((|X|+|Y|)t) + \sin((|X|-|Y|)t)\}\ dt \\
= &\frac{1}{2}\int (\cos\sqrt{2}t + 1)\ dt + \frac{k}{2}\int (1 - \cos\sqrt{2}t)\ dt \\
&+ \frac{-i+j}{2\sqrt{2}}\int \sin\sqrt{2}t\ dt + \frac{i+j}{2\sqrt{2}}\int \sin\sqrt{2}t\ dt \\
= &\frac{1}{2\sqrt{2}}\sin\sqrt{2}t + \frac{t}{2} - \frac{j}{2}\cos\sqrt{2}t + k\left(\frac{t}{2} - \frac{1}{2\sqrt{2}}\sin\sqrt{2}t\right).
\end{aligned}
$$

Thus, the solution ξ is obtained as follows:

$$
\begin{aligned}
\xi &= \exp(qt)\int \exp(-qt)\exp((\alpha-q)t)\ dt \\
&= \left(\cos\frac{t}{\sqrt{2}} - \frac{i+j}{\sqrt{2}}\sin\frac{t}{\sqrt{2}}\right)\int \exp(-qt)\exp((\alpha-q)t)\ dt \\
&= \frac{1}{2\sqrt{2}}\sin\frac{t}{\sqrt{2}} + \frac{t}{2}\cos\frac{t}{\sqrt{2}} - i\frac{t}{\sqrt{2}}\sin\frac{t}{\sqrt{2}} \\
&\quad + j\frac{1}{2}\cos\frac{t}{\sqrt{2}} + k\left(\frac{t}{2}\cos\frac{t}{\sqrt{2}} - \frac{1}{2\sqrt{2}}\sin\frac{t}{\sqrt{2}}\right) \\
&= t\left(\frac{1}{2}\cos\frac{t}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\sin\frac{t}{\sqrt{2}} + \frac{k}{2}\cos\frac{t}{\sqrt{2}}\right) \\
&\quad + \frac{1}{2}\left(\cos\frac{t}{\sqrt{2}} + \frac{-j-i}{\sqrt{2}}\sin\frac{t}{\sqrt{2}}\right)j \\
&= \left(\frac{(1+k)t}{2} - \frac{j}{2}\right)\left(\cos\frac{t}{\sqrt{2}} + \frac{-j-i}{\sqrt{2}}\sin\frac{t}{\sqrt{2}}\right) \\
&= \left(\frac{(1+k)t}{2} - \frac{j}{2}\right)\exp\left(\frac{-j-i}{2}t\right).
\end{aligned}
$$

4. Conclusion

The set $\mathbb{H}$ of quaternions is generated by $\{1, i, j, k\}$ and $M(\mathbb{C}, 2)$ is the set generated by $\{e_0, e_1, e_2, e_3\}$ with a relationship equal to $\{1, i, j, k\}$. Moreover, $\mathbb{H}$ is isomorphic to $M(\mathbb{C}, 2)$, where each e_r $(r = 0, 1, 2, 3)$ is a matrix with 2×2 complex entities. By substituting the expression in $M(\mathbb{C}, 2)$ for the quaternionic functions of one real variable, this paper defines the quaternionic Wronskian which determine the independence and dependency of solutions of a quaternionic homogeneous second-order equation. This substituting makes it easy to construct the quaternionic Wronskian, and it is easy to use, since the calculation proceeds using the complex-valued function.

In addition, we propose a step of reduction of order applicable to quaternionic differential equations(QDE). This is a method of deriving an applicable solution if QDE corresponds to the form of a homogeneous second-order equation without having to distinguish whether or not this method can be applied.

Appendix 1. Equation (2.5) can be obtained from the following calculation.

$$W = \begin{vmatrix} \cos t - i\sin t & 0 & \cos\sqrt{2}t + i\frac{\sin\sqrt{2}t}{\sqrt{2}} & -\frac{\sin\sqrt{2}t}{\sqrt{2}} \\ 0 & \cos t + i\sin t & \frac{\sin\sqrt{2}t}{\sqrt{2}} & \cos\sqrt{2}t - i\frac{\sin\sqrt{2}t}{\sqrt{2}} \\ -\sin t - i\cos t & 0 & -\sqrt{2}\sin\sqrt{2}t + i\cos\sqrt{2}t & -\cos\sqrt{2}t \\ 0 & -\sin t + i\cos t & \cos\sqrt{2}t & -\sqrt{2}\sin\sqrt{2}t - i\cos\sqrt{2}t \end{vmatrix}.$$

Appendix 2. We show that

$$\frac{d}{dt}\exp(qt) = q\exp(qt).$$

For a quaternion $q = q_0 + X$ with $X = iq_1 + jq_2 + kq_3$, we can write

$$\exp(qt) = e^{q_0 t}(\cos(|X|t) + \frac{X}{|X|}\sin(|X|t)).$$

So, we obtain

$$\begin{aligned}
&\frac{d}{dt}\exp(qt) \\
&= \frac{d}{dt}\left\{e^{q_0 t}(\cos(|X|t) + \frac{X}{|X|}\sin(|X|t))\right\} \\
&= q_0 e^{q_0 t}\left(\cos(|X|t) + \frac{X}{|X|}\sin(|X|t)\right) + e^{q_0 t}(-|X|\sin(|X|t) + X\cos(|X|t)) \\
&= q_0 e^{q_0 t}\left(\cos(|X|t) + \frac{X}{|X|}\sin(|X|t)\right) + e^{q_0 t}X\left(\frac{X}{|X|}\sin(|X|t) + \cos(|X|t)\right) \\
&= (q_0 + X)e^{q_0 t}\left(\cos(|X|t) + \frac{X}{|X|}\sin(|X|t)\right) \\
&= q\exp(qt).
\end{aligned}$$

Appendix 3. Let us consider the expression of $\int e^{qt}e^{pt}\ dt$. For two quaternions $q = q_0+q_1i+q_2j+q_3k$ and $p = p_0+p_1i+p_2j+p_3k$, let $X = q_1i+q_2j+q_3k$ and $Y = p_1i+p_2j+p_3k$. Then we can write $q = q_0+X$ and $p = p_0+Y$. Also, we have

$$|X| = \sqrt{q_1^2+q_2^2+q_3^3},\ |Y| = \sqrt{p_1^2+p_2^2+p_3^3},\ X\cdot Y = q_1p_1+q_2p_2+q_3p_3,$$

and

$$X\times Y = (q_2p_3-q_3p_2)i+(q_3p_1-q_1p_3)j+(q_1p_2-q_2p_1)k.$$

Since the exponential function for quaternions is $\exp(qt) = e^{q_0t}(\cos(|X|t) + \frac{X}{|X|}\sin(|X|t))$, we can express the following:

$$\begin{aligned}\int e^{qt}e^{pt}\ dt = &\int e^{(q_0+p_0)t}\cos(|X|t)\ \cos(|Y|t)\ dt\\ &+\frac{-X\cdot Y+X\times Y}{|X||Y|}\int e^{(q_0+p_0)t}\sin(|X|t)\ \sin(|Y|t)\ dt\\ &+\frac{Y}{|Y|}\int e^{(q_0+p_0)t}\cos(|X|t)\ \sin(|Y|t)\ dt\\ &+\frac{X}{|X|}\int e^{(q_0+p_0)t}\sin(|X|t)\ \cos(|Y|t)\ dt.\end{aligned}$$

Also, each component of the above formulas can be expressed as follows:

$$\begin{aligned}&\int e^{(q_0+p_0)t}\cos(|X|t)\ \cos(|Y|t)\ dt\\ &=\frac{1}{2}\int e^{(q_0+p_0)t}\{\cos((|X|+|Y|)t)+\cos((|X|-|Y|)t)\}\ dt,\end{aligned}$$

$$\begin{aligned}&\int e^{(q_0+p_0)t}\sin(|X|t)\ \sin(|Y|t)\ dt\\ &=\frac{1}{2}\int e^{(q_0+p_0)t}\{\cos((|X|-|Y|)t)-\cos((|X|+|Y|)t)\}\ dt,\end{aligned}$$

$$\begin{aligned}&\int e^{(q_0+p_0)t}\cos(|X|t)\ \sin(|Y|t)\ dt\\ &=\frac{1}{2}\int e^{(q_0+p_0)t}\{\sin((|X|+|Y|)t)-\sin((|X|-|Y|)t)\}\ dt\end{aligned}$$

and

$$\begin{aligned}&\int e^{(q_0+p_0)t}\sin(|X|t)\ \cos(|Y|t)\ dt\\ &=\frac{1}{2}\int e^{(q_0+p_0)t}\{\sin((|X|+|Y|)t)+\sin((|X|-|Y|)t)\}\ dt.\end{aligned}$$

Hence, we can obtain

$$
\begin{aligned}
&\int e^{qt}e^{pt}\ dt\\
&= \frac{1}{2}\int e^{(q_0+p_0)t}\{\cos((|X|+|Y|)t)+\cos((|X|-|Y|)t)\}\ dt\\
&\quad + \frac{-X\cdot Y + X\times Y}{|X||Y|}\frac{1}{2}\int e^{(q_0+p_0)t}\{\cos((|X|-|Y|)t)-\cos((|X|+|Y|)t)\}\ dt\\
&\quad + \frac{Y}{|Y|}\frac{1}{2}\int e^{(q_0+p_0)t}\{\sin((|X|+|Y|)t)-\sin((|X|-|Y|)t)\}\ dt\\
&\quad + \frac{X}{|X|}\frac{1}{2}\int e^{(q_0+p_0)t}\{\sin((|X|+|Y|)t)+\sin((|X|-|Y|)t)\}\ dt.
\end{aligned}
$$

Moreover, since the following formulas

$$
\int e^{\alpha t}\cos(\beta t)\ dt = \frac{1}{\alpha^2+\beta^2}\ e^{\alpha t}\Big(\alpha\cos(\beta t)+\beta\sin(\beta t)\Big)
$$

and

$$
\int e^{\alpha t}\sin(\beta t)\ dt = \frac{1}{\alpha^2+\beta^2}\ e^{\alpha t}\Big(\alpha\sin(\beta t)-\beta\cos(\beta t)\Big)
$$

are satisfied, we obtain

$$
\begin{aligned}
&\int e^{(q_0+p_0)t}\cos((|X|+|Y|)t)\ dt\\
&= \frac{e^{(q_0+p_0)t}}{(q_0+p_0)^2+(|X|+|Y|)^2}\\
&\quad \times \Big\{(q_0+p_0)\cos((|X|+|Y|)t)+(|X|+|Y|)\sin((|X|+|Y|)t)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
&\int e^{(q_0+p_0)t}\cos((|X|-|Y|)t)\ dt\\
&= \frac{e^{(q_0+p_0)t}}{(q_0+p_0)^2+(|X|-|Y|)^2}\\
&\quad \times \Big\{(q_0+p_0)\cos((|X|-|Y|)t)+(|X|-|Y|)\sin((|X|-|Y|)t)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
&\int e^{(q_0+p_0)t}\sin((|X|+|Y|)t)\ dt\\
&= \frac{e^{(q_0+p_0)t}}{(q_0+p_0)^2+(|X|+|Y|)^2}\\
&\quad \times \Big\{(q_0+p_0)\sin((|X|+|Y|)t)-(|X|+|Y|)\cos((|X|+|Y|)t)\Big\}
\end{aligned}
$$

and

$$\int e^{(q_0+p_0)t}\sin((|X|-|Y|)t)\ dt$$
$$= \frac{e^{(q_0+p_0)t}}{(q_0+p_0)^2+(|X|-|Y|)^2}$$
$$\times \Big\{(q_0+p_0)\sin((|X|-|Y|)t)-(|X|-|Y|)\cos((|X|-|Y|)t)\Big\}.$$

Thus, we can rewrite $\int e^{qt}e^{pt}\ dt$ as follows:

$$\int e^{qt}e^{pt}\ dt$$
$$= \frac{|X||Y|+X\cdot Y-X\times Y}{2|X||Y|}\frac{e^{(q_0+p_0)t}}{(q_0+p_0)^2+(|X|+|Y|)^2}$$
$$\times \Big((q_0+p_0)\cos((|X|+|Y|)t)+(|X|+|Y|)\sin((|X|+|Y|)t)\Big)$$
$$+ \frac{|X||Y|-X\cdot Y+X\times Y}{2|X||Y|}\frac{e^{(q_0+p_0)t}}{(q_0+p_0)^2+(|X|-|Y|)^2}$$
$$\times \Big((q_0+p_0)\cos((|X|-|Y|)t)+(|X|-|Y|)\sin((|X|-|Y|)t)\Big)$$
$$+ \frac{X|Y|+|X|Y}{2|X||Y|}\frac{e^{(q_0+p_0)t}}{(q_0+p_0)^2+(|X|+|Y|)^2}$$
$$\times \Big((q_0+p_0)\sin((|X|+|Y|)t)-(|X|+|Y|)\cos((|X|+|Y|)t)\Big)$$
$$+ \frac{X|Y|-|X|Y}{2|X||Y|}\frac{e^{(q_0+p_0)t}}{(q_0+p_0)^2+(|X|-|Y|)^2}$$
$$\times \Big((q_0+p_0)\sin((|X|-|Y|)t)-(|X|-|Y|)\cos((|X|-|Y|)t)\Big)$$
$$= \frac{e^{(q_0+p_0)t}\cos((|X|+|Y|)t)}{2|X||Y|\{(q_0+p_0)^2+(|X|+|Y|)^2\}}$$
$$\times \Big((|X||Y|+X\cdot Y-X\times Y)(q_0+p_0)-(X|Y|+|X|Y)(|X|+|Y|)\Big)$$
$$+ \frac{e^{(q_0+p_0)t}\sin((|X|+|Y|)t)}{2|X||Y|\{(q_0+p_0)^2+(|X|+|Y|)^2\}}$$
$$\times \Big((|X||Y|+X\cdot Y-X\times Y)(|X|+|Y|)+(X|Y|+|X|Y)(q_0+p_0))\Big)$$
$$+ \frac{e^{(q_0+p_0)t}\cos((|X|-|Y|)t)}{2|X||Y|\{(q_0+p_0)^2+(|X|-|Y|)^2\}}$$
$$\times \Big((|X||Y|-X\cdot Y+X\times Y)(q_0+p_0)-(X|Y|-|X|Y)(|X|-|Y|)\Big)$$

$$+ \frac{e^{(q_0+p_0)t}\sin((|X|-|Y|)t)}{2|X||Y|\{(q_0+p_0)^2+(|X|-|Y|)^2\}}$$
$$\times \Big((|X||Y| - X\cdot Y + X\times Y)(|X|-|Y|) + (X|Y| - |X|Y)(q_0+p_0)\Big).$$

Letting,

$$C_1 = \frac{(|X||Y| + X\cdot Y - X\times Y)(q_0+p_0) - (X|Y| + |X|Y)(|X|+|Y|)}{2|X||Y|\{(q_0+p_0)^2+(|X|+|Y|)^2\}},$$

$$C_2 = \frac{(|X||Y| + X\cdot Y - X\times Y)(|X|+|Y|) + (X|Y| + |X|Y)(q_0+p_0)}{2|X||Y|\{(q_0+p_0)^2+(|X|+|Y|)^2\}},$$

$$C_3 = \frac{(|X||Y| - X\cdot Y + X\times Y)(q_0+p_0) - (X|Y| - |X|Y)(|X|-|Y|)}{2|X||Y|\{(q_0+p_0)^2+(|X|-|Y|)^2\}}$$

and

$$C_4 = \frac{(|X||Y| - X\cdot Y + X\times Y)(|X|-|Y|) + (X|Y| - |X|Y)(q_0+p_0)}{2|X||Y|\{(q_0+p_0)^2+(|X|-|Y|)^2\}},$$

then, we have

$$\int e^{qt}e^{pt}\,dt = e^{(q_0+p_0)t}\{C_1\cos((|X|+|Y|)t) + C_2\sin((|X|+|Y|)t) + C_3\cos((|X|-|Y|)t) + C_4\sin((|X|-|Y|)t)\}.$$

Appendix 4. Let's look at the expression $e^{qt}\int e^{-qt}e^{(\alpha-q)t}\,dt$. We replace the preceding p with $\alpha - q$, that is, replace p_0 with $a_0 - q_0$ and Y with $i(a_1-q_1) + j(a_2-q_2) + k(a_3-q_3)$, then we have

$$\begin{aligned} e^{qt}\int e^{-qt}e^{(\alpha-q)t}\,dt = {} & \frac{C_1}{2}(\cos(|Y|t) + \cos((2|X|+|Y|)t)) \\ & + \frac{C_2}{2}(\sin((2|X|+|Y|)t) + \sin(|Y|t)) \\ & + \frac{C_3}{2}(\cos(|Y|t) + \cos((2|X|-|Y|)t)) \\ & + \frac{C_4}{2}(\sin((2|X|-|Y|)t) - \sin(|Y|t)) \end{aligned}$$

$$+\frac{X}{|X|}\{\frac{C_1}{2}(\sin(|Y|t)-\sin((2|X|+|Y|)t))$$
$$+\frac{C_2}{2}(\cos((2|X|+|Y|)t)-\cos(|Y|t))$$
$$-\frac{C_3}{2}(\sin((2|X|-|Y|)t)+\sin(|Y|t))$$
$$+\frac{C_4}{2}(\cos((2|X|-|Y|)t)-\cos(|Y|t))\}.$$

Putting

$$A_\pm=\frac{1}{2}(C_1\pm\frac{X}{|X|}C_2),\ B_\pm=\frac{1}{2}(C_3\pm\frac{X}{|X|}C_4),$$
$$D_1=A_-+B_-,\ D_2=\frac{X}{|X|}(A_--B_-),\ D_3=A_+,$$
$$D_4=-\frac{X}{|X|}A_+,\ D_5=B_+,\ D_6=-\frac{X}{|X|}B_+,$$

then, we finally obtain

$$e^{qt}\int e^{-qt}e^{(\alpha-q)t}\,dt=\cos(|Y|t)D_1+\sin(|Y|t)D_2$$
$$+\cos((2|X|+|Y|)t)D_3+\sin((2|X|+|Y|)t)D_4$$
$$+\cos((2|X|-|Y|)t)D_5+\sin((2|X|-|Y|)t)D_6.$$

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