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SOME RESULTS CONCERNING CLOSURE PRODUCTS ON MEROMORPHIC HURWTIZ-ZETA FUNCTION DEFINED BY LINEAR OPERATOR

Hishyar Kh. Abdullah

Department of Mathematics College Sciences
University of Sharjah, Sharjah, P.O. Box 27272, U.A.E.
e-mail: hishyyar@sharjah.ac.ae

Abstract. In this paper we establish some results concerning closure products on class of Hurwitz-Lerch-Zeta functions meromorphic functions in terms of the Srivastava-Attiya operator. The generalization of the class of univalent functions have been explored and the proprieties such as disterortion theorem and radii theorem are the main interests of solving problems. In addition, some interesting properties depending on some integral representations are discussed.

1. Introduction

The theory of analytic univalent function is a classical problem of complex analysis which belongs to a geometric function theory.

A large number of generalization of the class of univalent functions have been explored and proprieties such as distortion theorem and radii theorem are the main interests of solving problems. To date, various methods have been used such as method of differential subordinations, method of differential inequalities and method of arising from the convolution theory. One of the important studies in the univalent functions is the integrals operator.

Let M denote the class of meromorphic functions $f(z)$ defined by

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$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n, \tag{1.1}$$

which is analytic in the punctured unit disk $U^*=\{z\in\mathbb{C}:0<|z|<1\}$. For $0\leq\beta<1$, we denote by $S^*(\beta)$ and $k(\beta)$, the subclasses of M consisting of all meromorphic functions which are respectively, starlike of order β and convex of order β in U^*.

For functions $f_j(z)$ $(j=1,2)$ defined by

$$f_j(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_{n,j}z^n, \tag{1.2}$$

we denote the hadamard product (or convolution) of f_1 and f_2 by

$$(f_1*f_2)(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_{n,1}a_{n,2}z^n. \tag{1.3}$$

Now, we recall a general Hurwitz-Lerch Zeta function which, as many authors do, see for example ([1],[6],[8]-[11]).

$$\Phi(z,t,a)=\frac{1}{a^t}+\sum_{n=1}^{\infty}\frac{z^n}{(n+a)^s}, \tag{1.4}$$

where $Z_0^{-1}=\{0,-1,-2,...\}$, $U=\{z\in\mathbb{C}:|z|<1\}$, $\partial U=\{z\in\mathbb{C}:|z|=1\}$. $a\in\mathbb{C}\backslash Z_0^-$, $t\in\mathbb{C}$ when $z\in U=U^*\cup\{0\}$; $\mathbb{R}(t)>1$ when $z\in\partial U$.

Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z,t,a)$ can be found in the recent investigations by Choi and Srivastava [2], Ferreira and Lopez [3], Garg et al. [5], Lin and Srivastava [6], and Lin et al. [7] and others. Recent results on $\Phi(z,t,a)$, can be found in the expositions [17], [18].

In [4] (see also [14] and [15]), Ghanim defined

$$\begin{aligned} G_{t,a}(z) &= (1+a)^s\left[\Phi(z,t,a)-a^s+\frac{1}{z\,(1+a)^s}\right] \\ &= \frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{1+a}{n+a}\right)^s z^n;\ (z\in U^*). \end{aligned} \tag{1.5}$$

Corresponding to the functions $G_{s,a}(z)$ and using the hadamard product for $f(z)\in M$, we define a new linear operator.

$$L_a^s(\alpha,\beta)f(z) = \Phi(z,t,a) * G_{t,a}(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{(\alpha)_{n+1}}{(\beta)_{n+1}} C_a^s(n) a_n z^n; \quad (z \in U^*). \tag{1.6}$$

for $\beta \neq 0,-1,-2,...$ and $\alpha \in \mathbb{C}\setminus\{0\}$. Here $C_a^s(n) = \left(\frac{1+a}{n+1+a}\right)^s$ and, unless indicated otherwise, throughout this paper the parameter a is constrained to $a \in \mathbb{C}\setminus\left\{Z_0^-\right\}$, and s belongs $\mathbb{C}$. Recently Meromorphic functions in terms of Gaussian and generalized hypergeometric functions were considered by many authors see for example ([11]-[18] and the authors therein).

Clearly, it follows from (1.6) that

$$z\left(L_a^s(\alpha,\beta)f(z)\right)' = \alpha\left(L_a^s(\alpha+1,\beta)f(z)\right) - (\alpha+1)\left(L_a^s(\alpha,\beta)f(z)\right). \tag{1.7}$$

Now, for univalently meromorphic function $f(z) \in M$ the normalization

$$z^2 f(z)\mid_{z=0} = 0 \quad \text{and} \quad zf(z)\mid_{z=0} = 1 \tag{1.8}$$

is classical. We can obtain interesting results by applying Montel's normalization [8] of the form

$$z^2 f(z)\mid_{z=0} = 0 \quad \text{and} \quad zf(z)\mid_{z=\rho} = 1, \tag{1.9}$$

where ρ is a fixed point from the unit disk U^*. Note that if $\rho = 0$ the normalization (1.9) is the classical normalization (1.8).

Meromorphic multivalent functions have been studied by Mogra et al. [8], Uralegaddi and Somanatha [18], Srivastava et al. [15], Srivastava and Choi [12].

In this article, we define the following new subclass $M_a^s(\alpha,\beta)$ of meromorphic starlike function in the parabolic region of function M by making use of the generalized operator L_a^t with Montel's normalization. We study its characteristic properties: for example coefficient inequalities, growth and distortion inequalities, radii of starlikeness are obtained, we also establish some new results concerning the convolution products.

For a fixed parameters $\alpha \geq \frac{1}{2+\beta}$; $0 \leq \beta < 1$, denote the set $M_a^s(\alpha,\beta)$ consisting of those meromorphic function $f(z) \in M$ with two fixed points (or classical normalization) which satisfy

$$\left|\frac{z\left(L_a^s(\alpha,\beta)f(z)\right)'}{L_a^s(\alpha,\beta)f(z)} + \alpha + \alpha\beta\right| \leq \mathbb{R}\left\{\frac{-z\left(L_a^s(\alpha,\beta)f(z)\right)'}{L_a^s(\alpha,\beta)f(z)} + \alpha - \alpha\beta\right\}, (n \in \mathbb{N}_0), \tag{1.10}$$

where $L_a^t(\alpha,\beta)f(z)$ given by (1.6). In addition the text further, more let the subclass $M_a^s(\alpha,\beta)$ satisfying the condition (1.10) with Montel's (1.9) is denoted by $M_a^s(\alpha,\beta,\rho)$.

2. Main results

Now we are in a position to prove the main theorems:

Theorem 2.1. *Let the function $f(z)$ given by* (1.1) *be in $M_a^s(\alpha,\beta)$. Then the integral operator*

$$F(z) = c\int_0^1 u^c f(uz)\ du\ ;(0<u\le 1,\ \ c\in(0,\infty))$$

is in $M_a^s(\alpha,\delta)$, where

$$\delta = \frac{1}{\alpha}\left\{\frac{(c+n+1)\,(n-1-\alpha\beta)+c\,(n-1)\,(1-\alpha\beta)}{(c+n+1)\,(n-1+\alpha\beta)+c\,(1-\alpha\beta)}\right\}.$$

Proof. Let $f(z)\in M_a^s(\alpha,\beta)$. Then

$$\begin{aligned} F(z) &= c\int_0^1 u^c f(uz)\ du\ = c\int_0^1\left\{\frac{u^{c-1}}{z}+\sum_{n=1}^{\infty}a_n u^{n+c}z^n\right\}du \\ &= \left\{\left[\frac{u^c}{cz}\right]_0^1+\left[\sum_{n=1}^{\infty}a_n\frac{u^{n+c+1}}{n+c+1}\right]_0^1\right\} = \frac{1}{z}+\sum_{n=1}^{\infty}\frac{ca_n}{n+c+1}z^n. \end{aligned}$$

It is sufficient to prove that

$$\sum_{n=1}^{\infty}\frac{(n-1+\alpha\delta)}{(1-\alpha\delta)}\left(\frac{c}{n+c+1}\right)a_n\le 1, \tag{2.1}$$

is satisfied if

$$\frac{(n-1+\alpha\delta)}{(1-\alpha\delta)}\left(\frac{c}{n+c+1}\right)\le\frac{(n-1+\alpha\beta)}{(1-\alpha\beta)}.$$

Solving this inequality for δ we will get

$$\delta = \frac{1}{\alpha}\left\{\frac{(c+n+1)\,(n-1-\alpha\beta)+c\,(n-1)\,(1-\alpha\beta)}{(c+n+1)\,(n-1+\alpha\beta)+c\,(1-\alpha\beta)}\right\} = F(n).$$

Hence

$$F(n+1)-F(n) = \frac{1}{\alpha}\left[\frac{c\,(1-\alpha\beta)}{(c+n-1+\alpha\beta)\,(c+n+\alpha\beta)}\right] > 0,$$

for every n, and this completes the proof. $\square$

Theorem 2.2. *Let the function $f(z)$ given by (1.1) be in $M_a^s(\alpha,\beta)$. Then*

$$F(z)=\frac{1}{c}\left[(c+1)\,f(z)+zf'(z)\right]=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{c+n+1}{c}a_nz^n,\ \ c>0,$$

is in $M_a^s(\alpha,\beta)$ for $|z|\le r(\alpha,\beta,\delta)$, where

$$r(\alpha,\beta,\delta)=\inf_n\left(\frac{c\,(1-\alpha\delta)\,(n-1+\alpha\beta)}{(1-\alpha\beta)\,(c+n+1)\,(n-1+\alpha\delta)}\right),\ \ n=1,2,3,...$$

Proof. Let

$$w=\left\{\frac{-z\,(L_a^s(\alpha,\beta)f(z))'}{L_a^s(\alpha,\beta)f(z)}\right\}.$$

Then it is sufficient to prove that

$$\left|\frac{w+1}{w-1+\alpha\beta}\right|\le 1. \tag{2.2}$$

By simplifying inequality (2.2) we find this is satisfied if

$$\sum_{n=1}^{\infty}\frac{(n-1+\alpha\delta)\,(c+n+1)}{c\,(1-\alpha\beta)}a_n\,|z|^n\le 1. \tag{2.3}$$

Since $f\in M_a^s(\alpha,\beta)$, by using Theorem 1 of Abdullah [1], we get

$$\sum_{n=1}^{\infty}\frac{(n-1+\alpha\beta)}{(1-\alpha\beta)}\,|a_n|\le 1.$$

Inequality (2.3) is satisfied if

$$\sum_{n=1}^{\infty}\frac{(n-1+\alpha\delta)\,(c+n+1)}{c\,(1-\alpha\beta)}\,|z|^n\le\sum_{n=1}^{\infty}\frac{(n-1+\alpha\beta)}{(1-\alpha\beta)},$$

solving this inequality for $|z|^n$ we get

$$|z|^n\le\frac{c\,(n-1+\alpha\beta)\,(1-\alpha\delta)}{(1-\alpha\beta)\,(n-1+\alpha\delta)\,(c+n+1)},$$

then we have

$$|z|\le\inf_n\left\{\frac{c\,(n-1+\alpha\beta)\,(1-\alpha\delta)}{(1-\alpha\beta)\,(n-1+\alpha\delta)\,(c+n+1)}\right\}^{\frac{1}{n}}.$$

This completes the proof. $\square$

Theorem 2.3. *Let the function* $f_i(z)$ $,(i=1,2,3,...,m)$ *defined by*

$$f_i(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_{n,i}z^n,\ \ (i=1,2,3,...,m,\ n\geq 1)\,,$$

be in class $M_a^s(\alpha,\beta,\rho)$. *Then the arithmetic mean of* $f_i(z)$ $,(i=1,2,3,...,m)$ *is defined by*

$$h(z)=\frac{1}{m}\sum_{i=1}^{m}f_i(z),$$

is also in the class $M_a^s(\alpha,\beta,\rho)$.

Proof. Since $f_i(z)\in M_a^s(\alpha,\beta,\rho),(i=1,2,3,...,m)$, by using Theorem 2.3 of Vijaya and Kasthuri [19] for $p=1$, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\left[d_n+(1-\alpha\beta)\,\rho^n\right]\left(\frac{1}{m}\sum_{i=1}^{m}a_{n,i}\right) &= \frac{1}{m}\sum_{i=1}^{m}\left(\sum_{n=1}^{\infty}\left[d_n+(1-\alpha\beta)\,\rho^n\right]a_{n,i}\right)\\ &\leq \frac{1}{m}\sum_{i=1}^{m}(1-\alpha\beta)\leq(1-\alpha\beta)\,.\end{aligned}$$

Which means that $h(z)\in M_a^s(\alpha,\beta,\rho)$. This completes the proof. □

Theorem 2.4. *Let the function* $f_j(z)$, $(j=1,2)$ *defined by*

$$f_j(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_{n,j}\ z^n,\ \ (a_{n,j}\ \geq 0,\ j=1,2)\,,$$

be in class $M_a^s(\alpha,\beta,\rho)$. *Then the weighted mean of* $f_j(z)$, $(j=1,2)$ *defined by*

$$W_c=\frac{1}{2}\left[(1-c)\,f_1(z)+(1+c)\,f_2(z)\right]\ \ , \tag{2.4}$$

is also in the class $M_a^s(\alpha,\beta,\rho)$.

Proof. Since $f_j(z)=\frac{1}{z}+\sum\limits_{n=1}^{\infty}a_{n,j}z^n$ is in the class $M_a^s(\alpha,\beta,\rho)$ for $(j=1,2)$, by (2.4), we have

$$W_c=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{1}{2}\left[(1-c)\,a_{n,1}+(1+c)\,a_{n,2}\right]z^n\leq 1.$$

Using Theorem 2.3 of Vijaya and Kasthuri [19] for $p=1$ we get

$$\sum_{n=1}^{\infty}\frac{\left[d_n+(1-\alpha\beta)\,\rho^n\right]}{(1-\alpha\beta)}\left|a_{n,1}\right|\leq 1 \tag{2.5}$$

and

$$\sum_{n=1}^{\infty} \frac{[d_n + (1-\alpha\beta)\,\rho^n]}{(1-\alpha\beta)} |a_{n,2}| \leq 1. \tag{2.6}$$

Using (2.5) and (2.6) in (2.4), we have

$$W_c = \frac{1}{2}(1-c)(1-\alpha\beta) + \frac{1}{2}(1+c)(1-\alpha\beta) \leq (1-\alpha\beta). \tag{2.7}$$

Therefore $W_c \in M_a^s(\alpha, \beta, \rho)$. This completes the proof. □

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