EXPLICIT ITERATIVE METHODS FOR MAXIMAL MONOTONE OPERATORS IN HILBERT SPACES

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Abstract. In this paper, we introduce new explicit iterative methods for solving a variational inequality problem over the set of zeros for a maximal monotone operator in Hilbert spaces. By using two resolvents of the monotone operator at each iterate, we prove strong convergence of the methods under a general condition on resolvent parameter. As special cases, new contraction and generalized proximal point methods are obtained. An example is given for illustration and comparison.

1. Introduction

Let H be a real Hilbert space with inner product and norm denoted, respectively, by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. Let A be a maximal monotone operator in H.

In this paper, we assume that the set of zeros, $\Gamma := \{p \in \mathcal{D}(A) : \ 0 \in Ap\}$, is nonempty, where $\mathcal{D}(A)$ denotes the domain of A.

Finding a zero of a maximal monotone operator A, that is,

$$\text{finding a point} \quad p \in \mathcal{D}(A) \quad \text{such that } 0 \in Ap, \tag{1.1}$$

is an important part of the theory of monotone operators. A fundamental method for finding a zero point of a maximal monotone operator A in Hilbert space H, we can cite the proximal point one [27]. This method generates a sequence $\{x^k\}$ defined for each $k \geq 1$

$$x^{k+1} = J_{r_k}^A x^k + e^k \quad \text{or} \quad x^{k+1} = J_{r_k}^A(x^k + e^k), \tag{1.2}$$

where $x^1 \in H$, $J_{r_k}^A = (I + r_k A)^{-1}$, I is the identity mapping of H, $\{r_k\}$ is a sequence of real numbers such that $r_k \geq \varepsilon > 0$ for all $k \geq 1$ and e^k is an error vector satisfying

$$\text{either} \quad \|x^{k+1} - J_{r_k}^A x^k\| \leq \varepsilon_k \quad \text{with} \quad \sum_{k=1}^{\infty} \varepsilon_k < \infty$$

$$\text{or} \quad \|x^{k+1} - J_{r_k}^A x^k\| \leq \eta_k \|x^{k+1} - x^k\| \quad \text{with} \quad \sum_{k=1}^{\infty} \eta_k < \infty. \tag{1.3}$$

Methods (1.2)-(1.3) converge only weakly to a zero of A in infinite-dimensional Hilbert spaces, in general(see, [15]). In order to have a strong convergence sequence $\{x^k\}$ from the method, several modifications of (1.2) were proposed in [18, 21, 28, 39]. Kamimura et al. [18] introduced a method, in there two sequences $\{x^k\}$ and $\{y^k\}$ are built from a starting point $x^1 \in H$ as follows:

$$\begin{cases} y^k \approx J_{r_k}^A x^k, \quad \|y^k - J_{r_k}^A x^k\| \leq \delta_k, \\ x^{k+1} = t_k u + (1 - t_k) y^k, \quad k \geq 1, \end{cases} \tag{1.4}$$

where u is a fixed point in H. The sequence $\{x^k\}$ so generated, as $k \to \infty$, is strongly convergent to $P_\Gamma u$, the metric projection of u on the set Γ, under the following conditions:

(C1) $t_k \in (0, 1)$ for all $k \geq 1$, $\lim_{k\to\infty} t_k = 0$ and $\sum_{k=1}^{\infty} t_k = \infty$;
(C2) $r_k \in (0, \infty)$ for all $k \geq 1$ and $\lim_{k\to\infty} r_k = \infty$; and
(C3) $\sum_{k=1}^{\infty} \delta_k < \infty$.

At this time, Solodov et al. [28] suggested another modification of (1.2). Their method generates a sequence $\{x^k\}$, satisfying $x^{k+1} = P_{H^k \cap W^k} x^1$, where

$$\begin{cases} H^k = \{z \in H : \langle z - y^k, v^k \rangle \leq 0\}, \\ W^k = \{z \in H : \langle z - x^k, x^1 - x^k \rangle \leq 0\}, \end{cases}$$

and $(y^k, v^k) \in H \times H$ is an inexact solution of the inclusion $0 \in Ax + \mu(x - x^k)$.

Next, Xu [39] extended the prox-Tikhonov method of Lehdili et al. [21] in the following way

$$x^{k+1} = J^A_{r_k}(t_k u + (1 - t_k)x^k + e^k). \tag{1.5}$$

Further, Boikanyo et al. [1] showed that (1.5) is equivalent to

$$y^{k+1} = t_k u + (1 - t_k)J^A_{r_k}y^k + e^k, \tag{1.6}$$

and proved a strong convergence result for $\{y^k\}$, defined by (1.6), to $P_\Gamma u$, if there hold conditions (C1), (C2) and

(C3′) either (C3) with $\delta_k = \|e^k\|$ or $\lim_{k\to\infty}(\|e^k\|/t_k) = 0$.

Recently, Tian et al. [33] proved strong convergence of (1.6) under conditions (C1), (C3′) and

(C2′) $\liminf_{k\to\infty} r_k > 0$.

An extension of (1.6), the contraction proximal point method, was introduced by Yao et al. [42] and defined by

$$x^{k+1} = t_k u + \beta_k x^k + \gamma_k J^A_{r_k}x^k + e^k, \tag{1.7}$$

where the parameters t_k, β_k, $\gamma_k \in (0, 1)$ and $t_k + \beta_k + \gamma_k = 1$ for all $k \geq 1$. They proved that $\{x^k\}$, defined by (1.7), converges strongly to $P_\Gamma u$, if there hold conditions (C1), (C3) with $\delta_k = \|e^k\|$,

(C2″) $r_k \geq \varepsilon$, $r_{k+1} - r_k \to 0$, and
(C4) $\beta_k \in [a, b] \subset (0, 1)$ for all $k \geq 1$,

that was dropped by Wang et al. [36].

In [42], Yao et al. studied also another modification of (1.2), Eckstein et al. generalized proximal point method [13],

$$x^{k+1} = (1 - \beta_k)x^k + \beta_k J^A_{r_k}x^k + e^k. \tag{1.8}$$

They showed that the sequence $\{x^k\}$, defined by (1.8), converges weakly to a point in Γ, under conditions (C2″), (C3), and (C4).

The above methods with similar conditions on r_k and the second condition on errors e^k in (1.3) or their improvements were investigated in [2, 7, 12, 16, 34, 37] and references therein.

Clearly, if r_k satisfies condition (C2) or (C2′), then $r_k \geq \varepsilon$ for some positive constant ε and for all $k \geq 1$. Consequently, $\sum_{k=1}^{\infty} r_k = +\infty$.

Very recently, Buong et al. [6] proposed the method

$$x^{k+1} = J^A_{r_1}J^A_{r_2}\cdots J^A_{r_k}\big(t_k u + (1 - t_k)x^k + e^k\big) \tag{1.9}$$

and its equivalent form

$$z^{k+1} = t_k u + (1 - t_k) J_{r_1}^A J_{r_2}^A \cdots J_{r_k}^A z^k + e^k, \tag{1.10}$$

strong convergence of which is guaranteed by conditions (C1), (C3′), and

(C2‴) $\sum_{k=1}^{\infty} r_k < +\infty$.

Thus, $r_k \to 0$ as $k \to \infty$. Obviously, these methods are very complicated in computation. So, two questions are naturally posed:

(1) *Can we replace the composition* $J_{r_1}^A J_{r_2}^A \cdots J_{r_k}^A$ *in* (1.9) *and* (1.10) *by a simpler one?*

(2) *Is it possible, instead of* (C2), (C2′) *and* $r_k \to 0$, *to use a general condition on* r_k, *that is,*

(C2$^+$) $\{r_k\}$ *is any sequence of numbers in* $(0, \infty)$?

It means that $\{r_k\}$ either is bounded or contains a subsequence, converging to 0, or/and a subsequence, converging to $+\infty$. In order to answer the above questions, we introduce new modifications of (1.2), simple variants of (1.9) and (1.10),

$$x^{k+1} = J_{r_k}^A J_c^A (t_k' u + (1 - t_k') x^k) + e^k \tag{1.11}$$

and

$$x^{k+1} = t_k' u + (1 - t_k')(J_{r_k}^A J_c^A x^k + e^k), \tag{1.12}$$

where $J_c^A = (I + cA)^{-1}$ and c is any fixed positive real number, and the new following method

$$x^{k+1} = t_k' u + \beta_k' J_c^A x^k + \gamma_k' J_{r_k}^A x^k + e^k \tag{1.13}$$

with similar conditions on t_k', β_k' and γ_k' as the above on t_k, β_k and γ_k.

We will show that methods (1.11)–(1.13) are particular cases of the following methods,

$$z^{k+1} = J_{r_k}^A J_c^A (I - t_k \mu F) z^k + e^k \tag{1.14}$$

and

$$z^{k+1} = (1 - \beta_k)(I - t_k \mu F) J_c^A z^k + \beta_k J_{r_k}^A z^k + e^k, \tag{1.15}$$

proposed to solve a problem of finding a point

$$p_* \in \Gamma \quad \text{such that} \quad \langle F p_*, p_* - p \rangle \leq 0, \quad \forall p \in \Gamma, \tag{1.16}$$

where $\mu \in (0, 2\eta/L^2)$ is a constant and $F : H \to H$ is an η-strongly monotone and L-Lipschitz continuous operator with $\eta, L > 0$. The problem has been recently studied in [6] and references therein.

We will prove that the sequence $\{z^k\}$, defined by (1.14) or (1.15), converges strongly to the point p_* in (1.16), under several conditions from (C1), (C2$^+$), (C3$'$) and (C4).

The paper is organized as follows. In Sec. 2, we list some related facts that will be used in our results. In Sec. 3, we prove strong convergence of our main methods and show that their special cases are new contraction and generalized proximal point methods, that converge strongly to a zero under a general condition on the resolvent parameter. Sec. 4 is devoted for giving a numerical example in order to illustrate the convergence analysis of the proposed methods.

2. Preliminaries

In this section, we introduce some mathematical symbols, definitions, and lemmas which can be used in the proof of our main result.

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$. In what follows, we write $x^k \rightharpoonup x$ to indicate that the sequence $\{x^k\}$ converges weakly to x while $x^k \to x$ indicate that the sequence $\{x^k\}$ converges strongly to x.

First, we know that, for any Hilbert space H,

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H.$$

Let $F : H \to H$ be a mapping. F is said to be L-Lipschitz continuous and η-strongly monotone when the following conditions are satisfied:

$$\|Fx - Fy\| \leq L\|x - y\| \quad \text{and} \quad \langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2$$

for all $x, y \in H$, where L and η are some positive constants. F is said to be contraction operator, if $0 \leq L < 1$ and nonexpansive, if $L = 1$.

Lemma 2.1. ([40]) *Let H be a real Hilbert space and let F be an η-strongly monotone and L-Lipschitz continuous operator on H with some positive constants η and L. Then, for a fixed number $\mu \in (0, 2\eta/L^2)$ and any $t \in (0, 1)$, $I - t\mu F$ is a contraction with contractive constant $1 - t\tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu L^2)}$.*

Let C be a nonempty, closed and convex subset of H. We know that, for each $x \in H$, there is a unique $P_C x \in C$ such that

$$\|x - P_C x\| = \inf_{u \in C} \|x - u\| \tag{2.1}$$

and the mapping $P_C : H \to C$ defined by (2.1) is called the metric projection from H onto C. Moreover, we have

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \quad \forall x \in H, \ y \in C, \tag{2.2}$$

(see, for example, [14, Section 3]).

We introduce some definitions and propositions about set–valued mappings. Let A be a set–valued operator of H into 2^H with domain

$$\mathcal{D}(A) = \{x \in H : \ A(x) \neq \emptyset\},$$

range

$$\mathcal{R}(A) = \cup_{x \in \mathcal{D}(A)} Ax,$$

and the inverse of A is

$$A^{-1}(y) = \{x \in H : y \in A(x)\}.$$

$A : H \to 2^H$ is said to be

(i) monotone if

$$\langle u - v, x - y \rangle \geq 0 \quad \forall x, y \in \mathcal{D}(A), \ u \in A(x), \ v \in A(y);$$

(ii) maximal monotone if it is monotone and the graph

$$\mathcal{G}(A) = \{(x, y) \in H \times H : \ x \in \mathcal{D}(A), \ y \in A(x)\}$$

of A is not properly contained in the graph of any other monotone operator on $\mathcal{D}(A)$.

For a monotone mapping A, we define its resolvent J_r^A by

$$J_r^A := (I + rA)^{-1} : \mathcal{R}(I + rA) \subset H \to \mathcal{D}(A),$$

where $r > 0$, I is the identity operator on H.

A fixed point of the mapping $F : C \to C$ is a point $x \in C$ such that $Fx = x$. The set of all fixed points of the mapping F is denoted by $\text{Fix}(F)$.

Lemma 2.2. ([24]) *Let H be a real Hilbert space. If $A : H \to H$ is a maximal monotone operator, then*

(i) *J_r^A is nonexpansive, single-valued mapping and $\text{Fix}(J_r^A) = A^{-1}(0)$ for each $r > 0$, and*

(ii) *$\|J_r^A x - p\|^2 \leq \|x - p\|^2 - \|J_r^A x - x\|^2, \quad \forall x \in H, \ p \in A^{-1}(0)$.*

Lemma 2.3. ([38]) *Let $\{a_k\}$, $\{b_k\}$ and $\{c_k\}$ be sequences of real numbers such that, for all $k \geq 1$,*

$$a_{k+1} \leq (1 - b_k)a_k + b_k c_k, \quad a_k \geq 0,$$

(i) *b_k satisfies a condition of type (C1); and*

(ii) *either $\sum_{k=1}^{\infty} b_k |c_k| < \infty$ or $\limsup_{k \to \infty} c_k \leq 0$.*

Then, $\lim_{k \to \infty} a_k = 0$.

Lemma 2.4. ([22]) *Let $\{a_k\}$ be a sequence of real numbers with a subsequence $\{k_l\}$ of $\{k\}$ such that $a_{k_l} < a_{k_l+1}$ for all $l \in \mathbb{N}_+$. Then, there exists a nondecreasing sequence $\{m_k\} \subseteq \mathbb{N}_+$ such that $m_k \to \infty, a_{m_k} \leq a_{m_k+1}$ and $a_k \leq a_{m_k+1}$ for all (sufficiently large) numbers $k \in \mathbb{N}_+$. In fact, $m_k = \max\{l \leq k : a_l \leq a_{l+1}\}$.*

Proposition 2.5. ([4, 5]) *Let H and F be as in Lemma 2.1 and let T be a nonexpansive operator on H such that $\mathrm{Fix}(T) \neq \emptyset$. Then, for any bounded sequence $\{x^k\} \subset H$ such that $\lim_{k\to\infty} \|Tx^k - x^k\| = 0$, we have*

$$\limsup_{k\to\infty} \langle Fp_*, p_* - x^k \rangle \leq 0, \tag{2.3}$$

where $p_* \in \mathrm{Fix}(T)$, solving (1.16) with Γ replaced by $\mathrm{Fix}(T)$.

3. Main results

Now, we introduce the following main results.

Theorem 3.1. *Let A be a maximal monotone operator in a real Hilbert space H such that $\Gamma := \{p \in \mathcal{D}(A) : 0 \in Ap\} \neq \emptyset$ and let F with μ be as in Lemma 2.1. Assume that there hold conditions (C1), (C2$^+$) and (C3$'$). Then, the sequence $\{z^k\}$, defined by (1.14), as $k \to \infty$, converges strongly to the unique solution p_*, solving (1.16).*

Proof. We consider an exact variant of (1.14), that is,

$$x^{k+1} = J^A_{r_k} J^A_c (I - t_k \mu F) x^k. \tag{3.1}$$

Clearly, from (1.14), (3.1), the nonexpansive property of $J^A_{r_k}$ with (C2$^+$) and Lemma 2.1, we get the following inequality:

$$\|z^{k+1} - x^{k+1}\| = \|J^A_{r_k} J^A_c (I - t_k \mu F) z^k + e^k - J^A_{r_k} J^A_c (I - t_k \mu F) x^k\|$$
$$\leq (1 - t_k \tau)\|z^k - x^k\| + \|e^k\|.$$

According to conditions (C1) and (C3$'$), we apply Lemma 2.3 to conclude that $\|z^k - x^k\| \to 0$ as $k \to \infty$. Hence, to show the desired result, it suffices to prove that $\{x^k\}$ converges strongly to p_* as $k \to \infty$. For this purpose, we first prove that $\{x^k\}$ is bounded. Indeed, for a fixed point $p \in \Gamma$, by Lemma 2.1, we have

$$\|x^{k+1} - p\| = \|J^A_{r_k} J^A_c (I - t_k \mu F) x^k - J^A_{r_k} J^A_c p\|$$
$$\leq (1 - t_k \tau)\|x^k - p\| + t_k \mu \|Fp\|$$
$$\leq \max\left\{ \|x^1 - p\|, \mu\|Fp\|/\tau \right\},$$

by mathematical induction. Therefore, the sequence $\{x^k\}$ is bounded, and so are the sequences $\{Fx^k\}$ and $\{y^k\}$ where $y^k := (I - t_k\mu F)x^k$. Without loss of generality, we can assume that they are bounded by a positive constant M_1. Further, we estimate the value $\|x^{k+1} - p\|^2$ as follows.

$$
\begin{aligned}
\|x^{k+1} - p\|^2 &= \|J^A_{r_k} J^A_c y^k - J^A_{r_k} p\|^2 \\
&\leq \|J^A_c y^k - p\|^2 \\
&\leq \|y^k - p\|^2 - \|J^A_c y^k - y^k\|^2 \\
&= \|(I - t_k\mu F)x^k - p\|^2 - \|J^A_c y^k - y^k\|^2 \\
&\leq (1 - t_k\tau)\|x^k - p\|^2 + 2t_k\mu\langle Fp, p - y^k\rangle - \|J^A_c y^k - y^k\|^2.
\end{aligned}
\tag{3.2}
$$

We need only consider two cases.

Case 1. There exists an integer $k_0 \geq 1$ such that $\|x^{k+1} - p\| \leq \|x^k - p\|$ for all $k \geq k_0$. Then, $\lim_{k\to\infty} \|x^k - p\|$ exists. From (3.2), we can write that

$$
\|J^A_c y^k - y^k\|^2 \leq \|x^k - p\|^2 - \|x^{k+1} - p\|^2 + 2t_k\mu M_2,
\tag{3.3}
$$

where $M_2 \geq \|Fp\|(\|p\| + M_1)$. Since $\lim_{k\to\infty} \|x^k - p\|$ exists and $t_k \to 0$, letting k tend to infinity in (3.3), we get that $\lim_{k\to\infty} \|J^A_c y^k - y^k\| = 0$. This together with $\|y^k - x^k\| \leq t_k\mu M_1$ implies that $\lim_{k\to\infty} \|J^A_c x^k - x^k\| = 0$. By using Proposition 2.5 with $T = J^A_c$, we obtain inequality (2.3). Now, from (3.2) with $p = p_*$, we know that

$$
\begin{aligned}
\|x^{k+1} - p_*\|^2 &\leq (1 - t_k\tau)\|x^k - p_*\|^2 + 2t_k\mu\langle Fp_*, p_* - x^k + t_k\mu Fx^k\rangle \\
&= (1 - b_k)\|x^k - p_*\|^2 + b_k c_k \quad \text{for all} \quad k \geq k_0,
\end{aligned}
$$

where $b_k = t_k\tau$ and $c_k = \frac{2\mu}{\tau}[\langle Fp_*, p_* - x^k\rangle + t_k\mu\|Fp_*\|M_1]$, from which and Lemma 2.3 we obtain that $\|x^k - p_*\| \to 0$.

Case 2. There exists a subsequence $\{k_l\}$ of $\{k\}$ such that $\|x^{k_l} - p\| < \|x^{k_l+1} - p\|$ for all $l \in \mathbb{N}_+$. Hence, by Lemma 2.4, there exists a nondecreasing sequence $\{m_k\} \subseteq \mathbb{N}_+$ such that $m_k \to \infty$,

$$
\|x^{m_k} - p\| \leq \|x^{m_k+1} - p\| \quad \text{and} \quad \|x^k - p\| \leq \|x^{m_k+1} - p\|
\tag{3.4}
$$

for each $k \in \mathbb{N}_+$. Then, from (3.2) and the first inequality in (3.4), we know that

$$
\|x^{m_k} - p\|^2 \leq \frac{2\mu}{\tau}\langle Fp, p - y^{m_k}\rangle.
\tag{3.5}
$$

On the other hand, again from (3.2) the first inequality in (3.4), we have

$$
\|J^A_c y^{m_k} - y^{m_k}\|^2 \leq 2t_{m_k}\mu M.
$$

Therefore, $\lim_{k\to\infty} \|J_c^A y^{m_k} - y^{m_k}\| = 0$, and hence, by Proposition 2.5,

$$\limsup_{k\to\infty} \langle Fp_*, p_* - y^{m_k} \rangle \le 0,$$

from which and (3.5) with p replaced by p_*, it follows that

$$\lim_{k\to\infty} \|x^{m_k} - p_*\| = 0. \tag{3.6}$$

Finally, from (3.2) with k and p replaced, respectively, by m_k and p_*, we can write that

$$\|x^{m_k+1} - p_*\|^2 \le (1 - t_{m_k}\tau)\|x^{m_k} - p_*\|^2 + 2t_{m_k}\mu\langle Fp_*, p_* - y^{m_k}\rangle.$$

By virtue of (3.6) and $t_{m_k} \to 0$, $\lim_{m\to\infty} \|x^{m_k+1} - p_*\|^2 = 0$, which together with the second inequality in (3.4) implies that $\lim_{k\to\infty} \|x^k - p_*\| = 0$. This completes the proof. $\qquad\square$

Theorem 3.2. *Let H, F, A, t_k, r_k and e^k be as in Theorem 3.1. In addition, assume that the parameter β_k satisfies conditions (C4). Then, the sequence $\{x^k\}$, defined by (1.15) strongly converges to the unique solution p_* of the variational inequality (1.16) as $k \to \infty$.*

Proof. As in the proof for Theorem 3.1, we consider an exact variant of (1.15),

$$x^{k+1} = \beta_k(I - t_k\mu F)J_c^A x^k + (1 - \beta_k)J_{r_k}^A x^k. \tag{3.7}$$

It follows from Lemma 2.2(i) that $J_c^A p = J_{r_k}^A p = p$ for any point $p \in \Gamma$, with (C2$^+$). From the nonexpansive property of $J_{r_k}^A$, (3.7) and Lemma 2.3, we have

$$\begin{aligned}
\|x^{k+1} - p\| &= \|\beta_k[(I - t_k\mu F)J_c^A x^k - p] + (1 - \beta_k)(J_{r_k}^A x^k - J_k^A p)\| \\
&= \|\beta_k[(I - t_k\mu F)J_c^A x^k - (I - t_k\mu F)J_c^A p - t_k\mu Fp] \\
&\quad + (1 - \beta_k)(J_k^A x^k - J_{r_k}^A p)\| \\
&\le \beta_k[(1 - t_k\tau)\|x^k - p\| + t_k\mu\|Fp\|] + (1 - \beta_k)\|x^k - p\| \\
&= (1 - \beta_k t_k\tau)\|x^k - p\| + \beta_k t_k\mu\|Fp\| \\
&\le \max\{\|x^1 - p\|, \mu\|Fp\|/\tau\}.
\end{aligned}$$

Therefore, the sequence $\{x^k\}$ is bounded, and so are the sequences $\{J_{r_k}^A x^k\}$, $\{J_c^A x^k\}$ and $\{FJ_c^A x^k\}$. So, there exists a positive constant M_3 such that $\{x^k\}$, $\{J_{r_k}^A x^k\}$, $\{J_c^A x^k\}$ and $\{FJ_c^A x^k\}$ belong to $S(0, M_3)$, the closed ball with center 0 and radius M_3. Next, as in the proof of Theorem 3.1, using the properties of the convex functional $\|.\|^2$, Hilbert space H, the resolvent operators J_c^A, $J_{r_k}^A$

and Lemma 2.3, we get that

$$
\begin{aligned}
\|x^{k+1} - p\|^2 &= \|\beta_k\big[(I - t_k\mu F)J_c^A x^k - p\big] + (1 - \beta_k)(J_{r_k}^A x^k - J_{r_k}^A p)\|^2 \\
&\le \beta_k\|(I - t_k\mu F)J_c^A x^k - p\|^2 + (1 - \beta_k)\|J_{r_k}^A x^k - J_{r_k}^A p\|^2 \\
&\le \beta_k\big[\|(I - t_k\mu F)J_c^A x^k - (I - t_k\mu F)p\|^2 \\
&\quad + 2t_k\mu\langle Fp, p - J_c^A x^k + t_k\mu F J_c^A x^k\rangle\big] + (1 - \beta_k)\|x^k - p\|^2 \\
&\le \beta_k(1 - t_k\tau)\|J_c^A x^k - p\|^2 + (1 - \beta_k)\|x^k - p\|^2 \\
&\quad + 2\beta_k t_k\mu\langle Fp, p - J_c^A x^k + t_k\mu F J_c^A x^k\rangle \\
&\le \beta_k(1 - t_k\tau)\big[\|x^k - p\|^2 - \|J_c^A x^k - x^k\|^2\big] + \beta_k\|x^k - p\|^2 \\
&\quad + 2\beta_k t_k\mu\langle Fp, p - J_c^A x^k + t_k\mu F J_c^A x^k\rangle.
\end{aligned}
$$

Since $t_k \to 0$ as $k \to \infty$ and $\beta_k \in [a, b] \subset (0, 1)$, there exists a positive constant c_1 such that $\beta_k(1 - t_k\tau) \ge c_1$ for all $k \in \mathbb{N}_+$. Consequently, from the last inequality we can write that

$$
\begin{aligned}
\|x^{k+1} - p\|^2 &\le (1 - \beta_k t_k\tau)\|x^k - p\|^2 - c_1\|J_c^A x^k - x^k\|^2 \\
&\quad + 2\beta_k t_k\mu\langle Fp, p - J_c^A x^k + t_k\mu F J_c^A x^k\rangle
\end{aligned} \tag{3.8}
$$

for all $k \in \mathbb{N}_+$. As in the proof of Theorem 3.1, we consider two cases.

Case 1. There exists an integer $k_0 \ge 1$ such that $\|x^{k+1} - p\| \le \|x^k - p\|$ for all $k \ge k_0$. Then, $\lim_{k\to\infty}\|x^k - p\|$ exists. From (3.8), we can write that

$$
c_1\|J_c^A x^k - x^k\|^2 \le \|x^k - p\|^2 - \|x^{k+1} - p\|^2 - \beta_k t_k\tau\|x^k - p\|^2 + 2t_k M_4,
$$

where $M_4 \ge \|Fp\|\|p - J_c^A x^k + t_k\mu F J_c^A x^k\|$ and get that

$$
\lim_{k\to\infty}\|J_c^A x^k - x^k\| = 0.
$$

According to Proposition 2.5 with $T = J_c^A$, we obtain inequality (2.3). Now, from (3.8) with $p = p_*$, we know that

$$
\begin{aligned}
\|x^{k+1} - p_*\|^2 &\le (1 - \beta_k t_k\tau)\|x^k - p_*\|^2 + 2\beta_k t_k\mu\langle Fp_*, p_* - x^k\rangle \\
&\quad + 2\beta_k t_k\mu\langle Fp_*, x^k - J_c^A x^k + t_k\mu F J_c^A x^k\rangle,
\end{aligned}
$$

which together with Lemma 2.3, (2.3), $\|x^k - J_c x^k\| \to 0$, $t_k \to 0$ and the boundedness of $\{F J_c^A x^k\}$ implies that $\|x^k - p_*\| \to 0$.

Case 2. This case is similar to that in the proof of Theorem 3.1. $\qquad\square$

Remark 3.3.   (1) Clearly, the mapping $F = I - f$, where $f = \tilde{a}I + (1 - \tilde{a})u$ with a fixed number $\tilde{a} \in (0, 1)$ and a fixed point $u \in H$, is an η-strongly monotone and L-Lipschitz continuous operator with $\eta = 1 - \tilde{a}$ and $L = 1 + \tilde{a}$. Then, replacing F in (1.14) by $I - f$ and re-denoting $z^k := x^k$

with $t'_k := t_k\mu(1-\tilde{a})$, we obtain (1.11). Now, putting $y^k = (I-t_k\mu F)z^k$ in (1.14), we get that

$$y^{k+1} = (I - t_{k+1}\mu F)z^{k+1} = (I - t_{k+1}\mu F)(J^A_{r_k}J^A_c y^k + e^k).$$

Now, replacing F by $I - f$ with the above f and re-denoting $x^k := y^k$ and $t_k := t_{k+1}$, we obtain method (1.12).
 (2) Replacing F in (1.15) by $I-f$ with the above f, $t'_k = (1-\beta_k)t_k\mu(1-\tilde{a})$, $\beta'_k = (1 - \beta_k)(1 - t_k\mu(1 - \tilde{a}))$ and $\gamma'_k = \beta_k$, we obtain method (1.13).
 (3) The above results can be extended to the problem of finding zeros of m-accretive operators in uniformly convex Banach spaces, having a uniformly Gâteaux differentiable norm. The proximal point method for the class of operators are presented in [3, 8, 9, 10, 11, 17, 19, 29, 30, 31, 32, 35, 41, 43].

4. An illustrative numerical example

Given Euclidian space E^2 with the standard inner product $\langle x, y\rangle = x_1y_1 + x_2y_2$ and the norm $\|x\| = (x_1^2 + x_2^2)^{1/2}$ for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in E^2$.
  We consider a convex optimization problem:

$$\text{find a point } p_* \in E^2 \text{ such that } f(p_*) = \inf_{x\in E^2} f(x), \qquad (4.1)$$

where $f(x) = 0$ for all $x = (x_1, x_2) \in E^2$ with $x_2 \leq 0.5$ and $f(x) = x_2 - 0.5$ otherwise. It is well known that (see [20, pp. 15-16, Propositions 5.1 and 5.2]) if F is the gradient of a convex function f, then the variational inequality problem (1.16) is equivalent to the optimization problem (4.1). Hence, we have $p_* = (1; 0.5)$ is a unique solution of (1.16). It is not difficult to verify that

$$(I + r\partial f)^{-1}(x) = \begin{cases} (x_1, x_2), & x_2 \leq 0.5, \\ (x_1, x_2/(1 + r)), & \text{otherwise.} \end{cases}$$

Take $u = (1; 4)$. The numerical results, by using method (1.11) with data: $r_k = 1/(k + 1)$, $\tilde{a} = 0.5$, $\mu = 0.2$, $t_k = 10/(k + 1)$ and hence $t'_k = 1/(k + 1)$, $c = 0.05$, $e^k = (1; 1)/k^2$ or $e^k \equiv 0$ for all $k \geq 1$ and the initial guess $x^1 = (2; 3)$ are presented in numerical Tables 1 and 2.

k	x_1^{k+1}	x_2^{k+1}	k	x_1^{k+1}	x_2^{k+1}
10	1.4980669070	1.7686553410	100	1.0774491229	0.6322463834
20	1.2949477572	1.4612690118	200	1.0423779975	0.4991780648
30	1.2131334597	1.2645439136	300	1.0296487381	0.4924066379
40	1.1682430976	1.1142094108	400	1.0229734835	0.4891775358
50	1.1396929034	0.9936536310	500	1.0188338513	0.4791958109

Table 1. Numerical test for iteration method (1.11) with $e^k = (1; 1)/k^2$

k	x_1^{k+1}	x_2^{k+1}	k	x_1^{k+1}	x_2^{k+1}
10	1.0909090909	1.6426632502	100	1.0099009901	0.6301305040
20	1.0476190476	1.4180375560	200	1.0049751244	0.4988091351
30	1.0322580645	1.2430261962	300	1.0033222591	0.4815641525
40	1.0243902439	1.1014876372	400	1.0024937656	0.4802031494
50	1.0196078431	0.9853216016	500	1.0019960080	0.4936253750

Table 2. Numerical test for iteration method (1.11) with $e^k \equiv 0$

While, computing by method (1.5) with $r_k = 0.01 + 1/(k+1)$, $e^k \equiv 0$ and the same values for the rest parameters as above, we obtain numerical Table 3.

k	x_1^{k+1}	x_2^{k+1}	k	x_1^{k+1}	x_2^{k+1}
10	1.0909090909	1.9396774054	100	1.0099009901	1.4660031041
20	1.0476190476	1.8679177999	200	1.0049751244	1.1319834805
30	1.0322580645	1.8073814888	300	1.0033222591	0.9088635230
40	1.0243902439	1.7509738827	400	1.0024937656	0.7531136428
50	1.0196078431	1.7975720435	500	1.0019960080	0.6400290762

Table 3. Numerical test for iteration method (1.5) with $e^k \equiv 0$

We do not calculate by (1.12), because it is equivalent to (1.11). Further, calculating by method (1.13) with $\beta_k = 0.3 + 1/(k+1)$, the same values c, μ and x^1 as the above,

$$t'_k = (1 - \beta_k)(1 - t_k\mu(1 - \tilde{a})) = (1 - \beta_k)(1 - 1/(k+1)),$$

$$\beta'_k = (1 - \beta_k)t_k\mu(1 - \tilde{a})) = (1 - \beta_k)/(k+1),$$

and $\gamma'_k = \beta_k$, we obtain numerical Tables 4 and 5 for the cases $e^k = (1;1)/k^2$ and $e^k \equiv 0$.

k	x_1^{k+1}	x_2^{k+1}	k	x_1^{k+1}	x_2^{k+1}
10	2.1965684668	2.3838875762	100	1.3109126696	0.8079276796
20	1.8354366949	2.0271657196	200	1.1969511572	0.4977841669
30	1.6610105505	1. 7574683442	300	1.1501629597	0.4935501136
40	1.5556429410	1.5379928171	400	1.1237079356	0.4922345887
50	1.4839979612	1.3568425512	500	1.1063744997	0.4920375363

Table 4. Numerical test for iteration method (1.13) with with $e^k = (1;1)/k^2$

k	x_1^{k+1}	x_2^{k+1}	k	x_1^{k+1}	x_2^{k+1}
10	1.4075639968	2.0073473524	100	1.0942876649	0.8010551458
20	1.2716245286	1.8506617313	200	1.0585601097	0.5120987740
30	1.2103094495	1.6587322283	300	1.0442216475	0.4914171749
40	1.1744254202	1.4778094352	400	1.0362099307	0.4892355522
50	1.1505014800	1.3181382307	500	1.0310013837	0.4889589524

Table 5. Numerical test for iteration method (1.13) with with $e^k \equiv 0$

Further, computing by method (1.7) with $r_k = 0.01 + 1/(k+1)$, $e^k \equiv 0$ and the same values for the rest parameters as the above, we obtain numerical Table 6.

k	x_1^{k+1}	x_2^{k+1}	k	x_1^{k+1}	x_2^{k+1}
10	1.4075639968	2.2377671304	100	1.0942876649	2.3075950529
20	1.2716245286	2.3522492757	200	1.0585601097	2.0517792565
30	1.2103094495	2.3938575042	300	1.0442216475	1.8156557355
40	1.1744254202	2.4055089639	400	1.0362099307	1.6123752550
50	1.1505014800	2.4022300127	500	1.0310013837	1.4394415850

Table 6. Numerical test for iteration method (1.7) with with $e^k \equiv 0$

The numerical tables shows that our methods (1.11) and (1.13) work well and they enjoy a faster rate of convergence than (1.5) and (1.7), respectively.

5. Conclusion

We have presented two iterative methods for finding a point in the zero set of a maximal monotone mapping in Hilbert spaces, that solves a variational inequality problem, involving an η-strongly monotone and L-Lipschitz continuous operator on H for some positive constants η and L. As consequences, new generalized and contraction proximal point algorithms with any sequence of positive numbers for the resolvent parameter have been obtained. We also have given a numerical example for illustrating the presented methods.

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