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OPERATORS LOWER RATE ANALYSIS AND ITS APPLICATIONS IN SPACES

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Abstract. In this paper, two new concepts, (y, f)-rate and lower rate of an operator are introduced, and some properties and basic principles of the lower rate are studied in spaces. As the application of operator lower rate, a new fixed point existence theorem is constructed. The obtaining results and process seem to be general in nature.

1. Introduction

The rate of change of ordinary function and the rate of correlation between function are two classic and fashion concepts, and the rate of change of ordinary function is expressed as the derivative of function, and the rate of change of ordinary transformation(operator) is given the norm of operator [5]. It is well known that they are widely applied, and they are indispensable in many

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fields. However, they have certain limitations in theoretical research and practical application. Therefore, it is necessary to discuss the rate of change of each aspect of the operator in each background.

In 1979, Bhatia et al. introduced the rate of change of spectra of operators, and obtained a bound for the distance between the spectra of any two linear operators [1]. On the other hand, fuzzy set theory and its applications have been extensively developed since the 1970s [10] and industrial interest in fuzzy control has dramatically increased since 1990, and it has found many useful applications to problems of artificial intelligence, decision-making, operation research and many others [2]-[4], [9], for example, from 1989, Chang and Huang introduced and investigated a class of variational inequalities for fuzzy mappings [3]. And afterwards, Chang and Huang, Ding and Park, Li et al. studied several kinds of variational inequalities (inclusions) for fuzzy mappings [4], [6], [7], [9]. During that period, we found out that this kind of problem, which is the rate of fuzzy degree of between an element and it's image under an operator in fuzzy space, or more generally, the rate of mapping in a space, is interesting. We believe that the rate of change of the operator image in various forms at the original image is the ratio of the corresponding functionals acting on the original image and image. For more literature, we recommend to the reader [1]-[10].

In this paper, two new concepts, (y, f)-rate and lower rate of an operator are introduced, and some properties and basic principles of the lower rate are studied in spaces. As the application of operator lower rate, a new fixed point existence theorem is constructed. The obtaining results and process seem to be general in nature.

2. (y, f)-rate of operators and basic properties

Let X be a space, R be a real set, a real functional $\widehat{F} : X \to [0, 1]$ be a fuzzy set over X where $\widehat{F}(y)$ is the membership function of y in $\widehat{F}$, and $P(X) = \{\widehat{F}|\widehat{F} : X \to [0, 1]\}$ be a collection of all fuzzy sets over X, $B(X) = \{ B|B\colon X \to X\}$ be a set of all operators on X. Then we will consider: Fuzzy influence degree of an operator B which changes a $y \in X$ to the $B(y)$ on $F \subseteq P(X)$, that is,

$$\frac{\widehat{F}(B(y))}{\widehat{F}(y)} \quad \text{and} \quad \inf_{\widehat{F}\in F} \frac{\widehat{F}(B(y))}{\widehat{F}(y)}.$$

It is interesting thing because they maybe express some impaction and contribution of operator B when the operator B changes y to $B(y)$ to relate to arbitrarily given fuzzy subset F(or some system) for $y \in X$.

Definition 2.1. Let X be a space, $B(X)$=$\{B|B : X \to X\}$ be a set of all operators on X, $P(X) = \{f|f : X \to (-\infty, +\infty)\}$ be a set of all functionals on X. Let $B \in B(X)$, for $f \in P(X)$ and $y \in X$, if set

$$\sigma(B(y))_f = \begin{cases} \frac{f(B(y))}{f(y)} & \text{as} \quad f(y) \neq 0 \quad \text{for} \quad y \in lX, \\ \\ 0 & \text{as} \quad f(y) = 0 \quad \text{for} \quad y \in X. \end{cases} \tag{2.1}$$

Then $\sigma(B(y))_f$ is called a (y, f)-rate of the operator B at a point $y \in X$ with respect to f.

If set $f = d$ is a differential and B is a differentiable function on real set R, then

$$\sigma(B(y))_d = \frac{d(B(y))}{dy}$$

expresses the derivative at point $y \in R$.

Let's discuss some basic properties of the $\sigma(B(y))_f$ at $y \in X$ under these fundamental assumptions which X is a space, lR is the real set, $lB(X)$=$\{B|B : lX \to lX\}$ is a set of all operators on X and $lP(X) = \{f|f : lX \to (-\infty, +\infty)\}$ is a set of all functionals on lX.

Proposition 2.2. *Let $A, B \in B(X)$ and I be an identity operator on X, and $f \in P(X)$ and $y \in X$. Then the following properties hold.*

(1) *if $f(y) \neq 0$, $B = I$, then $\sigma(B(y))_f \equiv 1$;*
(2) *if $f(y) \equiv 0$, then $\sigma(B(y))_f \equiv 0$;*
(3) *(Chain Rule) if let $(AB)(y) = A(B(y))$ and $f(B(y)) \neq 0$ for $y \in X$, then*
$$\sigma((AB)(y))_f = \sigma((A(B((y)))_f \sigma(B(y))_f; \tag{2.2}$$
(4) *if set $B^0 = I$ and $B^n = B(B^{n-1})$ and $f(B^k(y)) \neq 0 (0 \leq k \leq n-1, n = 1, 2, ...)$, then*
$$\sigma(B^n(y))_f = \prod_{1 \leq k \leq n} \sigma(B(B^{k-1}(y))). \tag{2.3}$$

Proof. Let $A, B \in B(X)$, I be an identity operator on X, $f \in P(X)$ and $y \in X$. Then (1)-(2) hold, it follows directly from the Definition 2.1. Let $(AB)(y) = A(B(y))$ and $f(B(y)) \neq 0$ for $y \in X$. By using (2.1), we have

$$\begin{aligned} \sigma((AB)(y))_f &= \frac{f(A(B(y)))}{f(y)} \\ &= \frac{f(A(B(y)))}{f(B(y))} \frac{f(B(y))}{f(y)} \\ &= \sigma(A(B(y)))_f \sigma(B(y))_f. \end{aligned} \tag{2.4}$$

It follows that (3) is true. It is not difficult to prove the correctness of (4) by (3) and mathematical induction. This completes the proof. $\square$

3. F-lower rate of operators and basic properties

Definition 3.1. Let X be a space, $B(X)$=$\{B|B : X \to X\}$ be a set of all operators on X, $P(X) = \{f|f : X \to (-\infty, +\infty)\}$ be a set of all functionals on X. Let $B \in B(X)$, for $F \in P(X)$ and $y \in X$, if set

$$\sigma(B(y))_{F_L} = \inf_{f \in F} \sigma(B(y))_f, \tag{3.1}$$

then $\sigma(B(y))_{F_L}$ is called a F-lower rate of operator B at a point $y \in X$ with respect to F, for short, lower rate of operator B at a point $y \in X$.

Obviously, we can see

$$\sigma(B(y))_{F_L} = \inf_{f \in F} \frac{f(B(y))}{f(y)}, \tag{3.2}$$

and if F=$\{f\}$, then

$$\sigma(B(y))_{F_D} = \sigma(B(y))_f. \tag{3.3}$$

Theorem 3.2. *Let $B \in B(X)$, F be a nonempty uniformly bounded functional subset of $P(X)$ on X, that's,*

F=$\{f|f \in P(X)$, there exist a real $M > 0$ such that $|f(y)| \le M, \forall y \in X$ $\}$. If $\sigma(B(y))_{F_L}$ exists for any $y \in X$, then there exist two functionals $f, g \in F$ such that

$$\sigma(B(y))_{F_L} f(y) = g(B(y)), \tag{3.4}$$

for all $y \in X$.

Proof. Let F be a nonempty uniformly bounded functional subset of $P(X)$ on X and $\sigma(B(y))_{F_L}$ exist for any given $y \in X$, then since

$$\sigma(B(y))_{F_L} = \inf_{f \in F} \frac{f(B(y))}{f(y)}$$

be a real number so that for any natural number n, there exists a $g_n \in F$ such that

$$\sigma(B(y))_{F_L} \le \frac{g_n(B(y))}{g_n(y)} < \sigma(B(y))_{F_L} + \frac{1}{n}$$

and

$$\sigma(B(y))_{F_L} = \lim_{n \to \infty} \frac{g_n(B(y))}{g_n(y)}$$

for the $y \in X$.

Let F be a nonempty uniformly bounded functional subset of $P(X)$ on X. Then there exist a real $M > 0$ such that for all $f \in F$, $|f(y)| \le M$ for all

$y \in X$ and $\max\{|g_n(y)|, |g_n(B(y))|\} \le M(n = 1, 2, ...)$, and ulteriorly, there exist $\lim_{k\to\infty} g_{n_k}(y)$ and also $\lim_{m\to\infty} g_{n_{k_m}}(B(y))$, $|\lim_{k\to\infty} g_{n_k}(y)| \le M$ and $|\lim_{m\to\infty} g_{n_{k_m}}(B(y))| \le M$ for $y \in X$.

If $\lim_{k\to\infty} g_{n_k}(y) \ne 0$ then set

$$f(y) = \lim_{k\to\infty} g_{n_k}(y)$$

and

$$g(B(y)) = \lim_{m\to\infty} g_{n_{k_m}}(B(y)),$$

it follows that

$$\begin{aligned} \sigma(B(y))_{F_L} &= \lim_{n\to\infty} \frac{g_n(B(y))}{g_n(y)} \\ &= \lim_{m\to\infty} \frac{g_{n_{k_m}}(B(y))}{g_{n_{k_m}}(y)} \\ &= \frac{\lim_{m\to\infty} g_{n_{k_m}}(B(y))}{\lim_{m\to\infty} g_{n_{k_m}}(y)} \\ &= \frac{g(B(y))}{f(y)}, \end{aligned}$$

that is, $\sigma(B(y))_{F_L} f(y) = g(B(y))$, and $\max\{|f(y)|, |g(B(y))|\} \le M$ for $y \in X$.

If $\lim_{k\to\infty} g_{n_k}(y) = 0$, then let $f(y) = \lim_{m\to\infty} g_{n_{k_m}}(y) = 0$, and since

$$\sigma(B(y))_{F_L} = \lim_{n\to\infty} \frac{g_n(B(y))}{g_n(y)} = \lim_{m\to\infty} \frac{g_{n_{k_m}}(B(y))}{g_{n_{k_m}}(y)}$$

is a real number and $\lim_{m\to\infty} g_{n_{k_m}}(B(y)) = 0$, then set $g(B(y)) = 0$ [8], and also $\sigma(B(y))_{F_L} f(y) = g(B(y))$ and $\max\{|f(y)|, |g(B(y))| \le M\}$ for $y \in X$. Therefore, there exist $f(y), g(B(y)) \in F$ such that $\sigma(B(y))_{F_L} f(y) = g(B(y))$ for all $y \in X$. The proof of the theorem is completed. □

Lemma 3.3. *Let $\emptyset \ne A, B \subseteq R$, $A \cdot B=\{xy|x \in A, y \in B\}$ be a set of x times y, if $A \subseteq \{x|x \ge 0\}$ and $B \subseteq \{y|y \ge 0\}$ then* $\inf A \cdot B = \inf A \cdot \inf B$.

Proof. It follows that the above conclusions from the operation of inf and sup directly. □

Theorem 3.4. *Let $A, B \in B(X)$ and $y \in X$, $\emptyset \ne F \in P(X)$. Then we have the following chain rules:*

(1) *(Chain Rule): Let $F=\{f|f : X \to (0, +\infty)\}$ be a set of positive functionals, $(AB)(y) = A(B(y))$ for $y \in X$, and there exist $\sigma(A(B(y)))_{F_L}$ and $\sigma(B(y))_{F_L}$. Then $\sigma(B(y))_{F_L} \ge 0$, $\sigma(A(B(y)))_{F_L} \ge 0$, and*

$$\sigma((AB)(y))_{F_L} = \sigma(A(B(y)))_{F_L}\sigma(B(y))_{F_L}$$

for any $y \in X$;

(2) *(Chain Rule Corollary): If $B^n(y) = B(B^{n-1}(y))$, then there exists $\sigma(B(B^{k-1}(y)))_{F_L} \geq 0$ for $k = 1, 2, \cdots, n$ such that*

$$\sigma(B^n(y))_{F_L} = \prod_{1 \leq k \leq n} \sigma(B(B^{k-1}(y)))_{F_L}, \tag{3.5}$$

where $n = 1, 2, \cdots$ and $B^0 = I$ is identity operator.

Proof. (1) Let $(AB)(y) = A(B(y))$ for any $y \in X$. There exist $\sigma(A(B(y)))_{F_L} \geq 0$ and $\sigma(B(y))_{F_L} \geq 0$ because F=$\{f|f : X \to (0, +\infty)\}$ is a set of positive functionals. Then by Lemma 3.3, we have

$$\begin{aligned} \sigma((AB)(y))_{F_L} &= \inf_{f \in F} \frac{f(A(B(y)))}{f(y)} \\ &= \inf_{f \in F} \frac{f(A(B(y)))}{f(B(y))} \frac{f(B(y))}{f(y)} \\ &= \inf_{f \in F} \frac{f(A(B(y)))}{f(B(y))} \inf_{f \in F} \frac{f(B(y))}{f(y)} \\ &= \sigma(A(B(y)))_{F_L} \sigma(B(y))_{F_L}. \end{aligned}$$

(2) By using the assumptions and (1), we have

$$\begin{aligned} \sigma(B^n(y))_{F_L} &= \inf_{f \in F} \frac{f(B(B^{n-1}(y)))}{f(y)} \\ &= \inf_{f \in F} \frac{f((B(B^{n-1}(y)))}{f(B^{n-1}(y))} \frac{f(B^{n-1}(y))}{f(y)} \\ &= \inf_{f \in F} \frac{f((B(B^{n-1}(y))))}{f(B^{n-1}(y))} \inf_{f \in F} \frac{f(B^{n-1}(y))}{f(y)} \\ &= \sigma(B(B^{n-1}(y)))_{F_L} \inf_{f \in F} \frac{f(B^{n-1}(y))}{f(y)} \\ &\vdots \\ &= \prod_{1 \leq k \leq n} \sigma(B(B^{k-1}(y)))_{F_L}. \end{aligned}$$

Hence the result (3.5) holds. And distinctly, there exists $\sigma(B(B^{k-1}(y)))_{F_L} \geq 0$ for $k = 1, 2, \cdots, n$ and F=$\{f|f : X \to (0, +\infty)\}$. This completes the proof. $\square$

4. Applications–A new fixed point existence theorem

Lemma 4.1. *Let X be a space, $B(X)$=$\{B|B : X \to X\}$ be a set of all operators on X, $B \in B(X)$, and $P(X) = \{f|f : X \to (-\infty, +\infty)\}$ be a set of all functionals on X. Let $\emptyset \neq F \subseteq P(X)$ and F=$\{f|f : X \to (0, +\infty)\}$ be a set of positive functionals. If for $\Delta \geq 0$, there exists natural number N such that $\sigma(B^n(y))_{F_L} \leq \Delta$ as $n \geq N$, then*

(1) *there exists a functional sequence $\{f_q\}_{1\leq q\leq+\infty} \subseteq F$ such that*

$$\lim_{q\to\infty} \frac{f_q(B(B^{(k-1)}(y)))}{f_q(B^{(k-1)}(y)))} = 0 \tag{4.1}$$

for a natural number $k \geq 1$, or

(2) *there exists a functional sequence $\{f_q\}_{1\leq q\leq+\infty} \subseteq F$ such that*

$$\lim_{q\to\infty} \frac{f_q(B(B^{(k-1)}(y)))}{f_q(B^{(k-1)}(y)))} = e^{-\delta} \tag{4.2}$$

for a natural number $k \geq 1$ and $\delta > 0$, or

(3) *there exists a $f_0 \in F$ and natural number K such that*

$$f_0(B^k(y)) = f_0(B^{k-1}(y)) \tag{4.3}$$

for $k \geq K$, that is,

$$f_0(B(B^{k-1}(y))) = f_0(B^{k-1}(y)) \tag{4.4}$$

for $k \geq K$.

Proof. Let $\sigma(B(y))_{F_L} = \inf_{f\in F} \frac{f(B(y))}{f(y)}$ for $\emptyset \neq F \subseteq P(X)$ and $B^n(y) = B(B^{n-1}(y))$ for $n = 1, 2, \cdots$. If for $\Delta \geq 0$, there exists natural number N such that $\sigma(B^n(y))_{F_L} \leq \Delta$ as $n \geq N$, then by Definition 3.1 and (3.5), we have

$$0 \leq \prod_{1\leq k\leq n} \sigma(B(B^{k-1}(y)))_{F_L} = \sigma(B^n(y))_{F_L} \leq \Delta,$$

for $n \geq N$, and then there is a convergence subsequence in the sequence $\{\sigma(B^n(y))_{F_L}\}_{1\leq n\leq+\infty}$. Without loss of generality, let $\lim_{n\to\infty} \sigma(B^n(y))_{F_L} = \beta$ and $\beta \geq 0$.

(I) Let $\beta = 0$. Then there exists a $\sigma(B^k(y))_{F_L} = 0$ or all $\sigma(B^n(y))_{F_L} > 0$ for $n = 1, 2, \cdots$.

If some one, $\sigma(B^k(y))_{F_L} = 0$, then

$$\sigma(B^k(y))_{F_L} = \inf_{f\in F} \frac{f(B(B^{(k-1)}(y)))}{f(B^{(k-1)}(y))} = 0,$$

and there exists a functional sequence $\{f_q\}_{1\leq q\leq +\infty} \subseteq F$ such that

$$\lim_{q\to\infty} \frac{f_q(B(B^{(k-1)}(y)))}{f_q(B^{(k-1)}(y)))} = 0$$

for a natural number $k \geq 1$.

If all $\sigma(B^n(y))_{F_L} > 0$ for $n = 1, 2, \cdots$ and $\beta = \lim_{n\to\infty} \sigma(B^n(y))_{F_L} = 0$, then

$$\lim_{n\to\infty} \sum_{k=1}^{n} \ln \sigma(B(B^{k-1}(y)))_{F_L} = \lim_{n\to\infty} \ln \sigma(B^n(y))_{F_L} = -\infty,$$

and there exists an add item $\ln \sigma(B(B^{k-1}(y)))_{F_L} = -\delta$ for some $\delta > 0$. And $\sigma(B(B^{k-1}(y)))_{F_L} = e^{-\delta} < 1$. Hence, there exists a functional sequence $\{f_q\}_{1\leq q\leq +\infty} \subseteq F$ such that

$$\lim_{q\to\infty} \frac{f_q(B(B^{(k-1)}(y)))}{f_q(B^{(k-1)}(y)))} = e^{-\delta}$$

for a natural number $k \geq 1$ and $\delta > 0$.

(II) Let $\beta > 0$. Then the series

$$\sum_{k=1}^{\infty} \ln \sigma(B(B^{k-1}(y)))_{F_L} = \ln \beta$$

is convergent, and

$$\lim_{k\to+\infty} \ln \sigma(B(B^{k-1}(y)))_{F_L} = 0$$

and

$$\lim_{k\to+\infty} \sigma(B(B^{k-1}(y)))_{F_L} = 1.$$

It follows that for any natural number m there exists a M, such that for $k > K (K = \max\{M, N\})$,

$$1 - \frac{1}{m} < \sigma(B(B^{k-1}(y)))_{F_L} < 1 + \frac{1}{m}.$$

Since

$$\sigma(B(B^{k-1}(y)))_{F_L} = \inf_{f\in F} \frac{f(B(B^{k-1}(y)))}{f(B^{k-1}(y))} < +\infty,$$

for the number m,there exists a $f_0 \in F$ such that

$$1-\frac{1}{m} < \sigma(B(B^{k-1}(y)))_{F_L} \leq \frac{f_0(B(B^{k-1}(y)))}{f_0(B^{k-1}(y))} < \sigma(B(B^{k-1}(y)))_{F_L} + \frac{1}{m} < 1 + \frac{2}{m}.$$

As $m \to +\infty$, we have

$$\frac{f_0(B(B^{k-1}(y)))}{f_0(B^{k-1}(y))} = 1,$$

that is $f_0(B(B^{k-1}(y))) = f_0(B^{k-1}(y))$. This completes the proof. □

Definition 4.2. Let X be a space, $B(X)$=$\{B|B : X \to X\}$ be a set of all operators on X, $P(X) = \{f|f : X \to (-\infty, +\infty)\}$ be a set of all functionals on X and $B \in B(X)$. For $y \in X$, if exists a $g \in P(X)$ such that

$$g(B(B^{k-1}(y))) = g(B^{k-1}(y)),$$

where k is the smallest positive integer that satisfies this equation, then the element y is called an k-ordered quasi-fixed point of the operator B with respect to g, where $k = 1, 2, \cdots$.

We know that from the process of the proof in Lemma 4.1, the element $B^{k-1}(y)$ is an 1-ordered quasi-fixed point of the operator B with respect to f_0, or the element y is an $(K+1)$-ordered quasi-fixed point of the operator B with respect to g.

Theorem 4.3. (A New Fixed Point Existence Theorem) *Let X be a space, $B(X)$=$\{B|B : X \to X\}$ be a set of all operators on X, $P(X) = \{f|f : X \to (-\infty, +\infty)\}$ be a set of all functionals on X. Let $B \in B(X)$ and $\lim_{k\to+\infty} \sigma(B(B^{k-1}(y)))_{F_L} = 1$. If exists an injection functional $f_0 \in F$ and smallest positive integer K such that*

$$f_0(B^K(y)) = f_0(B^{(K-1)}(y)), \tag{4.5}$$

then the element $B^{(K-1)}(y)$ is a fixed point of the operator B.

Proof. It follows directly that the result holds from Lemma 4.1 and the injective condition of functional $f_0 \in F$. □

5. Conclusion

In this work, we have obtained the following results:

- (y, f)-rate of an diagonal matrix B in a plane is discussed.
- The F-lower rate of an operator in abstract spaces is introduced and given some properties of it, and basic principles of the down rate is studied.
- A new fixed point existence theorem of operator is proved.

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