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EXISTENCE AND CONTROLLABILITY OF MILD SOLUTION OF IMPULSIVE INTEGRO-DIFFERENTIAL INCLUSIONS

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Abstract. In this work, existence results for impulsive integro-differential inclusion by using Martelli and Covitz-Nadler fixed point theorems (FPT) has been studied and proposed improvement for some of the results with the help of impulsive inequality. Also controllability results has been investigated for impulsive integro-differential inclusion problems. This study will provide useful insights for design problems in engineering leading to controllability solutions of Integro differential equation subjected to impulsive perturbations taking into consideration of nonlocal and delay conditions.

1. Introduction

Most of natural systems has evoloved through dynamic process subjected to impulsive changes. These processes are modeled using integro differential

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equations with impulsive conditions and widely used in engineering design. Although results on integro-differential inclusion with impulsive condition has not been investigated completely. Control theory gives the motivation to study integro-differential inclusions. For more details see [1], [2], [3], [6], [16], [17].

Moreover, multivalued map play important role in studying inclusions because these kind of maps look for more than one selections for the systems. The dynamic processes which depends on multivalued states can be better modeled using integro-differential inclusions [4], [5], [12]. Many authors studied existence results of impulsive integro-differential inclusions and their particular cases. See [8], [9], [13], [14], [15].

Controllability has numerous applications in mathematics such as optimal control, stabilizability and in many branches of technical and physical sciences with help of various types of integro-differential equations. If dynamical system is controllable then the output of the system will be proper that is the system will have admissible controls from initial state to last state. Very little work has been towards controllability of inclusion problems.

To study the existence, following impulsive inclusion problem is taken under consideration,

$$w'(\xi) \in \mathcal{A}_1 w(\xi) + \mathcal{F}(\xi, w_\xi, \int_0^\xi k(\xi,\eta)h(\eta, w_\eta)d\eta), \xi \in (0,\mathbb{T}], \xi \neq \mathcal{T}_l, l = 1(1)m, \tag{1.1}$$

$$w(\xi) + (g(w_{\xi_1}, \cdots, w_{\xi_p}))(\xi) = \phi(\xi), \quad -r \leq \xi \leq 0, \tag{1.2}$$

$$\Delta w(\mathcal{T}_l) = I_l w(\mathcal{T}_l), \quad l = 1(1)m, \tag{1.3}$$

where $\mathcal{F} : [0,\mathbb{T}] \times C \times X \to 2^X$ is multivalued map. $0 < \xi_1 < \xi_2 < ... < \xi_p \leq T$, $p \in \mathbb{N}$, $\mathcal{F} : [0,\mathbb{T}] \times C \times X \to 2^X$ is multivalued map. $\mathcal{A}_1$ is the infinitesimal generator of strongly continuous semigroup of linear operators $\{T(\xi)\}_{\xi \geq 0}$ which are bounded. Linear operators $I_l (l = 1, 2, ..., m)$ acts on a Banach space X. Let the real valued function k be continuous on $[0,\mathbb{T}] \times [0,\mathbb{T}]$, $\phi \in C$ and the given functions f, h and g fullfill some assumptions. The impulsive moments τ_l are such that $0 \leq \tau_0 < \tau_1 < \tau_2 < ... < \tau_l < \tau_{m+1} \leq \mathbb{T}$, $m \in \mathbb{N}$, $\Delta w(\tau_l) = w(\tau_l + 0) - w(\tau_l - 0)$, where $w(\tau_l + 0)$ and $w(\tau_l - 0)$ are right and left limits of w at τ_l, respectively.

We prove controllability result of the problem:

$$w'(\xi) \in \mathcal{A}_1 w(\xi) + Bx(\xi) + \mathcal{F}(\xi, w_\xi, \int_0^\xi k(\xi,\eta)h(\eta, w_\eta)d\eta),$$
$$\xi \in (0,\mathbb{T}], \xi \neq \mathcal{T}_l, l = 1(1)m, \tag{1.4}$$

$$w(\xi) + (g(w_{\xi_1}, ..., w_{\xi_p}))(\xi) = \phi(\xi), \quad -r \leq \xi \leq 0, \tag{1.5}$$

$$\Delta w(\mathcal{T}_l) = I_l w(\mathcal{T}_l), \quad l = 1(1)m, \tag{1.6}$$

where $B : Y \to X$ is a continuous linear operator satisfying $\|B\| \leq N_1$, for $N_1 > 0$, Y is a Banach space with $x(\cdot) \in L^2((0, \mathbb{T}], Y)$, $x(\cdot)$ is a control function, and g is a completely continuous function.

In [8], Benchohra et al. studied the existence result of following first order inclusion problem,

$$y' \in F(\xi, y), \quad \xi \in J/J',$$
$$y(\xi_k^+) = I_k(y(\xi_k^-), \quad k = 1, .., m,$$
$$y(0) = y_0.$$

In [9], authors studied existence result of the following impulsive inclusion model using fixed point theorem (FPT) for condensing map:

$$y'(\xi) - Au(\xi) \in F(t, y_\xi),\ a.e. \quad \xi \in J = [0, b], \quad \xi \neq \xi_k, k = 1, 2, .., m,$$
$$\Delta y_{\xi=t_k)} = I_k y(t_k^-), \quad k = 1, 2, ..., m,$$
$$y(\xi) + (g(y_{\xi_{\eta_1}}, ..., y_{\eta_p}))(t) = \phi(\xi), \quad t \in [-r, 0].$$

The structure of the paper is organized as: Section 2 includes preliminaries, hypotheses required to prove results. Sections 3 gives the existence results. Controllability result is proved in Section 4.

2. Preliminaries

Let X be a Banach space with the norm $\|\cdot\|$. Let $C = \mathcal{C}([-r, 0], X), 0 < r < \infty$, be the Banach space which contains all continuous functions $\phi : [-r, 0] \to X$ endowed with supremum norm $\|\phi\|_C = \sup\{\|\phi(\xi)\| : -r \leq \xi \leq 0\}$ and $PC\big[[-r, \mathbb{T}], X\big] = \{w : [-r, \mathbb{T}] \to X | w(\xi)$ is piecewise continuous at $\xi \neq \tau_l$, left continuous at $\xi = \tau_l$, and the right limit $w(\tau_l + 0)$ exists for $l = 1, 2, ..., m\}$. $PC\big[[-r, \mathbb{T}], X\big]$ represents Banach space with the supremum norm $\|w\|_{PC} = \sup\{\|w(\xi)\| : \xi \in [-r, \mathbb{T}] \setminus \{\tau_1, \tau_2, ..., \tau_m\}\}$. For any $w \in PC\big[[-r, \mathbb{T}], X\big]$ and $\xi \in [0, \mathbb{T}] \setminus \{\tau_1, \tau_2, ..., \tau_m\}$, we denote w_ξ the element of C given by $w_\xi(\theta) = w(\xi + \theta)$ for $\theta \in [-r, 0]$ and ϕ is a given element of C. The existence of constants $K_0 \geq 1, D, L > 0$ are assummed such that $\|T(\xi)\| \leq K_0$, $\|\phi\| \leq D$, $|k(\xi, \eta)| \leq L$.

(1) Let $AC(E, X)$ be the space of all absolutely continuous functions $w : E \to X$, where E is any closed interval in $\mathbb{R}$,
(2) $L^1(E, X) = \{w : E \to X : w$ is Bochner integrable$\}$,
(3) Assume (Z, d) is a metric space,
(4) $\mathcal{P}_{cl}(Z) = \{Y \subset \mathcal{P}(Z) : Y$ is closed$\}$,
(5) $\mathcal{P}_b(Z) = \{Y \subset \mathcal{P}(Z) : Y$ is bounded$\}$,
(6) $\mathcal{P}_{cv}(Z) = \{Y \subset \mathcal{P}(Z) : Y$ is convex$\}$,
(7) $\mathcal{P}_{cp}(Z) = \{Y \subset \mathcal{P}(Z) : Y$ is compact$\}$,

(8) $\mathcal{P}_{cp,cv}(Z) = \mathcal{P}_{cp}(Z) \cap \mathcal{P}_{cv}(Z)$.

Definition 2.1. A function $w \in PC\big([-r, \mathbb{T}], X\big) \cap AC\big((\mathcal{T}_l, \mathcal{T}_{l+1}), X\big)$ is said to be a mild solution of the impulsive inclusion problem (1.1)-(1.3), if there exists a function $v \in L^1((0, \mathbb{T}], X)$ such that

$$v(\xi) \in \mathcal{F}(\xi, w_\xi, \int_0^\xi k(\xi, \eta) h(\eta, w_\eta) d\eta),$$

$\xi \in (0, \mathbb{T}], \xi \neq \mathcal{T}_l, l = 1(1)m$ a.e. on $(0, \mathbb{T}]$ and satisfies the equation

$$w(\xi) = T(\xi)[\phi(0) - (g(w_{\xi_1}, ..., w_{\xi_p}))(0)] + \int_0^\xi T(\xi - \eta) v(\eta) d\eta + \sum_{0<\mathcal{T}_l<\xi} T(\xi - \mathcal{T}_l) I_l w(\mathcal{T}_l).$$

Definition 2.2. For every $\phi \in C, w_1 \in X$, the system (1.4)-(1.6) is called controllable on $(0, \mathbb{T}]$, if there exists a control $x \in L^2((0, \mathbb{T}], X)$ such that the mild solution w of (1.4)-(1.6) satisfies $w(\mathbb{T}) = w_1$ satisfying

$$w(\xi) = T(\xi)[\phi(0) - (g(w_{\xi_1}, ..., w_{\xi_p}))(0)] + \int_0^\xi T(\xi - \eta) v(\eta) d\eta + \int_0^\xi T(\xi - \eta) B x(\eta) d\eta + \sum_{0<\mathcal{T}_l<\xi} T(\xi - \mathcal{T}_l) I_l w(\mathcal{T}_l).$$

Theorem 2.3. (Martelli fixed point theorem [18]) *Let X be a Banach space and $G : X \to P_{cp,cv}(X)$ be an upper semi-continuous and condensing map. If the set $\mathrm{M} = \{y \in X : \lambda y \in G(y),$ for some $\lambda > 1\}$ is bounded, then G has a fixed point.*

Lemma 2.4. ([7]) *Let for $t \geq t_0$, the following inequality holds*

$$u(t) \leq a(t) + \int_{t_0}^{t} b(t,s) u(s) ds + \int_{t_0}^{t} (\int_{t_0}^{s} k(t,s,\tau) u(\tau) d\tau) ds + \sum_{t_0<\tau_k<t} \beta_k(t) u(t_k)$$

where, $u, a \in PC([t_0, \infty), \mathbb{R}_+)$, a is nondecreasing, $b(t,s)$ and $k(t,s,\tau)$ are continuous and non-negative functions for $t, s, \tau \geq t_0$ and are nondecreasing with respect to t, $\beta_k(t)(k \in \mathbb{N})$ are nondecreasing for $t \geq t_0$. Then for $t \geq t_0$ the following inequality hold:

$$u(t) \leq a(t) \prod_{t_0<\tau_k<t} (1 + \beta_k(t)) exp(\int_{t_0}^{t} b(t,s) ds) + \int_{t_0}^{t} \int_{t_0}^{s} k(t,s,\tau) d\tau) ds.$$

Theorem 2.5. (Closed Graph Theorem [19]) *Let X be a Banach space, $F : J \times X \to P_{cp,c}(X)$ be an L^1 Caratheodary multivalued map with*

$$S_F(y) = \{g \in L^1(J, X) : g(\xi) \in F(t, y(t)) \; for \; a.e. \quad \xi \in J = [0, \mathbb{T}]\} \neq \phi$$

and Γ be a linear continuous mapping from $L^1(J, X)$ to $C(J, X)$. Then the operator

$$\Gamma \circ S_F : C(J, X) \to P_{pc,c}(C(J, X))$$
$$y \to (\Gamma \circ S_F)(y) = \Gamma(S(F(y)))$$

is a closed graph operator in $C(J, X) \times C(J, X)$.

Theorem 2.6. (Covitz and Nadler [11]) *Let (X,d) be a complete metric space. If $G : X \to P_{cl}(X)$ is a contraction, then G has a fixed point in X.*

2.1. **Hypotheses:**

(H_1) Let $\mathcal{F} : [0, \mathbb{T}] \times C \times X \to \mathcal{P}_{b,cp,cv}(X)$ be a measurable function with respect to ξ for each $w \in X$, upper semi-continuous with respect to w and the set $S_{\mathcal{F},w}$ is nonempty.

(H_2) Let $f : [0, \mathbb{T}] \times C \times X \to X$ and $h : [0, \mathbb{T}] \times C \to X$ be continuous functions. There exists continuous nondecreasing functions p and q : $[0, \mathbb{T}] \to \mathbb{R}_+$ such that

$$\|f(\xi, \psi, w)\| \leq p(\xi)(\|\psi\|_C + \|w\|),$$
$$\|h(\xi, \psi)\| \leq q(\xi)(\|\psi\|_C),$$

for all $\xi \in [0, \mathbb{T}]$, $\psi \in C$ and $w \in X$, $f \in \mathcal{F}(\xi, w_\xi, \int_0^\xi k(\xi, \eta)h(\eta, w_\eta)d\eta)$.

(H_3) For every positive integer l, there exist functions $h_l \in L^1[(0, \mathbb{T}], \mathbb{R}_+]$ satisfying

$$\sup_{\|\psi\|_C, \|u\| \leq l} \|f(\xi, \psi, u)\| \leq h_l(\xi), \quad a.e. \quad \xi \in (0, \mathbb{T}],$$

where $f \in \mathcal{F}(\xi, w_\xi, \int_0^\xi k(\xi, \eta)h(\eta, w_\eta)d\eta)$.

(H_4) For $g : C^p \to C$, there exists a constant $G \geq 0$ such that

$$\max_{\xi \in [-r,0]} \| g(w_{\xi_1}, w_{\xi_2}, ..., w_{\xi_p}) \| \leq G.$$

(H_5) Let $I_l : X \to X$ be functions which grantees the existence of constants L_l satisfying

$$\|I_l(w)\| \leq L_l \|w\|, \; w \in X, \; l = 1, 2, ..., m.$$

(H_6) The operator $W : L^2((0,\mathbb{T}],Y) \to X$ is linear and defined as

$$Wx = \int_0^{\mathbb{T}} T(\mathbb{T}-\eta)Bx(\eta)d\eta$$

has an inverse bounded operator W^{-1} in $\frac{L^2((0,\mathbb{T}],Y)}{kerW}$. Then there exist positive constants N_1, N_2 satisfying $\|B\| \leq N_1$, $\|W^{-1}\| \leq N_2$.

$(H_1^{'})$ There exists a constant $G^{'}$ which satisfies

$$\|(g(w_{\xi_1},...,w_{\xi_p}))(0) - (g(\bar{w}_{\xi_1},...,\bar{w}_{\xi_p}))(0)\| \leq G^{'}\|w_{\xi_p} - \bar{w}_{\xi_p}\|.$$

$(H_2^{'})$ $\mathcal{F} : [0,\mathbb{T}] \times C \times X \to P_{cp,cv}(X)$ is measurable and for $\bar{l} \in L^1[(0,\mathbb{T}],\mathbb{R}]$, it satisfies

$$\mathcal{H}_d(\mathcal{F}(w), \mathcal{F}(\overline{w})) \leq \bar{l}(\xi)\|w - \bar{w}\|, \ \textit{for each } w, \bar{w} \in PC([-r,\mathbb{T}],X).$$

$(H_3^{'})$ Let $h : [0,\mathbb{T}] \times C \to X$ be a continuous function. Then there exists a positive constants p^* such that

$$\|h(\eta,w) - h(\eta,v)\| \leq p^*\|w - v\|.$$

3. Existence results

Theorem 3.1. *Assume the hypotheses* (H_1)-(H_5) *hold. Then the impulsive inclusion problem* (1.1)-(1.3) *has at least one mild solution* w *on* $[-r,T]$.

Proof. Since the proof of Theorem 3.1 goes on similar line with Theorem 4.1, we will give the proof in Theorem 4.1. □

Theorem 3.2. *Assume the hypotheses* $(H_1^{'})$-$(H_3^{'})$, (H_4), (H_5) *hold and the following condition satisfied*

$$\left(K_0(G' + l^{**}(1 + Lp^*) + \sum_{0<\mathcal{T}_l<\xi} L_l)\right) < 1.$$

Then the impulsive inclusion problem (1.1)-(1.3) *has at least one mild solution.*

Proof. Consider the multivalued operator

$$F : PC([-r,\mathbb{T}],X) \to \mathcal{P}(PC([-r,\mathbb{T}],X)),$$

where,

$F(w) = m \in PC([-r, \mathbb{T}], X)$:

$$m(\xi) = \begin{cases} \phi(\xi) - (g(w_{\xi_1}, ..., w_{\xi_p}))(\xi), & \xi \in [-r, 0]; \\ T(\xi)[\phi(0) - (g(w_{\xi_1}, ..., w_{\xi_p}))(0)] + \int_0^\xi T(\xi - \eta) v(\eta) d\eta & \\ + \sum\limits_{0 < \mathcal{T}_l < \xi} T(\xi - \mathcal{T}_l) I_l w(\mathcal{T}_l), & \xi \in (0, \mathbb{T}]. \end{cases}$$

where $v \in S_{\mathcal{F},w}$,

$$S_{\mathcal{F},w}$$

$$= \{w \in L^1((0, \mathbb{T}], X) : v(\xi) \in \mathcal{F}(\xi, w_\xi, \int_0^\xi k(\xi, \eta) h(\eta, w_\eta) d\eta) \text{ for a.e. } \xi \in (0, \mathbb{T}]\}.$$

Set

$$\Delta = \{w \in PC([-r, \mathbb{T}], X) : w(\xi) = \phi(\xi) - (g(w_{\xi_1}, ..., w_{\xi_p}))(\xi), \forall \xi \in [-r, 0]\}.$$

Let us consider that the multivalued operator $F : \Delta \to \mathcal{P}(\Delta)$ is defined as above. We prove that F satisfies the Covitz-Nadler fixed point theorem (see Theorem 2.6).

Step I: $F(w) \in \mathcal{P}_{cl}(\Delta)$, for each $w \in PC([-r, \mathbb{T}], X)$.

Let $\{w_n\}_{n \geq 0} \in F(w)$ be such that $w_n \to w$ in $PC([-r, \mathbb{T}], X)$. For every $\xi \in (0, \mathbb{T}]$ there exists $v_n \in S_{\mathcal{F},w}$ satisfying

$$w_n(\xi) = T(\xi)[\phi(0) - (g(w_{\xi_1}, ..., w_{\xi_p}))(0)] + \int_0^\xi T(\xi - \eta) v_n(\eta) d\eta \\ + \sum_{0 < \mathcal{T}_l < \xi} T(\xi - \mathcal{T}_l) I_l w_n(\mathcal{T}_l), \; i = 1, 2.$$

Since $\mathcal{F}$ has compact values and $\{v_n\}$ converges to v in $L^1((0, \mathbb{T}], X)$, $v \in S_{\mathcal{F},w}$. Then for each $\xi \in (0, \mathbb{T}]$, $w_n(\xi) \to w(\xi)$

$$w(\xi) = T(\xi)[\phi(0) - (g(w_{\xi_1}, ..., w_{\xi_p}))(0)] + \int_0^\xi T(\xi - \eta) v(\eta) d\eta \\ + \sum_{0 < \mathcal{T}_l < \xi} T(\xi - \mathcal{T}_l) I_l w(\mathcal{T}_l), \; i = 1, 2.$$

It means that $w \in F(w)$.

Step II: There exists $\delta < 1$ such that

$$\mathcal{H}_d(F(w), F(\overline{w})) \leq \delta \|w - \bar{w}\|, \; w, \bar{w} \in PC([-r, \mathbb{T}], X).$$

Let $w, \bar{w} \in PC([-r, \mathbb{T}], X)$ and $m \in F(w)$. Then there exists $v \in S_{\mathcal{F},w}$ such that for each $\xi \in (0, \mathbb{T}]$, we get,

$$m(\xi) = T(\xi)[\phi(0) - (g(w_{\xi_1}, ..., w_{\xi_p}))(0)] + \int_0^\xi T(\xi - \eta)v(\eta)d\eta + \sum_{0<\mathcal{T}_l<\xi} T(\xi - \mathcal{T}_l)I_l w(\mathcal{T}_l),\ i = 1, 2.$$

By using conditions (H_2') and (H_3'), we have

$$\mathcal{H}_d\bigg(\mathcal{F}\bigg(\xi, w_\xi, \int_0^\xi k(\xi,\eta)h(\eta, w_\eta)d\eta), \mathcal{F}(\xi, \bar{w_\xi}, \int_0^\xi k(\xi,\eta)h(\eta, \bar{w_\eta})d\eta\bigg)\bigg) \leq \bar{l}(\xi)\bigg[\|w_\xi - \bar{w_\xi}\| + Lp^*\|w_\eta - \bar{w_\eta}\|\bigg],$$

hence there is $y \in \mathcal{F}(\xi, \bar{w_\xi}, \int_0^\xi k(\xi,\eta)h(\eta, \bar{w_\eta})d\eta)$ such that

$$\|v(\xi) - y\| \leq \bar{l}(\xi)\bigg[\|w_\xi - \bar{w_\xi}\| + Lp^*\|w_\eta - \bar{w_\eta}\|\bigg].$$

Consider, the map $\sigma : (0, \mathbb{T}] \to \mathcal{P}(X)$ given by

$$\sigma(\xi) = \bigg\{y \in X : \|v(\xi) - y\| \leq \bar{l}(\xi)[\|w_\xi - \bar{w_\xi}\| + Lp^*\|w_\eta - \bar{w_\eta}\|]\bigg\}.$$

As the multivalued operator $V(\xi) = \sigma(\xi) \cap \mathcal{F}(\xi, w_\xi, \int_0^\xi k(\xi,\eta)h(\eta, w_\eta)d\eta)$ is measurable, there exists a function $\xi \to \bar{v}(\xi)$, is measurable choice for V. So we have

$$\bar{v}(\xi) \in \mathcal{F}(\xi, w_\xi, \int_0^\xi k(\xi,\eta)h(\eta, w_\eta)d\eta),$$

therefore

$$\|v(\xi) - \bar{v}(\xi)\| \leq \bar{l}(\xi)\big(\|w_\xi - \bar{w_\xi}\| + Lp^*\|w_\eta - \bar{w_\eta}\|\big).$$

We define,

$$\bar{m}(\xi) = T(\xi)[\phi(0) - (g(\bar{w}_{\xi_1}, ..., \bar{w}_{\xi_p}))(0)] + \int_0^\xi T(\xi - \eta)\bar{v}(\eta)d\eta + \sum_{0<\mathcal{T}_l<\xi} T(\xi - \mathcal{T}_l)I_l \bar{w}(\mathcal{T}_l),\ i = 1, 2.$$

Then we have

$$\begin{aligned}
\|m(\xi)-\bar{m}(\xi)\| &\leq K_0\|g(w_{\xi_1},...,w_{\xi_p})-g(\bar{w}_{\xi_1},...,\bar{w}_{\xi_p})\| \\
&\quad + K_0\int_0^{\xi}\|v(\eta)-\bar{v}(\eta)\|d\eta + \sum_{0<\mathcal{T}_l<\xi} K_0L_l\|w(\mathcal{T}_l)-\bar{w}(\mathcal{T}_l)\| \\
&\leq K_0G^{'}\|w_{\xi_p}-\bar{w}_{\xi_p}\| \\
&\quad + K_0\int_0^{\xi}\bar{l}(\eta)[\|w_\xi-\bar{w}_\xi\|+Lp^*\|w_\eta-\bar{w}_\eta\|]d\eta \\
&\quad + \sum_{0<\mathcal{T}_l<\xi} K_0L_l\|w(\mathcal{T}_l)-\bar{w}(\mathcal{T}_l)\| \\
&\leq K_0G^{'}\|w-\bar{w}\|_{PC} + K_0\int_0^{\mathbb{T}}\bar{l}(\eta)\|w-\bar{w}\|_{PC} \\
&\quad + K_0\int_0^{\mathbb{T}}\bar{l}(\eta)Lp^*\|w_\eta-\bar{w}_\eta\|_{PC}]d\eta \\
&\quad + \sum_{0<\mathcal{T}_l<\xi} K_0L_l\|w(\mathcal{T}_l)-\bar{w}(\mathcal{T}_l)\| \\
&\leq [K_0G^{'}+K_0l^{**}+K_0Lp^*l^{**}+\sum_{0<\mathcal{T}_l<\xi}K_0L_l]\|w-\bar{w}\|_{PC} \\
&\leq [K_0(G^{'}+K_0l^{**}+Lp^*l^{**}+\sum_{0<\mathcal{T}_l<\xi}L_l)]\|w-\bar{w}\|_{PC},
\end{aligned}$$

where, $l^{**}=\int_0^{\mathbb{T}}\bar{l}(\eta)d\eta$.

Now by exchanging the roles of the w and $\bar{w}$, we get

$$\mathcal{H}_d(F(w),F(\overline{w})) \leq \left[K_0(G^{'}+Kl^{**}+Lp^*l^{**}+\sum_{0<\mathcal{T}_l<\xi}L_l)\right]\|w-\bar{w}\|_{PC},$$

where $\|w-\bar{w}\|_{PC}=\sup\{\|w-\bar{w}\|,\ w\in[-r,\mathbb{T}]\}$. Therefore F is a contraction. By Covitz-Nadler fixed point theorem, F has a fixed point w, which is a mild solution of the impulsive inclusion problem (1.1)-(1.3). □

4. Controllability result

Theorem 4.1. *Suppose that the hypotheses* (H_1)-(H_6) *hold. Then the impulsive inclusion problem* (1.4)-(1.6) *is controllable on* $[-r,T]$.

Proof. For any $w_1\in X$, define control function $x(\xi)$ as

$$x(\xi) = W^{-1}\big\{w_1 - T(\xi)\big[(\phi(0) - (g(w_{\xi_1}, ..., w_{\xi_p})))(0)\big] \\ - \int_0^{\xi} T(\xi-\eta)v(\eta)d\eta - \sum_{0<\mathcal{T}_l<\xi} T(\xi - \mathcal{T}_l)I_l w(\mathcal{T}_l)\big]\big\}.$$

Define the multivalued operator $F : PC([-r,\mathbb{T}],X) \to \mathcal{P}(PC([-r,\mathbb{T}],X))$, such that

$$F(w) = m \in PC([-r,\mathbb{T}],X) :$$

$$m(\xi) = \begin{cases} \phi(\xi) - (g(w_{\xi_1}, ..., w_{\xi_p}))(\xi), & \xi \in [-r,0]; \\ T(\xi)[\phi(0) - (g(w_{\xi_1}, ..., w_{\xi_p}))(0)] + \int_0^{\xi} T(\xi-\eta)v(\eta)d\eta & \\ + \int_0^{\xi} T(\xi-\eta)Bx(\eta)d\eta + \sum\limits_{0<\mathcal{T}_l<\xi} T(\xi-\mathcal{T}_l)I_l w(\mathcal{T}_l), & \xi \in (0,\mathbb{T}]. \end{cases}$$

where $v \in S_{\mathcal{F},w}$,

$$S_{\mathcal{F},w}$$

$$= \{w \in L^1((0,\mathbb{T}],X) : v(\xi) \in \mathcal{F}(\xi, w_\xi, \int_0^{\xi} k(\xi,\eta)h(\eta,w_\eta)d\eta) \text{ for a.e.} \, \xi \in (0,\mathbb{T}]\}.$$

We prove that F satisfies the assumptions of Martelli's fixed point theorem (see Theorem 2.3).

Step I: We prove that $F(w)$ is convex, for each $w \in PC([-r,\mathbb{T}],X)$.

If m_1 and $m_2 \in F(w)$, then there exist $v_1, v_2 \in S_{\mathcal{F},w}$ such that for $\xi \in (0,\mathbb{T}]$, we have

$$m_i(\xi) = T(\xi)[\phi(0) - (g(w_{\xi_1}, ..., w_{\xi_p}))(0)] + \int_0^{\xi} T(\xi-\eta)v_i(\eta)d\eta \\ + \int_0^{\xi} T(\xi-\eta)Bx_i(\eta)d\eta + \sum_{0<\mathcal{T}_l<\xi} T(\xi-\mathcal{T}_l)I_l w(\mathcal{T}_l), \quad i = 1,2.$$

For $0 \le \lambda \le 1$, we get,

$$(\lambda m_1 + (1-\lambda)m_2)(\xi)$$

$$= T(\xi)[\phi(0) - (g(w_{\xi_1}, ..., w_{\xi_p}))(0)] + \int_0^{\xi} T(\xi-\eta)[\lambda v_1(\eta) + (1-\lambda)v_2(\eta)]d\eta \\ + \int_0^{\xi} T(\xi-\eta)[\lambda Bx_1(\eta) + (1-\lambda)Bx_2(\eta)]d\eta + \sum_{0<\mathcal{T}_l<\xi} T(\xi-\mathcal{T}_l)I_l w(\mathcal{T}_l), i = 1,2.$$

Since $S_{\mathcal{F},w}$ is convex, $\lambda m_1 + (1-\lambda)m_2 \in S_{\mathcal{F},w}$. Therefore, $\lambda m_1 + (1-\lambda)m_2 \in F(w)$.

Step II: F is a map from bounded sets into bounded sets in $PC([-r,\mathbb{T}],X)$.

Define $B_q = \{y \in PC([-r, \mathbb{T}], X) : \|w\| \leq q, w \in X\}$. Then for any positive constant $\bar{q}$, we show that

$$\|F(w)\| = \sup\{\|m\| : m \in F(w)\} \leq \bar{q}.$$

Let $w \in B_q$ and $m \in F(w)$. Then there exists a $v \in S_{\mathcal{F},w}$ such that

$$m(\xi) = T(\xi)[\phi(0) - (g(w_{\xi_1}, ..., w_{\xi_p}))(0)] + \int_0^{\xi} T(\xi - \eta)v(\eta)d\eta + \int_0^{\xi} T(\xi - \eta)Bx(\eta)d\eta + \sum_{0<\mathcal{T}_l<\xi} T(\xi - \mathcal{T}_l)I_l w(\mathcal{T}_l).$$

Therefore, by using hypotheses $(H_1]), (H_3), (H_4), (H_5)$ and (H_6), we get,

$$\begin{aligned} \|m(\xi)\| &\leq K_0(D + G) + K_0 \int_0^{\mathbb{T}} h_{l(\eta)} d\eta \\ &\quad + K_0 \int_0^{\mathbb{T}} \|Bx(\eta)\| d\eta + L_l \sum_{0<\mathcal{T}_l<\xi} \|w(\mathcal{T}_l)\| \\ &\leq K_0(D + G + N_1\mathbb{T}) + K_0 \|h_{l(\eta)}\|_{L^1} + L_l \sum_{0<\mathcal{T}_l<\xi} \|w(\mathcal{T}_l)\|. \end{aligned}$$

Then for each $m \in F(B_q)$, we obtain,

$$\|F(w)\| \leq K_0(D + G + N_1\mathbb{T}) + K_0 \|h_{l(\eta)}\|_{L^1} + L_l \sum_{0<\mathcal{T}_l<\xi} \|w(\mathcal{T}_l)\| := \bar{q}.$$

Step-III: F maps from bounded sets into equicontinuous sets of $PC([-r, \mathbb{T}], X)$.

Let $\xi_1, \xi_2 \in (0, \mathbb{T}] \setminus \{\mathcal{T}_l, \mathcal{T}_2, ..., \mathcal{T}_l\}$, $\xi_1 < \xi_2$. Let B_q be a bounded set of $PC([-r, \mathbb{T}], X)$. Let $w \in B_q$, $m \in F(w)$ and $v \in S_{\mathcal{F},w}$ be such that

$$\begin{aligned} \|m(\xi_2) - m(\xi_1)\| &\leq \|T(\xi_2) - T(\xi_1)\|[\|\phi(0) + \|g(w_{\xi_1}, ..., w_{\xi_p})\|] \\ &\quad + \int_0^{\xi_2} \|T(\xi_2 - \eta) - T(\xi_1 - \eta)\| h_{l(\eta)}(\eta) d\eta \\ &\quad + \int_{\xi_1}^{\xi_2} \|T(\xi_1 - \eta)\| h_{l(\eta)}(\eta) d\eta \\ &\quad + \int_0^{\xi_2} \|T(\xi_2 - \eta) - T(\xi_1 - \eta)\| \|Bx(\eta)\| d\eta \\ &\quad + \int_{\xi_1}^{\xi_2} \|T(\xi_1 - \eta)\| \|Bx(\eta)\| d\eta \sum_{0<\mathcal{T}_l<\xi_2-\xi_1} L_l \|w(\mathcal{T}_l)\| \\ &\quad + \sum_{0<\mathcal{T}_l<\xi_1} \|T(\xi_2 - \mathcal{T}_l) - T(\xi_1 - \mathcal{T}_l)\| L_l \|w(\mathcal{T}_l)\|. \end{aligned}$$

Since $\xi_2 \to \xi_1$ and ϵ is arbitrary small, $\|m(\xi_2) - m(\xi_1)\| \to 0$, since strongly continuous operators $T(\xi)$ are compact implies the continuity in uniform operator topology. This is the proof of equicontinuity for the intervals where $\xi \neq \mathcal{T}_l$, $l = 1$ to m. Equicontinuities for the cases $\xi_1 < \xi_2 \leq 0$ and $\xi_1 \leq 0 \leq \xi_2$ are obvious. So by Arzela-Ascoli theorem, the operator $F : PC([-r, \mathbb{T}], X) \to \mathcal{P}(PC([-r, \mathbb{T}], X))$ is completely continuous multivalued map and hence it proves F is a condensing operator.

Step IV: F has a closed graph.

Let $w_n \to w_*$, $m_n \in F(w_n)$ and $m_n \to m_*$. We prove that $m_* \in F(w_*)$. For $m_n \in F(w_n)$ there exist $v_n \in S_{\mathcal{F},w_n}$ such that for each $\xi \in (0, \mathbb{T}]$

$$m_n(\xi) = T(\xi)[\phi(0) - (g(w_n)_{\xi_1}, ..., (w_n)_{\xi_p})(0)] + \int_0^\xi T(\xi - \eta)v_n(\eta)d\eta + \int_0^\xi T(\xi - \eta)Bx_n(\eta)d\eta + \sum_{0<\mathcal{T}_l<\xi} T(\xi - \mathcal{T}_l)I_l w_n(\mathcal{T}_l).$$

Now we prove the existence of $v_* \in S_{\mathcal{F},w_*}$ such that

$$m_*(\xi) = T(\xi)[\phi(0) - (g(w_*)_{\xi_1}, ..., (w_*)_{\xi_p})(0)] + \int_0^\xi T(\xi - \eta)v_*(\eta)d\eta + \int_0^\xi T(\xi - \eta)Bx_*(\eta)d\eta + \sum_{0<\mathcal{T}_l<\xi} T(\xi - \mathcal{T}_l)I_l w_*(\mathcal{T}_l).$$

Since I_l, $l = 1(1)m$ are continuous and g is completely continuous, we get,

$$\begin{aligned}
&\Bigg\|\Bigg(m_n(\xi) - T(\xi)[\phi(0) - (g(w_n)_{\xi_1}, ..., (w_n)_{\xi_p})(0)] - \int_0^\xi T(\xi - \eta)Bx_n(\eta)d\eta \\
&\quad - \sum_{0<\mathcal{T}_l<\xi} T(\xi - \mathcal{T}_l)I_l w_n(\mathcal{T}_l)\Bigg) \\
&\quad - \Bigg(m_*(\xi) - T(\xi)[\phi(0) - (g(w_*)_{\xi_1}, ..., (w_*)_{\xi_p})(0)] - \int_0^\xi T(\xi - \eta)Bx_*(\eta)d\eta \\
&\quad - \sum_{0<\mathcal{T}_l<\xi} T(\xi - \mathcal{T}_l)I_l w_*(\mathcal{T}_l)\Bigg)\Bigg\| \\
&\to 0 \ \text{ as } \ n \to \infty.
\end{aligned}$$

Consider, $\Gamma : L^1((0, \mathbb{T}], X) \to C((0, \mathbb{T}], X)$, a linear continuous operator and $v \to \Gamma(v)(\xi) = \int_0^\xi T(\xi - \eta)v(\eta)d\eta$.

From Theorem 2.5, $\Gamma \circ S_{\mathcal{F}(w)}$ is closed graph operator. Furthermore, we have,

$$m_n(\xi) - T(\xi)[\phi(0) - (g(w_n)_{\xi_1}, ..., (w_n)_{\xi_p})(0)] - \int_0^{\xi} T(\xi-\eta)Bx_n(\eta)d\eta$$
$$- \sum_{0<\mathcal{T}_l<\xi} T(\xi-\mathcal{T}_l)I_l w_n(\mathcal{T}_l) \in \Gamma(S_{\mathcal{F}(w_n)}).$$

Since $w_n \to w_*$, by using closed graph lemma, we get

$$m_*(\xi) - T(\xi)[\phi(0) - (g(w_*)_{\xi_1}, ..., (w_*)_{\xi_p})(0)] - \int_0^{\xi} T(\xi-\eta)Bx_*(\eta)d\eta$$
$$- \sum_{0<\mathcal{T}_l<\xi} T(\xi-\mathcal{T}_l)I_l w_*(\mathcal{T}_l) = \int_0^{\xi} T(\xi-\eta)v_*(\eta)d\eta,$$

for some $v_* \in S_{\mathcal{F}(w_*)}$.

Step V: We now prove the set

$$\mathbf{M} := \{w \in PC([-r,\mathbb{T}], X) : \lambda w \in F(w), \text{ for some } \lambda > 1\}$$

is bounded.

Let $w \in \mathbf{M}$. Then $\lambda w \in F(w)$, for some $\lambda > 1$ and for each $\xi \in (0,\mathbb{T}]$

$$w(\xi) = \lambda^{-1}\Big[T(\xi)[\phi(0) - (g(w_{\xi_1}, ..., w_{\xi_p}))(0)] + \int_0^{\xi} T(\xi-\eta)v(\eta)d\eta$$
$$+ \int_0^{\xi} T(\xi-\eta)Bx(\eta)d\eta + \sum_{0<\mathcal{T}_l<\xi} T(\xi-\mathcal{T}_l)I_l w(\mathcal{T}_l)\Big].$$

Since $v(\xi) \in \mathcal{F}(\xi, w_\xi, \int_0^{\xi} k(\xi,\eta)h(\eta, w_\eta)d\eta)$, by using the hypotheses (H_2), (H_4), (H_5), we obtain that,

$$\|w(\xi)\| \le K_0\lambda^{-1}(D + G + N_1\mathbb{T}) + \int_0^{\xi} \lambda^{-1}K_0 p(\eta)[\|w_\eta\|$$
$$+ \int_0^{\eta} Lq(\mathcal{T})(\|w_{\mathcal{T}}\|)\|d\mathcal{T}]d\eta + \sum_{0<\mathcal{T}_l<\xi} K_0 L_l \lambda^{-1}\|w(\mathcal{T}_l)\|.$$

In [15], Authors obtained bound for $\|w(\xi)\|$ and $\|w(\xi)\| \le Q$, for some constant Q using impulsive inequality given in Lemma 2.4. Therefore $\mathbb{M}$ is bounded. By Martelli's fixed point theorem, F has a fixed point. Hence the impulsive inclusion problem (1.4)-(1.6) is controllable. □

5. Conclusion

Existence and Controllability of solutions of impulsive integro-differential inclusions have been proved successfully with the aid of Martelli and Covitz-Nadler fixed point theorems. We have improved the results using impulsive inequality and nonlocal condition which is more precise than usual initial condition. Further it can be extended these results for fractional, mixed impulsive integro-differential equations which has wide applicability in Engineering and Biology.

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