Nonlinear Functional Analysis and Applications
Vol. 25, No. 4 (2020), pp. 769-797
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2020.25.04.10
http://nfaa.kyungnam.ac.kr/journal-nfaa

NEW GERAGHTY TYPE CONDENSING OPERATORS AND SOLVABILITY OF NONLINEAR QUADRATIC VOLTERRA-STIELTJES INTEGRAL EQUATION

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Abstract. The true motivation of this article is to provide sufficient conditions with the aid of Geraghty type condensing operators that guarantee the existence of a solution of non-linear quadratic Volterra-Stieltjes integral equation. We also address several new fixed point theorems that ensure the existence of a fixed point for Geraghty type condensing operators in real Banach spaces. An example and numerical approximations are presented to justify the basis of our results.

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[0]Received June 17, 2020. Revised September 13, 2020. Accepted September 17, 2020.
[0]2010 Mathematics Subject Classification: 47H09, 47H10, 37C25.
[0]Keywords: Monotone operator, ξ_μ-admissible, ξ_μ-β condensing operators, measure of noncompactness, non-linear quadratic Volterra-Stieltjes integral equation.
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1. Introduction

Integral equations are extensively employed in an optimum theory of control queuing, radiative transfer, engineering, biology, mechanics, economics and mathematical physics (see, e.g., [7, 9, 10, 15, 16, 17] and the references therein). The augmentation of techniques of measures of noncompactness (due to Darbo [11]) opened a new direction of research to study the solvability of non-linear integral equations. Recently, many researchers worked on the generalizations of the Darbo′s fixed point theorem and its applicability in solving various classes of integral equations that are based on the techniques of measure of non-compactness (see, e.g., [2, 6, 14, 18, 19, 21]).

Inspired by the above findings, in this article, we first describe ξ_μ-β, ξ_μ-β-ψ condensing operators and to prove related fixed point theorems. In addition, it was used to examine the solvability of the following non-linear quadratic Volterra-Stieltjes integral equation:

$$\begin{aligned} u(l) =&\pi_1(l, u(a_1(l)), ..., u(a_m(l))) + \pi_2(l, u(b_1(l)), ..., u(b_n(l))) \\ &\times \int_0^{\phi(l)} g(l, \tau, u(c_1(l)), ..., u(c_p(l))) d_\tau K(\phi(l), \phi(\tau)), \end{aligned} \tag{1.1}$$

where $l \in \mathbb{R}_+$ and the integral equation (1.1) is considered in $BC(\mathbb{R}_+)$. The functions $a_r (1 \leq r \leq m),\ b_t(1 \leq t \leq n),\ c_s(1 \leq s \leq p), \pi_i\ (i = 1, 2),\ \phi$ and $K(l, \tau)$ defined on their respective domains and verifies the assumptions given in Section 4. The concept of measure of non-compactness is adopted in this paper and allows us not merely to look at the existence of a solution of the mentioned integral equation but also characterize the asymptotic stability of the solution.

The remainder of the paper was made up in the following way. In Section 2, we contribute some definitions and essential theorems related to non-compactness measures. Section 3, includes new generalizations of the Darbo′s fixed point theorem and its consequences. Section 4, deals with the resolvability of the integral equation (1.1) by using the obtained result in Section 3. Finally, an illustrative example with some numerical estimations is provided.

2. Background

In this paper, X denotes the Banach space with the norm $\|.\|$ and the zero element by θ. The closed ball centred at x with radius r is denoted by $B(x, r)$. An Algebraic operation on the sets is denoted by λU and $U + V$. Next, the closure of a set U is denoted by the symbol $\overline{U}$ and coU, $\overline{co}U$ denotes the convex hull and closed convex hull of U, respectively. The family of all bounded subsets of the space X is denoted by $\mathfrak{M}_X$ whereas $\mathfrak{N}_X$ is the subfamily comprising all pre-compact subsets of X. Finally, for the sake of writing assume that Λ denote the class of nonempty, bounded, closed and convex subsets a Banach space X, and I.E. used to denote integral equation and MNC stands for a measure of noncompactness.

First of all, we recall the axiomatic concept of a measure of non-compactness.

Definition 2.1. ([8]) Let a function $\mu : \mathfrak{M}_X \longrightarrow \mathbb{R}_+$ is called a measure of noncompactness if satisfies the following axioms:

(i) The family $ker\mu = \{U \in \mathfrak{M}_X : \mu(X) = 0\}$ is a nonempty set and $ker\mu \subseteq \mathfrak{N}_X$;
(ii) $U \subset V \implies \mu(U) \leq \mu(V)$;
(iii) $\mu(U) = \mu(\overline{U})$;
(iv) $\mu(coU) = \mu(U)$;
(v) $\mu(\lambda U + (1-\lambda))V \leq \lambda\mu(U) + (1-\lambda)\mu(V), \quad$ for all $\lambda \in [0,1]$;
(vi) If $U_n \in \mathfrak{M}_X$ for $n = 1, 2, \cdots$ is decreasing sequence of closed subsets of X and $\lim_{n\to\infty} \mu(U_n) = 0$ then $U_\infty := \bigcap_{n=1}^{\infty} U_n$ is nonempty.

The family defined in axiom (i) is called the kernel of the measure of non-compactness μ and denoted by $ker\mu$. In fact, by the virtue of axiom (vi) we have $\mu(U_\infty) \leq \mu(U_n)$ for any n, thus $\mu(U_\infty) = 0$. This yields that $U_\infty \in ker\mu$.

Theorem 2.2. (Schauder′s fixed point theorem [20]) *Let Ω be the member of the family Λ. If P is a continuous and compact mapping on Ω, then P has at least one fixed point in Ω.*

Definition 2.3. (μ-Condensing operator) Let Ω be the member of the family Λ and P be a self-mapping defined on Ω. A mapping P is said to be a μ-condensing if

$$\mu(P(A)) \leq \lambda\mu(A)$$

for some $\lambda \in [0, 1)$ and every nonempty subset A of Ω.

Now, the Darbo′s fixed point theorem concerning the measure of non-compactness can be described as follows.

Theorem 2.4. (Darbo′s fixed point theorem [11]) *Let Ω be the member of the family Λ and P be a continuous self-mapping defined on Ω. If P is μ-condensing, then P has at least one fixed point in Ω.*

Definition 2.5. ([12]) Let Υ denote the class of those functions $\beta : \mathbb{R}^+ \to [0, 1)$ which satisfies the condition $\beta(t_n) \to 1$ implies $t_n \to 0$.

Definition 2.6. ([19]) Let P be a self-mapping on X and $\xi : 2^X \to [0, +\infty)$. Then P is said to be ξ_μ-admissible if

$$\xi(E) \geq 1 \implies \xi(\overline{co}P(E)) \geq 1,$$

for every $E \in 2^X$.

Example 2.7. Let $P : BC(\mathbb{R}_+) \to BC(\mathbb{R}_+)$ and there exist $\xi : 2^{BC(\mathbb{R}_+)} \to \mathbb{R}_+$ such that

$$Pu(s) = 2u(s) \quad \text{and} \quad \xi(E) = diam(E),$$

for every $u \in BC(\mathbb{R}_+)$ and $E \subset BC(\mathbb{R}_+)$. Then P is a ξ_μ-admissible operator.

Example 2.8. Let $P : BC(\mathbb{R}_+) \to BC(\mathbb{R}_+)$ and there exist $\xi : 2^{BC(\mathbb{R}_+)} \to \mathbb{R}_+$ such that

$$Pu(s) = e^{u(s)} \quad \text{and} \quad \xi(E) = \sup\{\|u\| : u \in E\},$$

for every $u \in BC(\mathbb{R}_+)$, $E \subset BC(\mathbb{R}_+)$. Then P is a ξ_μ-admissible operator.

3. Main results

In this section, we provide ξ_μ-β and ξ_μ-β-ψ condensing operators and prove their corresponding fixed point theorems.

Definition 3.1. Let X be a Banach space and P be a self-mapping defined on X. Then P is said to be a ξ_μ-β condensing operator if there exist a function $\xi : 2^X \to [0, +\infty)$ and $\beta \in \Upsilon$ such that

$$\xi(A)\mu(P(A)) \leq \beta(\mu(A))\mu(A) \tag{3.1}$$

for any bounded set $A \in 2^X$.

Theorem 3.2. *Let Ω be the member of a class Λ and P be a continuous self-mapping defined on Ω. Moreover, if P is ξ_μ-admissible and ξ_μ-β condensing operator satisfying the following condition:*

(C) *there exist closed and convex $A_0 \subseteq \Omega$ such that*

$$P(A_0) \subseteq A_0, \quad \xi(A_0) \geq 1, \tag{3.2}$$

where μ is an arbitrary measure of noncompactness.

Then, P has at least one fixed point in Ω.

Proof. Consider the sequence of the sets $\{A_n\}$ defined by

$$A_n = \overline{co}(PA_{n-1}), \quad n = 1, 2, \cdots .$$

Since $PA_0 \subseteq A_0$, we have

$$A_1 = \overline{co}(PA_0) \subseteq A_0$$

and

$$A_2 = \overline{co}(PA_1) \subseteq \overline{co}(PA_0) = A_1.$$

Continuing in this way, we obtain

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots A_n \supseteq A_{n+1} \supseteq \cdots$$

and also

$$PA_n \subseteq PA_{n-1} \subseteq \overline{co}(PA_{n-1}) = A_n.$$

If there exists a non-negative integer k such that $\mu(A_k) = 0$, then A_k is pre-compact set and $PA_k \subseteq A_k$. Thus from Theorem 2.2, we can say that P has at least one fixed point. If $\mu(A_k) \neq 0$ for all $k \in \mathbb{N}$. Since P is a ξ_μ-admissible operator, we have

$$\xi(A_1) = \xi(\overline{co}PA_0) \geq 1.$$

After some finite number of iteration, we get

$$\xi(A_n) \geq 1, \ \forall n \geq 0. \tag{3.3}$$

Now, since P is a ξ_μ-β condensing operator, from (3.3), we have

$$\begin{aligned}\mu(A_{n+1}) &\leq \xi(A_n)\mu(A_{n+1}) \\ &= \xi(A_n)\mu\big(\overline{co}(PA_n)\big) \\ &= \xi(A_n)\mu(PA_n) \\ &\leq \beta(\mu(A_n))\mu(A_n) \\ &\leq \mu(A_n),\end{aligned} \tag{3.4}$$

which implies that $\mu(A_n)$ is a non-increasing sequence of positive real numbers, thus there is $r \geq 0$ so that $\mu(A_n) \to r$ as $n \to \infty$. We claim that $r = 0$, assume the contradiction that $r \neq 0$. Then from (3.4) we have

$$\frac{\mu(A_{n+1})}{\mu(A_n)} \leq \beta(\mu(A_n)) < 1, \tag{3.5}$$

which yields

$$\beta(\mu(A_n)) \to 1 \text{ as } n \to \infty.$$

Since $\beta \in \Upsilon$, $\mu(A_n) \to 0$ as $n \to \infty$. Since $A_{n+1} \subseteq A_n$ from the fact that $PA_n \subseteq A_n$ for all $n \geq 1$. From the above discussion we conclude that $\{A_n\}$ is a nested sequence of sets, on the base of Definition 2.1(axiom vi), we conclude that $A_\infty = \bigcap_{n=1}^{\infty} A_n$ is a nonempty, closed, convex, and compact subset of the set A_0. Finally from Theorem 2.2, we get the desired result. □

Next, we have some definitions in order to prove next results.

Definition 3.3. ([3]) Let $\varPsi$ be a family of functions $\varrho : [0, \infty) \to [0, \infty)$ satisfying the following condition:

(i) ϱ is a non-decreasing function.

(ii) for all $s > 0$, $\lim_{n\to\infty} \varrho^n(s) = 0$ where $\varrho^n(s)$ is n-th iterate of ϱ.

Definition 3.4. Let X be a Banach space and P be a self-mapping defined on X. Then P is said to be a ξ_μ-β-ψ-condensing operator if there exists $\xi : 2^X \to [0, +\infty)$, $\beta \in \Upsilon$ and $\psi \in \Psi$ such that

$$\xi(A)\psi\big(\mu(PA)\big) \leq \beta\big(\psi(\mu(A))\big)\psi(\mu(A)) \tag{3.6}$$

for any bounded subset A of Ω.

Theorem 3.5. *Let Ω be a member of a class Λ and P be a continuous self-mapping defined on Ω. Further, if P is a ξ_μ-admissible and ξ_μ-β-ψ-condensing operator satisfying the following condition:*

(C) *there exists aclosed and convex subset A_0 of Ω such that*

$$PA_0 \subseteq A_0 \quad \textit{and} \quad \xi(A_0) \geq 1, \tag{3.7}$$

where μ is an arbitrary measure of noncompactness.

then P has at least one fixed point in Ω.

Proof. Consider the sequence of the sets $\{A_n\}$ defined by

$$A_n = \overline{co}(PA_{n-1}) \quad \text{for} \quad n = 1, 2, \cdots.$$

Since $PA_0 \subseteq A_0$, it implies that

$$A_1 = \overline{co}(PA_0) \subseteq A_0$$

and

$$A_2 = \overline{co}(PA_1) \subseteq \overline{co}(PA_0) = A_1.$$

Continuing in this process we get

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots A_n \supseteq A_{n+1} \supseteq \cdots$$

and

$$PA_n \subseteq PA_{n-1} \subseteq \overline{co}(PA_{n-1}) = A_n.$$

If there exists a non-negative integer k such that $\mu(X_k) = 0$, then A_k is pre-compact set and $PA_k \subseteq A_k$. Thus, from Theorem 2.2, P has at least one fixed point.

On the other hand, if $\mu(A_k) \neq 0$ for all n then $\mu(A_k) \geq 0$ for $k \in \mathbb{N}$. Since P is ξ_μ-admissible operator, from (3.7), we obtain

$$\xi(A_0) = \xi(\overline{co}TA_0) \geq 1.$$

Recursively, we have following inequality

$$\xi(A_n) \geq 1, \ \forall\, n \geq 0. \tag{3.8}$$

Now, since P is ξ_μ-β-ψ condensing operator, from (3.8) we have

$$\begin{aligned}\psi(\mu(A_{n+1})) &\leq \xi(A_n)\psi(\mu(A_{n+1}))\\ &= \xi(A_n)\psi\big(\mu(\overline{co}(PA_n))\big)\\ &= \xi(A_n)\psi\big(\mu(PA_n)\big)\\ &\leq \beta\big(\psi(\mu(A_n))\big)\psi\big(\mu(A_n)\big)\\ &\leq \psi\big(\mu(A_n)\big),\ \forall n \in \mathbb{N}.\end{aligned} \tag{3.9}$$

As we know that ψ is a non-decreasing function, then $\mu(A_{n+1}) \leq \mu(A_n)$ for all $n \in \mathbb{N}$, which implies that $\mu(A_n)$ is a decreasing sequence of positive real numbers. Thus, there is $r \geq 0$ such that $\mu(A_n) \to r$ as $n \to \infty$. Hence, we claim that $r = 0$, assume the contradiction that $r \neq 0$, so from (3.9) we can write

$$\frac{\psi(\mu(A_{n+1}))}{\psi(\mu(A_n))} \leq \beta(\psi(\mu(A_n))) < 1, \tag{3.10}$$

which yields

$$\beta(\psi(\mu(A_n))) \to 1 \quad \text{as} \quad n \to \infty.$$

Since, $\beta \in \Upsilon$ we get $\psi\big(\mu(A_n)\big) \to 0$ as $n \to \infty$, also $\psi \in \Gamma$, therefore $\mu(A_n) \to 0$ as $n \to \infty$, and we assured that $r = 0$. Since $PA_n \subseteq A_n$ for all $n \geq 1$, $A_{n+1} \subseteq A_n$. Hence, we conclude that $\{A_n\}$ sequence is nested of sets, on the basis of Definition 2.1(axiom vi), we derive that $A_\infty = \bigcap_{n=1}^{\infty} A_n$ is a nonempty, closed and convex subset of the set A_0. Finally, from Theorem 2.2, we get the desired result. □

3.1. **Consequences.** In this section, we illustrate the existing results from the literature [2, 11, 19] which certainly conclude from our main results of Section 3.

Corollary 3.6. *Let Ω be a member of a class Λ and P be a continuous self-mapping defined on Ω. If there exist $\xi : 2^{\Omega} \to [0, +\infty)$ such that P is a ξ_μ-admissible operator satisfying the followings conditions:*

- *for any $A \subset \Omega$ we have*

$$\xi(A)\mu(PA) \leq \psi(\mu(A)), \tag{3.11}$$

- *there exists closed and convex $A_0 \subset \Omega$ such that*

$$PA_0 \subseteq A_0,\ \xi(A_0) \geq 1, \tag{3.12}$$

where μ is an arbitrary measure of noncompactness and $\psi : [0, \infty) \to [0, \infty)$ is non-decreasing function such that $\lim_{n\to\infty} \psi^n(s) = 0$ for all $s > 0$,

then P has at least one fixed point in Ω.

Proof. Let us define the function

$$\beta(s)=\begin{cases}\frac{1}{2} & \text{if} \quad s=0,\\ \frac{\psi(s)}{s} & \text{if} \quad 0<s\leq\mu(\Omega),\\ \frac{\psi(\mu(\Omega))}{s} & \text{if} \quad s>\mu(\Omega).\end{cases}$$

We need to show that $\beta\in\Upsilon$, suppose that $\beta(r_n)\to 1$, implies that $\{r_n\}$ must be bounded (otherwise, $\beta(r_n)\to 0$) and so has a convergent subsequence say r_{n_k}. Furthermore, suppose that $r_{n_k}\to r_0$ and since ψ is upper semi-continuous we get

$$r_0=\lim_{k\to\infty} r_{n_k}=\limsup_{k\to\infty}\psi(r_{n_k})\leq\psi(r_0).$$

Since $\psi(t)<t$ for $t>0$, this implies that $r_0=0$. So we get $r_{n_k}\to 0$, also $r_n\to 0$. It follows that $\beta\in\Upsilon$ and on the other hand form (3.11) we get

$$\xi(A)\mu(PA)\leq\psi(\mu(A))=\beta(\mu(A))\mu(A).$$

By Theorem 3.2, we get the desired result. □

Corollary 3.7. *Let Ω be a member of a class Λ and P be a continuous self-mapping defined on Ω satisfying*

$$\mu(TA)\leq\beta\left(\mu\left(A\right)\right)\mu(A)$$

for any subset A of Ω, where μ is an arbitrary measure of noncompactness and $\beta\in\Upsilon$. Then P has at least one fixed point in Ω.

Proof. Let us substitute $\xi(A)=1$ in Theorem 3.2, we get the desired result. □

Corollary 3.8. *Let Ω be a member of a class Λ and P be a continuous self-mapping defined on Ω satisfying*

$$\mu(TA)\leq\lambda\mu(A)$$

for any subset A of Ω, where μ is an arbitrary measure of noncompactness and $\lambda\in[0,1)$. Then P has at least one fixed point in Ω.

Proof. Let us assume that $\xi(A)=1$ and $\beta(t)=\lambda$ with $\lambda\in[0,1)$ in Theorem 3.2, we get the desired result. □

4. Application

In this section, we apply Theorem 3.2 to prove the solvability of nonlinear quadratic Volterra-Stieltjes integral equation (1.1). The integral equation defined on the Banach space $BC(\mathbb{R}_+)$ equipped with the norm

$$||u||=\sup\left\{|u(l)|:l\geq 0\right\}.$$

Let us recall the definition of a measure of noncompactness which is introduced in the paper of Banas [4], will be used throughout the this section. Consider an $X \neq \phi$ subset of $BC(\mathbb{R}_+)$ and M be a positive number. For $u \in X$ and $\epsilon > 0$, the modulus of continuity is defined on interval $[0, M]$ as follows:

$$\omega^M(u,\epsilon) = \{|u(l_1) - u(l_2)| : l_1, l_2 \in [0, M], |l_1 - l_2| \leq \epsilon\}.$$

Next, we have

$$\begin{aligned} \omega^M(X,\epsilon) &= \sup\left\{\omega^M(u,\epsilon) : u \in X\right\}, \\ \omega_0^M(X,\epsilon) &= \lim_{\epsilon\to 0}\omega^M(X,\epsilon), \\ \omega_0(X) &= \lim_{M\to\infty}\omega_0^M(X). \end{aligned}$$

Further, the function $D(X)$ defined in the following way:

$$D(X) = \lim_{M\to\infty}\left\{\sup_{u\in X}\{\sup\{|u(l_1) - u(l_2)| : l_1, l_2 \geq M\}\}\right\}.$$

Finally, the formula for $\mu(X)$ is given as

$$\mu(X) = \omega_0(X) + D(X) \tag{4.1}$$

is the measure of noncompactness in the space $BC(\mathbb{R}_+)$. We can verify that μ defined in equation (4.1) has maximum and sub-linearity property but not a full measure of noncompactness [5]. The kernel of this MNC denoted by $ker\mu$ and consists of nonempty and bounded subsets of $BC(\mathbb{R}_+)$ such that the functions tend to the finite limit at infinity and are locally equicontinuous on $\mathbb{R}_+$.

In order to discuss the existence of the solution of the quadratic Volterra-Stieltjes I.E, let us recall some information about bounded variation and Volterra-Stieltjes integral [1]. For a real-valued function x defined on $[a, b]$ the function of bounded variation will be denoted by the symbol $\bigvee_a^b x$. The function x is said to of bounded variation if $\bigvee_a^b x$ is finite on the interval $[a, b]$. Now consider the real-valued function of two variable $g(p, q) = g : [a, b] \times [a, b] \to \mathbb{R}$, in the sequel of above discussion, for fixed number $q \in [u, v]$ we have $\bigvee_{a_1}^{b_1} g(p, q)$ is the variation of the function $p \to g(p, q)$ on the sub-interval $[a_1, b_1]$ of $[a, b]$ and the quantity $\bigvee_{q=a}^{b} g(p, q)$ is total variation on $[a, b]$.

Let us assume that x and k are real-valued functions on the interval $[a,b]$. Then, we can define the Stieltjes integral (in the Riemann-Stieltjes sense)

$$\int_a^b x(t)dk(t) \tag{4.2}$$

of the function x with respect to the function k, under appropriate assumptions on the functions x and k [1]. If the integral (4.2) exist, we say that x is Stieltjes integrable on the interval $[a,b]$ with respect to k.

Let us assume that $k(t),\ \psi(t)$ are real valued functions on $[a,b]$, then under suitable consideration composition function $k(\psi(t))$, is function of bounded variation on $[\psi(a),\psi(b)]$ and the quantity $\bigvee\limits_{\psi(a)}^{\psi(b)} k(\psi(t))$ will be meaningful [13]. Further, we assume that $g(p,q) = g : [a,b]\times[a,b] \to \mathbb{R}$ and $\psi : [a,b] \to \mathbb{R}$ are real valued functions. For fixed $q \in [u,v]$, $\psi(q)$ will be a fixed number in $[\psi(u),\psi(v)]$ and $\bigvee\limits_{\psi(q)=\psi(a)}^{\psi(b)} g(\psi(p),\psi(q))$ is the variation of function $\psi(p) \to g(\psi(p),\psi(q))$ on corresponding subinterval. If we assume that $x(t)$ is continuous and the composition function $k(\psi(t))$ is bounded variation then the integral

$$\int_{\psi(a)}^{\psi(b)} x(t)dk(\psi(t)), \tag{4.3}$$

exists and x is Stieltjes integrable on $[\psi(a),\psi(b)]$, with respect to $k(\psi)$.

For the further discussion we are defining the concept of K-bounded variation. For a positive integer K, let

$J_K = \{X \subset I : X$ can be expressed as a union of K closed or open intervals$\}$.

We know that any interval is union of two sub-intervals, hence $J_K \subset J_{K+1}$. A function $f : I \to \mathbb{R}$ is called K-bounded variation if $f^{-1}([a,b]) \in J_K$ for all $[a,b] \subset R$. Consider the set of all K-bounded variation on interval I as $BV(K)$.

In order to establish our result, we consider the following basic results [1].

Lemma 4.1. *If k is a function of bounded variation on $[a,b]$ and ψ is K-bounded variation on $[a,b]$, then the composition function $k(\psi)$ is bounded variation on $[a,b]$.*

Lemma 4.2. *If x is Stieltjes integrable on the interval $[a,b]$ with respect to composition function $k(\psi)$ of bounded variation, then*

$$\left|\int_{\psi(a)}^{\psi(b)} x(t)dk(\psi(t))\right| \leq \int_{\psi(a)}^{\psi(b)} |x(t)|d_t\left(\bigvee_{\psi(a)}^{\psi(t)} k(\psi)\right),$$

where k is a function of bounded variation and ψ is a non-decreasing K-bounded variation on $[a,b]$.

Lemma 4.3. *Let x_1, x_2 be Stieltjes integrable on the interval $[a,b]$ with respect to a non-decreasing function $k(\psi(t))$ such that $x_1(t) \le x_2(t)$ for $t \in [a,b]$. Then*

$$\int_{\psi(a)}^{\psi(b)} x_1(t) d_t k(\psi(t)) \le \int_{\psi(a)}^{\psi(b)} x_2(t) d_t k(\psi(t)).$$

In the similar fashion, we can consider the Stieltjes integral of the form:

$$\int_{\psi(a)}^{\psi(b)} x(t) d_t k(\psi(s), \psi(t)), \tag{4.4}$$

where $k : [a,b] \times [a,b] \to \mathbb{R}$ and the symbol d_t indicates the integration with respect to the variable t.

Next, in order to prove the existence of solution of integral equation (1.1) we consider the following assumptions:

(A_1) The functions $a_r(1 \le r \le m), b_t(1 \le t \le n), c_s(1 \le s \le p)$ and ϕ are continuous functions on $\mathbb{R}_+$. Further, ϕ is a monotonic increasing K-bounded variation on every bounded interval of $\mathbb{R}_+$ such that $\phi(0) = 0$.

(A_2) Let $\pi_1 : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}$ and $\pi_2 : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ be continuous functions and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous, monotonic increasing function with $\psi(t) < t,\ \psi(0) = 0$ such that for each $r > 0$, we have

$$\begin{aligned} &|\pi_1(l, u_1, ..., u_m) - \pi_1(l, v_1, ...v_m)| \le \alpha\ \psi\left(\max_{1\le s\le m} |u_s - v_s|\right) \\ &\text{and} \\ &|\pi_2(l, u_1, ..., u_n) - \pi_2(l, v_1, ..., v_n)| \le \lambda\ \psi\left(\max_{1\le t\le n} |u_t - v_t|\right), \end{aligned} \tag{4.5}$$

where, α and λ are non-negative real numbers, and for all $u_t, v_s \in [-r, r]$ and any $l \in \mathbb{R}_+$.

(A_3) The function $g : \Theta \times \mathbb{R}^p \to \mathbb{R}$ defined such that, for any fixed $r > 0$, g is uniformly continuous on $\Theta \times [-r, r]^p$, where,

$$\Theta = \{(l, \tau) : 0 \le \tau \le l\}. \tag{4.6}$$

(A_4) The function $u(l, \tau, u_1, u_2, ..., u_p) : \Theta \times \mathbb{R}^{\prime} \to \mathbb{R}$ is continuous and there exist continuous and non-decreasing function $b : \mathbb{R}^p \to \mathbb{R}^p$ such that,

$$u(l, \tau, u_1, u_2, ..., u_p) \le b\left(|u_1|, |u_2|, ..., |u_p|\right)$$

for all $l,\ \tau \in \Theta$ and $u_t\ \in \mathbb{R}$ for all $1 \le t \le p$.

(A_5) For any positive numbers $\epsilon_1,\ \epsilon_2$ there exist $M > 0$ such that for $l_2 > l_1 \geq M$ the functions $\pi_1 : \mathbb{R}_+ \times [-h,h]^m,\ \pi_2 : \mathbb{R}_+ \times [-h,h]^n$, satisfied the following conditions for all $h > 0$,

$$|\pi_1(l_1, u_1, u_2, \cdots, u_m) - \pi_1(l_2, u_1, u_2, \cdots, u_m)| < \epsilon_1,$$
$$|\pi_2(l_1, u_1, u_2, \cdots, u_n) - \pi_1(l_2, u_1, u_2, \cdots, u_n)| < \epsilon_2.$$

(A_6) The function $K = K(l,\tau)$ is continuous on Θ and $K(l,0) = 0$.

(A_7) The function $l \to K(l,\tau)$ has a bounded variation on the interval $[0,l]$ for fixed $l > 0$, and $l \to \vee_{\tau=0}^{l} K(l,\tau)$ is bounded on $\mathbb{R}_+$.

(A_8) For every $\epsilon > 0$ there exist $\delta > 0$ such that for all $l_1, l_2 \in \mathbb{R}^+, l_1 < l_2,\ l_2 - l_1 \leq \delta$, the following inequality holds:

$$\bigvee_{q=0}^{l_1}\Big[K(l_2,q) - K(l_1,q)\Big] \leq \epsilon.$$

(A_9) The following equations also true:

$$\lim_{M\to\infty}\left\{\sup\left[\bigvee_{\tau=l_2}^{l_1} K(l_1,\tau) : M \leq l_2 \leq l_1\right]\right\} = 0;$$

$$\lim_{M\to\infty}\left\{\sup\left[\bigvee_{\tau=l_2}^{l_1} K(l_1,\tau) - K(l_2,\tau : M \leq l_2 < l_1)\right]\right\} = 0;$$

$$\lim_{M\to\infty}\left\{\sup\Big[\big|g(l_1,\tau,v_1,...,v_p) - g(l_2,\tau,v_1,...,v_p)\big| : l_1,l_2 \geq M, \tau \in \mathbb{R}^+,\right.$$
$$\left.\tau \leq l_2, \tau \leq l_1, v_i \in [-h,h]\Big]\right\} = 0, \text{ for any } h > 0.$$

(A_{10}) The inequality

$$\alpha\ \psi(r) + M_{\pi_1} + (\beta\ \psi(r) + M_{\pi_2})B(\bar{r})K_\phi \leq r$$

has a positive solution with

$$\alpha + \beta B(\bar{r}_0)K_\phi\ \leq\ 1.$$

(A_{11}) If $\varsigma\,(u(t),v(t)) \geq 0$ for all $u,v \in X \subset BC(R_+)$ then $\varsigma\,(f(t),g(t)) \geq 0$ for all $f,g \in \bar{co}TX$. Moreover $\varsigma\,(u(t),v(t)) \geq 0$ for all $u,v \in B(0,r_0)$.

Remark 4.4. If $K(l,\tau)$ is a function of bounded variation and $\phi(l)$ is non-increasing, K-bounded variation on any bounded interval of $\mathbb{R}_+$, then the composition function $K(\phi(l),\phi(\tau))$ satisfies the assumptions (A_6), (A_7), (A_8) and (A_9) on respective domains [1].

Remark 4.5. From assumption (A_7) and Lemma 4.1, we have the real number $K_\phi < \infty$, where K_ϕ is defined below:

$$K_\phi = \sup\left\{ \bigvee_{\phi(\tau)=0}^{\phi(l)} K(\phi(l),\phi(\tau)) : l \in \mathbb{R}_+ \right\}, \tag{4.7}$$

where the function $\phi(l) \to \bigvee_{\phi(\tau)=0}^{\phi(l)} K(\phi(l),\phi(\tau))$ is bounded on $\mathbb{R}_+$.

Remark 4.6. By the virtue of assumptions (A_2) and (A_5) the functions $l \to \pi_1(l, 0, , 0, ..., 0)$ and $l \to \pi_2(l, 0, 0, ..., 0)$ are members of $BC(\mathbb{R}_+)$ with

$$\begin{aligned} M_{\pi_1} &= \sup\{\pi_1(l, 0, 0, ..., 0) : l \in \mathbb{R}_+\}\ , \\ M_{\pi_2} &= \sup\{\pi_2(l, 0, 0, ..., 0) : l \in \mathbb{R}_+\}. \end{aligned}$$

Next, the existence of solution of equation (1.1) is describe by the following theorem.

Theorem 4.7. *Under the assumptions* (A_1) *to* (A_{11}) *the integral equation* (1.1) *has at least one solution in the space* $BC(\mathbb{R}_+)$. *Moreover, the solution has finite limit at infinity.*

Proof. For $l \in \mathbb{R}_+$, define the operator T on $BC(\mathbb{R}_+)$ by:

$$\begin{aligned} (Tu)(l) =& \pi_1(l, u(a_1(l)), ..., u(a_m(l))) + \pi_2(l, u(b_1(l)), ..., u(b_n(l))) \\ &\times \int_0^{\phi(l)} g(l, \tau, u(c_1(l)), ..., u(c_p(l)))d_\tau K(\phi(l), \phi(\tau)). \end{aligned} \tag{4.8}$$

The existence of solution of the integral equation (1.1) is equivalent to fixed point of the operator (4.8). Let us split the operator in equation (4.8) by defining the operators Π_1, Π_2, and I on $BC(\mathbb{R}_+)$ in the following fashion:

$$\begin{aligned} (\Pi_1 u)(l) &= \pi_1(l, u(a_1(l)), ..., u(a_m(l))), \\ (\Pi_2 u)(l) &= \pi_2(l, u(b_1(l)), ..., u(b_n(l))), \\ (Iu)(l) &= \int_0^{\phi(l)} g(l, \tau, u(c_1(l)), ..., u(c_p(l)))d_\tau K(\phi(l), \phi(\tau)), \end{aligned} \tag{4.9}$$

Then we have

$$(Tu)(l) = (\Pi_1 u)(l) + (\Pi_2 u)(l)(Iu)(l).$$

Now, let us fix a random function $u \in BC(\mathbb{R}_+)$, and for $M > 0$, $\epsilon > 0$, the numbers $l_1, l_2 \in [0, M]$ such that $|l_1 - l_2| \leq \epsilon$.

Now we prove the continuity of $T(u)$ on R_+ by proving the continuity of $\Pi_1(u), \Pi_2(u)$ and $I(u)$.

From the assumptions (A_6)-(A_7) and Remarks 4.4–4.6, we have the following inequality:

$$
\begin{aligned}
&|(Iu)(l_1) - (Iu)(l_2)| \\
&\leq \bigg| \int_0^{\phi(l_1)} g(l_1, \tau, u(c_1(l_1)), ..., u(c_p(l_1))) d_\tau K(\phi(l_1), \phi(\tau)) \\
&\quad - \int_0^{\phi(l_2)} g(l_2, \tau, u(c_1(l_2)), ..., u(c_p(l_2))) d_\tau K(\phi(l_2), \phi(\tau)) \bigg| \\
&\leq \bigg| \int_0^{\phi(l_1)} g(l_1, \tau, u(c_1(l_1)), ..., u(c_p(l_1))) d_\tau K(\phi(l_1), \phi(\tau)) \\
&\quad - \int_0^{\phi(l_2)} g(l_1, \tau, u(c_1(l_1)), ..., u(c_p(l_1))) d_\tau K(\phi(l_1), \phi(\tau)) \bigg| \\
&\quad + \bigg| \int_0^{\phi(l_2)} g(l_1, \tau, u(c_1(l_1)), ..., u(c_p(l_1))) d_\tau K(\phi(l_1), \phi(\tau)) \\
&\quad - \int_0^{\phi(l_2)} g(l_1, \tau, c_1(l_1)), ..., u(c_p(l_1))) d_\tau K(\phi(l_2), \phi(\tau)) \bigg| \\
&\quad + \bigg| \int_0^{\phi(l_2)} g(l_1, \tau, u(c_1(l_1)), ..., u(c_p(l_1))) d_\tau K(\phi(l_2), \phi(\tau)) \\
&\quad - \int_0^{\phi(l_2)} g(l_2, \tau, u(c_1(l_2)), ..., u(c_p(l_2))) d_\tau K(\phi(l_2), \phi(\tau)) \bigg| \\
&\leq \int_{\phi(l_2)}^{\phi(l_1)} \bigg| g(l_1, \tau, u(c_1(l_1)), ..., u(c_p(l_1))) \bigg| d_\tau K(\phi(l_1), \phi(\tau)) \\
&\quad + \int_0^{\phi(l_2)} \bigg| g(l_1, \tau, u(c_1(l_1)), ..., u(c_p(l_1))) d_\tau \left[K(\phi(l_1), \phi(\tau)) - K(\phi(l_2), \phi(\tau)) \right] \bigg| \\
&\quad + \int_0^{\phi(l_2)} \bigg| g(l_1, \tau, u(c_1(l_1)), ..., u(c_p(l_1))) \\
&\quad - g(l_1, \tau, u(c_1(l_2)), ..., u(c_p(l_2))) \bigg| d_\tau K(\phi(l_2), \phi(\tau)) \\
&\leq \int_{\phi(l_2)}^{\phi(l_1)} b(|u(c_1(l_1))|, ..., |u(c_2(l_1))|) \ d_\tau \left(\bigvee_{p=0}^{\phi(\tau)} K(\phi(l_1), p) \right) \\
&\quad + \int_0^{\phi(l_2)} b(|u(c_1(l_1))|, ..., |u(c_2(l_1))|) \ d_\tau \left(\bigvee_{p=0}^{\phi(\tau)} \left[K(\phi(l_1), p) - K(\phi(l_2), p) \right] \right) \\
&\quad + \int_0^{\phi(l_2)} \omega^{1,M}_{||u||}(g, \epsilon) \ d_\tau \left(\bigvee_{p=0}^{\phi(\tau)} K(\phi(l_2), p) \right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq b(||u||,...,||u||)\int_{\phi(l_2)}^{\phi(l_1)} d_\tau \left(\bigvee_{p=0}^{\phi(\tau)} K(\phi(l_1),p)\right) \\
&\quad + b(||u||,...,||u||)\int_{0}^{\phi(l_2)} d_\tau \left(\bigvee_{p=0}^{\phi(\tau)} K\big[\phi(l_1),p) - K(\phi(l_2),p)\big]\right) \\
&\quad + \omega^{1,M}_{||u||}(g,\epsilon)\int_{0}^{\phi(l_2)} d_\tau \left(\bigvee_{p=0}^{\phi(\tau)} K(\phi(l_2),p)\right),
\end{aligned}
$$

where

$$
\begin{aligned}
&\omega^{1,M}_{r}(g,\epsilon) \\
&= \sup\big\{\big|g(l_1,\tau,u(c_1(l_1)),...,u(c_1(l_2))) - g(l_2,\tau,u(c_1(l_2)),...,u(c_2(l_2)))\big| : \\
&\qquad l_1,l_2,\tau\in[0,M], |l_1-l_2|\leq\epsilon, u\in[-r,r]\big\},
\end{aligned}
$$

for an arbitrary number $r > 0$. Therefore, the above expression can be written as;

$$
\begin{aligned}
|I(u)(l_1) - I(u)(l_2)| &\leq B(\bar{r})\int_{\phi(l_2)}^{\phi(l_1)} d_\tau \left(\bigvee_{p=0}^{\phi(\tau)} K(\phi(l_1),p)\right) \\
&\quad + B(\bar{r})\int_{0}^{\phi(l_2)} d_\tau \left(\bigvee_{p=0}^{\phi(\tau)} \big[K(\phi(l_1),p) - K(\phi(l_2),p)\big]\right) \\
&\quad + \omega^{1,M}_{||u||}(g,\epsilon)\int_{0}^{\phi(l_2)} d_\tau \left(\bigvee_{p=0}^{\phi(\tau)} K(\phi(l_2),p)\right). \\
&\leq B(\bar{r})\left(\bigvee_{\phi(l_2)}^{\phi(l_1)} K(\phi(l_1),p)\right) \\
&\quad + B(\bar{r})\left(\bigvee_{p=0}^{\phi(l_2)} \big[K(\phi(l_1),p) - K(\phi(l_2),p)\big]\right) \\
&\quad + \omega^{1,M}_{||u||}(g,\epsilon)\left(\bigvee_{p=0}^{\phi(l_2)} K(\phi(l_2),p)\right),
\end{aligned}
\tag{4.10}
$$

where

$$B(\bar{r}) = \sup\{b(||u||, ||u||, ..., ||u||) : \sup\{||u||| \leq r : u \in BC(\mathbb{R}_+), \bar{r} = (r, r, \cdots, r)\}\}.$$

Since g is continuous function, $\omega^{1,M}_{||u||}(g, \epsilon) \to 0$ as $\epsilon \to 0$. Hence, from the assumptions (A_7) and (A_8), Remarks 4.4 - 4.6, we conclude that (Iu) is continuous on $[0, M]$, arbitrariness of M implies the continuity of (Iu) on $\mathbb{R}_+$.

Furthermore, in the view of assumption (A_2) we deduce

$$\begin{aligned}
&|(\Pi_1 u)(l_1) - (\Pi_1 u)(l_2)| \\
&\leq |\pi_1(l_1, u(a_1(l_1)), ..., u(a_m(l_1))) - \pi_1(l_2, u(a_1(l_2)), ..., u(a_m(l_2)))| \\
&\leq |\pi_1(l_1, u(a_1(l_1)), ..., u(a_m(l_1))) - \pi_1(l_1, u(a_1(l_2)), ..., u(a_m(l_2)))| \\
&\quad + |\pi_1(l_1, u(a_1(l_2)), ..., u(a_m(l_2))) - \pi_1(l_2, u(a_1(l_2)), ..., u(a_m(l_2)))| \\
&\leq \alpha\psi\left(\max_{1\leq s\leq m} |u(a_s(l_1)) - u(a_s(l_2))|\right) + \omega^{1,M}_{||u||}(\pi_1, \epsilon)
\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}
&|(\Pi_2 u)(l_1) - (\Pi_2 u)(l_2)| \\
&\leq |\pi_2(l_1, u(b_1(l_1)), ..., u(b_n(l_1))) - \pi_2(l_2, u(b_1(l_2)), ..., u(b_n(l_2)))| \\
&\leq |\pi_2(l_1, u(b_1(l_1)), ..., u(b_n(l_1))) - \pi_2(l_1, u(b_1(l_2)), ..., u(b_n(l_2)))| \\
&\quad + |\pi_2(l_1, u(b_1(l_2)), ..., u(b_n(l_2))) - \pi_2(l_2, u(b_1(l_2)), ..., u(b_n(l_2)))| \\
&\leq \lambda\psi\left(\max_{1\leq t\leq n} |u(b_t(l_1)) - u(b_t(l_2))|\right) + \omega^{1,M}_{||u||}(\pi_2, \epsilon).
\end{aligned} \tag{4.12}$$

For an arbitrary $r > 0$. we denote the following quantities:

$$\begin{aligned}
&\omega^{1,M}_r(\pi_1, \epsilon) \\
&= \sup\big\{\big|\pi_1(l_1, u(a_1(l_2)), \cdots, u(a_m(l_2))) - \pi_1(l_2, u(a_1(l_2)), ..., u(a_m(l_2)))\big| : \\
&\qquad l_1, l_2 \in [0, M], |l_1 - l_2| \leq \epsilon \; u \in [-r, r]\big\},
\end{aligned}$$

and

$$\begin{aligned}
&\omega^{1,M}_r(\pi_2, \epsilon) \\
&= \sup\big\{\big|\pi_2(l_1, u(b_1(l_2)), \cdots, u(b_n(l_2))) - \pi_2(l_2, u(b_1(l_2)), ..., u(b_n(l_2)))\big| : \\
&\qquad l_1, l_2 \in [0, M], |l_1 - l_2| \leq \epsilon \; u \in [-r, r]\big\}.
\end{aligned}$$

Now with regard to the fact that π_1 is uniformly continuous on $[0, M] \times [-r, r]^m$ and π_2 is uniformly continuous on $[0, M] \times [-r, r]^n$, we can conclude that $\omega^{1,M}_\delta(\pi_1, \epsilon) \to 0$ and $\omega^{1,M}_\delta(\pi_2, \epsilon) \to 0$ as $\epsilon \to 0$. Linking of assumption (A_5) with estimation (4.11) implies that $(\Pi_1 u)$, $(\Pi_2 u)$ are continuous on $[0, M]$ and by the arbitrariness of M we can say that $(\Pi_1 u)$, $(\Pi_2 u)$ are continuous on $\mathbb{R}_+$. Thus from equation (4.9), we have (Tu) is continuous on $\mathbb{R}_+$.

Next, let us prove the boundedness of $T(u)$ on $\mathbb{R}_+$ utilising the assumptions (A_1)-(A_7) and Lemmas 4.1 - 4.3. For any $l \in \mathbb{R}_+$ and fixed $u \in BC(\mathbb{R}_+)$, we have

$$
\begin{aligned}
&|(Tu)(l)| \\
&\leq |\pi_1(l, u(a_1(l)), ..., u(a_m(l)))| + |\pi_2(l, u(b_1(l)), ..., u(b_n(l)))| \\
&\quad \times \left| \int_0^{\phi(l)} g(l, \tau, u(c_1(l)), ..., u(c_p(l))) d_\tau K(\phi(l), \phi(\tau)) \right| \\
&\leq |\pi_1(l, u(a_1(l)), ..., u(a_m(l))) - \pi_1(l, 0, 0, ..., 0) + \pi_1(l, 0, 0, ..., 0)| \\
&\quad + |\pi_2(l, u(b_1(l)), ..., u(b_n(l))) - \pi_2(l, 0, 0, ..., 0) + \pi_2(l, 0, 0, ..., 0)| \\
&\quad \times \left| \int_0^{\phi(l)} u(l, \tau, u(c_1(l)), ..., u(c_p(l))) d_\tau K(\phi(l), \phi(\tau)) \right| \\
&\leq \alpha\psi \left(\max_{1 \leq s \leq m} |u(a_s(l))| \right) + M_{\pi_1} + \lambda\psi \left(\max_{1 \leq t \leq n} |u(b_t(l))| \right) + M_{\pi_2} \qquad (4.13) \\
&\quad \times b(|u(c_1(l))|, ..., |u(c_p(l))|) \left| \int_0^{\phi(l)} d_\tau \bigvee_{p=0}^{\phi(\tau)} K(\phi(l), p) \right| \\
&\leq \alpha\psi \left(\max_{1 \leq s \leq m} |u(a_s(l))| \right) + M_{\pi_1} \\
&\quad + \left(\lambda\ \psi \left(\max_{1 \leq t \leq n} |u(b_t(l))| \right) + M_{\pi_2} \right) B(\bar{r}) \bigvee_{p=0}^{\phi(l)} K(\phi(l), p) \\
&\leq \big(\alpha\psi(||u||) + M_{\pi_1}\big) + (\lambda\ \psi(||u||) + M_{\pi_2}) B(\bar{r}) K_\phi,
\end{aligned}
$$

where K_ϕ is total variation and $B(\bar{r})$ was mentioned earlier. This shows that (Tu) is bounded operator on $\mathbb{R}_+$. Thus from (4.11), (4.12) and (4.13) we conclude that (Tu) maps $BC(\mathbb{R}_+)$ into itself. Moreover, form the assumption (A_{10}), there exists r_0 such that T maps the ball $B_{r_0} \subset BC(\mathbb{R}_+)$ into itself.

Now, we prove that T is continuous operator on B_{r_0} by proving the continuity of the functions $\Pi_1,\ \Pi_2$ and I. For this, take arbitrary $u, v \in B_{r_0}$, $l \in \mathbb{R}_+$ and $\epsilon > 0$ such that $||u - v|| \leq \epsilon$. From the assumption (A_2) we can write

$$
\begin{aligned}
&|(\Pi_1 u)(l) - (\Pi_1 v)(l)| \\
&= |\pi_1(l, u(a_1(l)), ..., u(a_m(l))) - |\pi_1(l, v(a_1(l)), ..., v(a_m(l)))| \\
&\leq \alpha\ \psi \left(\max_{1 \leq s \leq m} |u(a_s(l)) - v(a_s(l))| \right)
\end{aligned}
$$

and

$$
\begin{aligned}
&|(\Pi_2 u)(l) - (\Pi_2 v)(l)| \\
&= |\pi_2(l, u(b_1(l)), ..., u(b_n(l))) - |\pi_2(l, v(b_1(l)), ..., v(b_n(l)))| \\
&\leq \lambda\ \psi\left(\max_{1\leq t\leq n} |u(b_t(l)) - v(b_t(l))|\right).
\end{aligned}
$$

Hence, we have

$$||\Pi_1 u - \Pi_2 v|| \leq \alpha\ \psi\left(||u - v||\right) \tag{4.14}$$

and

$$||\Pi_2 u - \Pi_2 v|| \leq \lambda\ \psi\left(||u - v||\right). \tag{4.15}$$

Hence, the functions Π_1 and Π_2 are continuous on B_{r_0}.

Now, from assumption $(A_3), (A_7)$ and Lemmas 4.1–4.2, we deduce that

$$
\begin{aligned}
&|(Iu)(l) - (Iv)(l)| \\
&\leq \left| \int_0^{\phi(l)} [g(l, \tau, u(c_1(l)), ..., u(c_p(l))) - g(l, \tau, v(c_1(l)), ..., v(c_p(l)))] \right. \\
&\qquad\qquad \left. d_\tau K(\phi(l), \phi(\tau)) \right| \\
&\leq \int_0^{\phi(l)} \left| g(l, \tau, u(c_1(l)), ..., u(c_p(l))) - g(l, \tau, v(c_1(l)), ..., v(c_p(l)))] \right| \\
&\qquad\qquad d_\tau \left(\vee_{p=0}^{\phi(\tau)} K(\phi(l), p)\right) \\
&\leq \int_0^{\phi(l)} \omega_{r_0}^3(g, \epsilon) d_\tau \left(\vee_{p=0}^{\phi(\tau)} K(\phi(l), p)\right) \\
&\leq \omega_{r_0}^3(g, \epsilon) \left(\vee_{p=0}^{\phi(l)} K(\phi(t), p)\right) \\
&\leq \omega_{r_0}^3(u, \epsilon) K_\phi,
\end{aligned}
\tag{4.16}
$$

where

$$
\begin{aligned}
\omega_{r_0}^3(g, \epsilon) = \sup\big\{ \big| g(l, \tau, u(c_1(l)), ..., u(c_p(l))) - g(l, \tau, v(c_1(l)), ..., v(c_p(l)) \big| : \\
l, \tau \in \mathbb{R}_+, u, v \in [-r_0, r_0], |u - v| \leq \epsilon \big\}.
\end{aligned}
$$

Since, for any fixed $r > 0$, g is uniformly continuous on $\Theta \times [-r_0, r_0]^p$, $\omega_{r_0}^3(g, \epsilon) \to 0$ as $\epsilon \to 0$. By similar argument $\omega_{r_0}^2(\pi_1, \epsilon)$, $\omega_{r_0}^2(\pi_2, \epsilon)$ tends to zero as $\epsilon \to 0$, thus, we can say that T is a continuous operator on B_{r_0}.

Now, consider X is any nonempty subset of B_{r_0} and $u \in X$. For $M > 0$ and $\epsilon > 0$ choose the numbers $l_1, l_2 \in [0, M]$ such that $|l_1 - l_2| \leq \epsilon$. Without loss of generality we can assume that $l_1 < l_2$, and $\varsigma\left(u(l_1), u(l_2)\right) \geq 0$ for arbitrary

$u \in X$, from equation (4.10) we can obtain:

$$\begin{aligned} &|(Iu)(l_1) - (Iu)(l_2)| \\ &\leq B(\bar{r}) \left(\vee_{p=0}^{\phi(l_1)} K(\phi(l_1), p)\right) + B(\bar{r}) \left(\vee_{p=0}^{\phi(l_1)} \left[K(\phi(l_1), p) - K(\phi(l_2), p)\right]\right) \\ &\quad + \omega_{||u||}^{1,M}(g, \epsilon) \left(\vee_{p=0}^{\phi(l_1)} K(\phi(l_2), p)\right). \end{aligned} \tag{4.17}$$

Let us define some characteristics functions $R(\epsilon)$ and $S(\epsilon)$ as follows:

$$R(\epsilon) = \sup\left\{ \vee_{p=0}^{\phi(l_2)} \left[K(\phi(l_1), p) - K(\phi(l_2), p)\right] : l_2,\ l_1 \in \mathbb{R}^+, l_2 < l_1, l_2 - l_1 \leq \epsilon \right\}, \tag{4.18}$$

$$S(\epsilon) = \sup\left\{ \vee_{p=\phi(l_2)}^{\phi(l_1)} K(\phi(l_1), p) : l_2,\ l_1 \in \mathbb{R}^+, l_2 < \ l_1, l_2 - l_1 \leq \epsilon \right\}. \tag{4.19}$$

Obviously, in the view of assumptions (A_8) and Remark 4.4, we have, $R(\epsilon) \to 0,\ S(\epsilon) \to 0$ as $\epsilon \to 0$ and the equation (4.17) becomes

$$|(Iu)(l_1) - (Iu)(l_2)| \leq B(\bar{r}) \left(R(\epsilon) + S(\epsilon)\right) + K_\phi \omega_{r_0}^{1,M}(g, \epsilon). \tag{4.20}$$

Also, from equations (4.11) and (4.12), we obtain:

$$\begin{aligned} |(\Pi_1 u)(l_1) - (\Pi_1 u)(l_2)| &\leq \alpha\ \psi\left(\omega^M(u, \epsilon)\right) + \omega_{||u||}^{1,M}(\pi_1, \epsilon), \\ |(\Pi_2 u)(l_1) - (\Pi_2 u)(l_2)| &\leq \lambda \psi\left(\omega^M(u, \epsilon)\right) + \omega_{||u||}^{1,M}(\pi_2, \epsilon). \end{aligned} \tag{4.21}$$

Now, from the equations (4.11)-(4.12), (4.18)-(4.20) and (4.21) for fixed $u \in X,\ l_1, l_2 \in [0, M],\ l_2 < l_1,\ l_1 - l_2 \leq \epsilon$, we can derive the following expression:

$$\begin{aligned} &(Tu)(l_1) - (Tu)(l_2) \\ &\leq |(\Pi_1 u)(l_1) - (\Pi_1 u)(l_2)| + |(\Pi_2 u)(l_1)(Iu)(l_1) - (\Pi_2 u)(l_2)(Iu)(l_2)| \\ &\leq |(\Pi_1 u)(l_1) - (\Pi_1 u)(l_2)| + |(\Pi_2 u)(l_1)(Iu)(l_1) - (\Pi_2 u)(l_1)(Iu)(l_2) \\ &\quad + (\Pi_2 u)(l_1)(Iu)(l_2) - (\Pi_2 u)(l_2)(Iu)(l_2)| \\ &\leq |(\Pi_1 u)(l_1) - (\Pi_1 u)(l_2)| + |(\Pi_2 u)(l_1)((Iu)(l_1) - (Iu)(l_2))| \\ &\quad + |(Iu)(l_2)((\Pi_2 u)(l_1) - (\Pi_2 u)(l_2))| \\ &\leq \left(\alpha\ \psi\left(\omega(u, \epsilon)\right) + \omega_{r_0}^{1,M}(\pi_1, \epsilon)\right) \\ &\quad + \left(\alpha \psi\left(||u||\right) + M_{\pi_1}\right) \left(B(\bar{r}_0) \left(R(\epsilon) + S(\epsilon)\right) + K_\phi \omega_{r_0}^{1,M}(g, \epsilon)\right) \\ &\quad + B(\bar{r}_0) K_\phi \left(\lambda\ \psi\left(\omega^M(g, \epsilon)\right) + \omega_{||u||}^{1,M}(\pi_2, \epsilon)\right). \end{aligned}$$

Hence, by using the properties of ψ, π_1, π_2, g, we have $\omega_{r_0}^{1,M}(\pi_1, \epsilon), \omega_{r_0}^{1,M}(\pi_2, \epsilon)$, $\omega_{r_0}^{1,M}(g, \epsilon),\ R(\epsilon)$ and $S(\epsilon)$ goes to zero as $\epsilon \to 0$. Thus the above expression becomes

$$\begin{aligned}\omega_0^M(TX) &\leq \alpha\ \psi(\omega_0^M(X)) + B(\bar{r})K_\phi\lambda\ \psi\left(\omega_0^M(X)\right)\\ &\leq (\alpha+\lambda\ B(\bar{r})K_\phi)\,\psi\left(\omega_0^M(X)\right),\end{aligned}$$

consequently, we get

$$\omega_0(TX) \leq (\alpha+\lambda\ B(\bar{r})K_\phi)\,\psi\left(\omega_0(X)\right). \tag{4.22}$$

Next, we choose that u is arbitrary function from the set $X \subset B_{r_0}$ and $l_1,\ l_2 \in \mathbb{R}_+$ are such that $M \leq l_1 < l_2$. For the further estimations, let us define some characteristic functions as follows:

$$\begin{aligned}P(M) &= \sup\left\{\bigvee_{p=\phi(l_2)}^{\phi(l_1)}[K(\phi(l_1),p)] : M \leq l_2 < l_1\right\},\\ Q(M) &= \sup\left\{\bigvee_{p=0}^{\phi(l_1)}[K(\phi(l_1),p) - K(\phi(l_2),p)] : M \leq l_2 < l_1\right\}.\end{aligned}$$

Moreover, for fixed $H > 0$ let us put:

$$\begin{aligned}&W_H(M)\\ &= \sup\Big\{|\pi_1(l_1,u(a_1(l_1)),...,u(a_m(l_1))) - \pi_1(l_2,u(a_1(l_2)),...,u(a_m(l_2)))| :\\ &\qquad M \leq l_2 < l_1,\ u \in [-H,H]\Big\},\end{aligned}$$

$$\begin{aligned}&F_H(M)\\ &= \sup\Big\{|\pi_2(l_1,u(b_1(l_1)),...,u(b_n(l_1))) - \pi_2(l_2,u(b_1(l_2)),...,u(b_n(l_2)))| :\\ &\qquad M \leq l_2 < l_1,\ u \in [-H,H]\Big\},\end{aligned}$$

$$\begin{aligned}&V_H(M)\\ &= \sup\big\{|g(l_1,\tau,u(c_1(l_1)),...,u(c_p(l_1))) - g(l_2,\tau,u(c_1(l_2)),...,u(c_p(l_2)))| :\\ &\qquad l_2,l_1 \geq M, \tau \in \mathbb{R}_+, \tau \leq l_2, \tau \leq l_1,\ u \in [-H,H]\big\}.\end{aligned}$$

Next, keeping in mind assumptions (A_9), we have that $P(M),\ Q(M)$ tends to 0 as $M \to \infty$. Moreover, for fixed $H > 0$ the quantities $F_H(M),\ W_H(M)$ and $V_H(M)$ also approaches to 0 as $M \to \infty$. Therefore, as per the expression

(4.20) and (4.21), for $M \leq l_2 < l_1$ and $u \in X$ we obtain:

$$
\begin{aligned}
&|(Tu)(l_1) - (Tu)(l_2)| \\
&\leq |\pi_1(l_1, u(a_1(l_1)), ..., u(a_m(l_1))) - \pi_1(l_1, u(a_1(l_2)), ..., u(a_m(l_2)))| \\
&\quad + \pi_1(l_1, u(a_1(l_2)), ..., u(a_m(l_2)))| - \pi_1(l_2, u(a_1(l_2)), ..., u(a_m(l_2)))| \\
&\quad + \pi_2(l_1, \tau, u(b_1(l_1)), ..., u(b_n(l_1))) - \pi_2(l_1, 0, 0, ..., 0) + \pi_2(l_1, 0, 0, ..., 0) \\
&\quad \times (Iu)(l_1) - (Iu)(l_2) + b(|u(c_1(l_1))|, |u(c_2(l_1))|, ...|u(c_p(l_1))|) \\
&\quad \times \bigvee_{p=0}^{\phi(l_2)} K(\phi(l_2), p)|(\Pi_2 u)(l_1) - (\Pi_2 u)(l_2)| \\
&\leq \alpha\psi\left(\max_{1\leq s\leq m} |u(a_s(l_1)) - u(a_s(l_2))| : M \leq l_2 < l_1\right) + W_{r_0}(M) \\
&\quad + \alpha\psi\left(\max_{1\leq s\leq m} |u(a_s(l_1))|\right) + M_{\pi_1}\bigg\{B(\bar{r_0})\left(\vee_{p=0}^{\phi(l_1)} K(\phi(l_2), p)\right) \\
&\quad + B(\bar{r_0})\bigg(\vee_{p=0}^{\phi(l_1)}\left[K(\phi(l_1), p) - K(\phi(l_2), p)\right]\bigg) \\
&\quad + V_{r_0}(M)\left(\vee_{p=0}^{\phi(l_1)} K(\phi(l_2), p)\right)\bigg\} \\
&\quad + B(\bar{r_0})K_\phi\left\{\lambda\ \psi\left(\max_{1\leq t\leq n} |u(b_t(l_1)) - u(b_t(l_2))| : M \leq l_2 < l_1\right) + F_{r_0}(M)\right\} \\
&\leq \alpha\psi\left(\max_{1\leq s\leq m}\Big[|u(a_s(l_1)) - u(a_s(l_2))| : M \leq l_2 < l_1\Big]\right) + W_{r_0}(M) \\
&\quad + \bigg\{\left(\alpha\ \psi\left(\max_{1\leq s\leq m} |u(a_s(l_1))|\right) + M_{\pi_1}\right)\Big(B(\bar{r_0})\left(P(M)\right) + B(\bar{r_0})\left(Q(M)\right)\Big) \\
&\quad + V_{r_0}(M)\left(\vee_{p=0}^{\phi(l_1)} K(\phi(l_2), p)\right)\bigg\} \\
&\quad + B(\bar{r_0})K_\phi\left\{\lambda\psi\left(\max_{1\leq t\leq n}\Big[|u(b_t(l_1)) - u(b_t(l_2))| : M \leq l_2 < l_1\Big]\right) + F_{r0}(M)\right\}.
\end{aligned}
$$

Taking limit as $M \to \infty$ the estimated quantities $P(M), Q(M), W_{r_0}(M)$, $F_{r_0}(M)$ and $V_{r_0}(M)$ tend to zero and the above inequality becomes:

$$
\begin{aligned}
D(TX) &\leq \Big(\alpha\ \psi\left(D(X)\right) + B(\bar{r_0})K_\phi\lambda\ \psi\left(D(X)\right)\Big), \\
D(TX) &\leq \left(\alpha + \lambda B(\bar{r_0})K_\phi\right)\psi\left(D(X)\right).
\end{aligned}
\tag{4.23}
$$

Finally, combining equation (4.22) and (4.23), we have the following inequality

$$
\begin{aligned}
\mu(TX) &\leq \left(\alpha + \lambda B(\bar{r_0})K_\phi\right)\psi\left(\omega_0(X)\right)\left(\alpha + \beta B(\bar{r_0})K_\phi\right)\psi\left(D(X)\right) \\
&\leq \left(\alpha + \lambda B(\bar{r_0})K_\phi\right)\psi\left(\mu(X)\right).
\end{aligned}
\tag{4.24}
$$

Let us define the two functions $\beta(s)$ and $\xi_\mu(t)$ in the following way:

$$\beta(s) = \begin{cases} \frac{1}{2} & \text{if } s = 0, \\ \frac{\psi(s)}{s} & \text{if } 0 < s \leq \mu(\Omega), \\ \frac{\psi(\mu(\Omega))}{s} & \text{if } s > \mu(\Omega) \end{cases}$$

and

$$\xi_\mu(C) = \begin{cases} \frac{1}{(\alpha+\lambda B(\bar{r_0})K_\phi)} & \text{if } \varsigma\left(x(t), y(t)\right) \geq 0, x, y \in C, \\ 0 & \text{otherwise.} \end{cases}$$

Then equation (4.24) becomes

$$\xi(X)\mu\left(TX\right) \leq \beta\left(\mu\left(X\right)\right)\mu\left(X\right). \tag{4.25}$$

Since by assumption (A_{11}), if $\xi_\mu(X) \geq 1$ then $\xi_\mu(TX) \geq 1$, hence, T is ξ_μ-admissible and ξ_μ-β condensing operator. Thus from Theorem 3.2 the Volterra-Stieltjes integral equation (1.1) has at least one solution in $BC\left(\mathbb{R}_+\right)$. This completes the proof. $\square$

Remark 4.8. (i) The integral equation defined in [6] is obtained from (1.1) by substituting $m = n = p = 1$, and define $\phi(t) = t$, $f(t,x) \approx a(t)$ for all x and $g(t,x) = 1$,

$$u(l) = a(l) + \int_0^l g(l,\tau,u(\tau))d_\tau K(l,\tau). \tag{4.26}$$

(ii) If $m = n = p = 1$, $\phi(t) = t$ and $f(t,x) \approx a(t)$ for all x, then we obtain class of integral equation defined in [14] ,

$$u(l) = a(l) + g(l,u(l))\int_0^l g(l,\tau,u(\tau))d_\tau K(l,\tau). \tag{4.27}$$

Example 4.9. Consider the following functional integral equation of Volterra-Stieltjes type in the Banach space $BC(R_+)$.

$$\begin{aligned} u(l) = \zeta \cdot \frac{\ln(1+l)}{1+l}\sin(u^2(l)) \\ + \delta \cdot \arctan\left(\frac{l^2+u^2(l)}{1+l^2}\right)\int_0^{\frac{3}{2}} \frac{(l^2+\tau^2)e^{-l^2} + \frac{\tau^2}{l^2+\tau^2}}{1+l^6+\tau^2}u(\tau)d\tau, \end{aligned} \tag{4.28}$$

where $\delta > 0$ and $\zeta > 0$ are real numbers. Then the Volterra-Stieltjes integral equation (4.28) has at least one solution in the Banach space $BC(\mathbb{R}_+)$.

In fact, the integral equation (4.28) is obtained from (1.1) by substituting:

$$\begin{aligned}
&m = n = p = 1,\ \phi(l) = l^{\frac{3}{2}};\\
&K(\phi(l), \phi(\tau)) = \frac{1}{\sqrt{1+l^6}} \arctan \frac{\tau}{\sqrt{1+l^6}};\\
&\pi_1(l, u) = \zeta\ \frac{\ln(1+l)}{1+l} \sin(u^2(l));\\
&\pi_2(l, u) = \delta \arctan\left(\frac{l^2+u^2(l)}{1+l^2}\right);\\
&g(l, \tau, u(\tau)) = \frac{(l^2+\tau^2)e^{-l^2} + \frac{\tau^2}{l^2+\tau^2}}{1+l^6+\tau^2} u(\tau).
\end{aligned} \tag{4.29}$$

The given Volterra-integral equation (4.28) can be written as

$$\begin{aligned}
u(l) = \zeta \cdot \frac{\ln(1+l)}{1+l} \sin(u^2) + \delta \cdot \arctan\left(\frac{l^2+u^2(l)}{1+l^2}\right)\\
\times \int_0^{\frac{3}{2}} (l^2+\tau^2)e^{-l^2} + \frac{\tau^2}{l^2+\tau^2} u(\tau) d_\tau \left(\frac{1}{\sqrt{1+l^6}} \arctan \frac{\tau}{\sqrt{1+\tau^6}}\right).
\end{aligned}$$

For an arbitrary $R > 0$ the function $\phi(l)$ is non-decreasing and satisfies Lipschitz condition on $[-R, R]$, hence $\phi(l)$ is K-bounded variation on $[-R, R]$, also we can easily check that

$$d_\tau K(\phi(l), \phi(\tau)) = \frac{\partial K(\phi(l), \phi(\tau))}{\partial \tau} = \frac{1}{1+\tau^2+l^6}\ d\tau.$$

Let us prove that the functions $\pi_1(l, u)$ and $\pi_2(l, u)$, satisfies the assumption (A_2). In fact,

$$\begin{aligned}
|\pi_1(l, u) - \pi_1(l, v)| &\leq \zeta \left| \frac{\ln(1+l)}{1+l} \sin(u^2(l)) - \frac{\ln(1+l)}{1+l} \sin(v^2(l)) \right|\\
&\leq \zeta \left| \frac{\ln(1+l)}{1+l} \right| |\sin(u^2(l)) - \sin(v^2(l))|\\
&\leq \frac{\zeta}{e} |u^2 - v^2|\\
&\leq \frac{\zeta r}{e} |u - v|
\end{aligned}$$

and

$$|\pi_2(l, u) - \pi_2(l, v)| \leq \delta \arctan\left(\frac{l^2+u^2(l)}{1+l^2}\right) - \delta \arctan\left(\frac{l^2+v^2(t)}{1+l^2}\right),$$

and hence

$$\begin{aligned}
|\pi_2(l,u)-\pi_2(l,v)| &\leq \delta \left| \frac{\sin\left(\frac{l^2+u^2}{1+l^2}\right)-\sin\left(\frac{l^2+v^2}{1+l^2}\right)}{1+\sin\left(\frac{l^2+u^2}{1+l^2}\right)\sin\left(\frac{l^2+v^2}{1+l^2}\right)} \right| \\
&\leq \delta \left| \left(\frac{l^2+u^2}{1+l^2}\right)-\left(\frac{l^2+v^2}{1+l^2}\right) \right| \\
&\leq \delta \left(\frac{1}{1+l^2}\right) \left|u^2-v^2\right| \\
&\leq \frac{\delta r}{2}|u-v|.
\end{aligned}$$

Hence, the function $\pi_1(l,u)$ and $\pi_2(l,u)$ satisfied the assumption (A_2).

Now for fixed $r>0$ and for any $u \in [-r,r]$ the limit of $\pi_1(l,u)$ and $\pi_2(l,u)$ as $t \to \infty$ is:

$$\lim_{l\to\infty} \pi_1(l,u) = \lim_{l\to\infty} \zeta \frac{\ln(1+l)}{1+l}\sin(u^2(l) = 0$$

and

$$\lim_{l\to\infty} \pi_2(l,u) = \lim_{l\to\infty} \delta \arctan\left(\frac{l^2+u^2}{1+l^2}\right) = \delta \cdot \frac{\pi}{4}.$$

Let us assume that $r>0$, $M>0$ be arbitrary real numbers such that $u \in [-r,r]$ and $l_1 > l_2 \geq M$, with this consideration let us verify the assumption (A_5).

$$\begin{aligned}
|\pi_1(l_1,u)-\pi_2(l_2,u)| &\leq \zeta \left| \frac{\ln(1+l_1)}{1+l_1}\sin(u^2) - \frac{\ln(1+l_2)}{1+l_2}\sin(u^2) \right| \\
&\leq \zeta \left| \frac{\ln(1+l_1)}{1+l_1} - \frac{\ln(1+l_2)}{1+l_2} \right| |\sin(u^2)| \\
&\leq \zeta \left(\frac{\ln(1+l_1)}{1+l_1} + \frac{\ln(1+l_2)}{1+l_2} \right) |\sin(u^2)| \\
&\leq \zeta\, 2\frac{\ln(1+M)}{1+M} \qquad \because |(u^2)| \leq 1
\end{aligned} \tag{4.30}$$

and

$$\begin{aligned}
|\pi_2(l_1,u)-g(l_2,u)| &\leq \delta \ \left| \arctan\left(\frac{l_1+u}{1+l_1^2}\right) - \arctan\left(\frac{l_2+u}{1+l_2^2}\right) \right| \\
&\leq \delta \left| \frac{\sin\left(\frac{l_1+u}{1+l_1^2}\right)-\sin\left(\frac{l_2+u}{1+l_2^2}\right)}{1+\sin\left(\frac{l_1+u}{1+l_1^2}\right)\sin\left(\frac{l_2+u}{1+l_2^2}\right)} \right| \\
&\leq \delta \left| \frac{\left(\frac{l_1^2+u^2}{1+l_1^2}\right)-\left(\frac{l_2^2+u^2}{1+l_2^2}\right)}{1} \right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \delta\left|\left(\frac{l_1^2+u^2}{1+l_1^2}\right)-\left(\frac{l_2^2+u^2}{1+l_2^2}\right)\right| \\
&\leq \left(\frac{M^2+r^2}{1+M^2}+\frac{M^2+r^2}{1+M^2}\right) \\
&\leq \delta(r^2+1)\frac{1}{1+M^2}.
\end{aligned}
$$

Thus, assumption (A_5) is verified. Now, let us verify (A_4), we can easily check the continuity of $g=g(l,\tau,u(\tau))$ on the set $\Theta\times\mathbb{R}$. Moreover, for the arbitrary $(l,\tau)\in\Theta$ and $u\in\mathbb{R}$, we get

$$
\begin{aligned}
|g(l,\tau,u(\tau))| &\leq \left|(l^2+\tau^2)e^{-l^2}+\frac{\tau^2}{l^6+\tau^2+1}u(\tau)\right| \\
&\leq (l^2+l^2)e^{-l^2}+\frac{l^2}{l^6+l^2+1}u(\tau) \\
&\leq (2l^2)e^{-l^2}+\frac{l^2}{l^6+l^2}u(\tau) \\
&\leq \frac{2}{e}+\frac{r}{1}.
\end{aligned}
$$

Thus g verifies the inequality of assumption (A_4) with $B(\bar{r})=\frac{2}{e}+\frac{r}{1}$. Now, fix $r>0,\ M>0$ and take $l_1,\ l_2,\tau\in\mathbb{R}_+$ such that $l_1,l_2\geq M,\tau\leq l_1,\tau\leq l_2$ and $u\in[-r,r]$. Without loss of generality, suppose that $M\geq 2$. Then we have:

$$
\begin{aligned}
&|g(l_1,\tau,u(\tau))-g(l_2,\tau,u(\tau))| \\
&\leq \left|\left((l_1^2+\tau^2)e^{-l_1^2}+\frac{\tau^2}{l_1^6+\tau^2+1}u(\tau)\right)-\left((l_2^2+l_2^2)e^{-l_2^2}+\frac{\tau^2}{l_2^6+\tau^2+1}u(\tau)\right)\right| \\
&\leq \left|\left((l_1^2+\tau^2)e^{-l_1^2}-(l_2^2+\tau^2)e^{-l_2^2}\right)\right|+\left|\left(\frac{\tau^2}{l_1^6+\tau^2+1}-\frac{\tau^2}{l_2^6+\tau^2+1}\right)\right||u(\tau)| \\
&\leq \left((l_1^2+\tau^2)e^{-l_1^2}+(l_2^2+\tau^2)e^{-l_2^2}\right)+\left(\frac{\tau^2}{l_1^6+\tau^2+1}+\frac{\tau^2}{l_2^6+\tau^2+1}\right)r \\
&\leq 4M^2e^{-M^2}+\frac{2M^2}{M^6+M^2}r \\
&\leq 4M^2e^{-M^2}+\frac{2}{M^4+1}r.
\end{aligned}
$$

Let us verify that $K=K(\phi(l),\phi(\tau))$ defined by (4.29) satisfies assumption (A_7). To show that function $K(\phi(l),\phi(\tau))$ is function of bounded variation, consider

$$
\frac{\partial K(\phi(l),\phi(\tau))}{\partial\tau}=\frac{1}{1+\tau^2+l^6}>0.
$$

We infer that the function $\phi(\tau) \to K(\phi(l), \phi(\tau))$ is increasing on every interval of the form $[0, l]$. Since $K(\phi(l), \phi(\tau))$ is bounded on the set Θ which guaranties that $\phi(\tau) \to K(\phi(l), \phi(\tau))$ is the function of bounded variation on the interval $[0,\ l]$. Moreover, we can have

$$\begin{aligned}\bigvee_{p=0}^{\phi(l)} K(\phi(l), p) &= K(\phi(l), \phi(l)) - K(\phi(l), 0)\\ &= \bigvee_{p=0}^{l^{\frac{3}{2}}} \sqrt{1+t^6}\arctan\frac{\tau}{\sqrt{1+t^6}}\\ &\leq \frac{\pi}{4}.\end{aligned}$$

From the above estimation we have $K_\phi \leq \frac{\pi}{4}$ and assumption (A_7) is satisfied. Let us prove that the function $K(\phi(l), \phi(\tau))$ defined by (4.29) satisfies the equalities in the assumption (A_9). For this fix the arbitrary $l_2 < l_1 \in [0, M]$, we get

$$\begin{aligned}&K(\phi(l_1), \phi(\tau)) - K(\phi(l_2), \phi(\tau))\\ &= \sqrt{1+l_1^6}\arctan\frac{\tau}{\sqrt{1+l_1^6}} - \sqrt{1+l_2^6}\arctan\frac{\tau}{\sqrt{1+l_2^6}}\\ &\leq \frac{\pi}{4}\frac{1}{\sqrt{1+M^6}}.\end{aligned}$$

The above expression shows that $K(\phi(l), \phi(\tau))$ satisfies the second equality assumption A_9. Similarly, we can easily verify first equality of assumption (A_9).

Let us proceed to assumption (A_{10}) we have

$$\pi_1(l, 0) = 0 \implies M_{\pi_1} = 0 \ \text{ and } \pi_2(l, 0) = \delta \implies M_{\pi_2} = \frac{\delta\ \pi}{4}.$$

Considering above estimated values we are easily establish the first inequality of assumption (A_{10}).

$$\frac{\zeta r}{e} + \left(\frac{\delta\ r}{2} + \frac{\delta\ \pi}{4}\right)\left(\frac{2}{e} + \frac{r}{1}\right)\frac{\pi}{4} \leq r,$$

it implies that

$$\left(\frac{\delta\ r}{2} + \frac{\delta\ \pi}{4}\right)\left(\frac{2}{e} + \frac{r}{1}\right)\frac{\pi}{4} \leq r\left(1 - \frac{\zeta}{e}\right). \tag{4.31}$$

By the suitable choice of $r,\ \delta$ and ζ, the inequality in (4.31) and second inequality of assumption (A_{10}) are simultaneously satisfied. Thus all the conditions of Theorem 4.7 are satisfied. Therefore the Volterra-Stieltjes integral equation (4.28) has at least one solution in the Banach space $BC(\mathbb{R}_+)$.

4.1. **Numerical estimations for r, ζ and δ.** From the inequality (4.31), the range of ζ is $0 < \zeta < e$. Let us discuss the relation between ζ, δ and r. From the inequality (4.31) for every value of $\zeta \in (0, e)$ there is a number $\kappa \in \mathbb{R}_+$ such that $\delta \leq \kappa$. The Table 1 represents numerical values for which the integral equation has solution for $r = \frac{1}{2}, 1, \frac{3}{2}$.

	$r = \frac{1}{2}$	$r = 1$	$r = \frac{3}{2}$
ζ	κ	κ	κ
0.25	0.452400	0.518779	0.505579
0.50	0.406593	0.466271	0.454407
0.75	0.360805	0.413763	0.403235
1.00	0.315018	0.361255	0.352063
1.25	0.269231	0.308747	0.30091
1.50	0.223443	0.256239	0.249719
1.75	0.177655	0.203731	0.198547
2.00	0.131868	0.151223	0.147375
2.25	0.086081	0.098715	0.096203
2.50	0.040293	0.046207	0.045031

TABLE 1. Relation between ζ and κ, $(\delta \leq \kappa)$

In the Figure 1 the graphical representation of Table 1 has given. We can observe that ζ and κ are inversely proportional to each other, moreover ζ lies in $(0, e)$ then δ lies in $(0, 0.6)$ for $r = \frac{1}{2}, 1, \frac{3}{2}$.

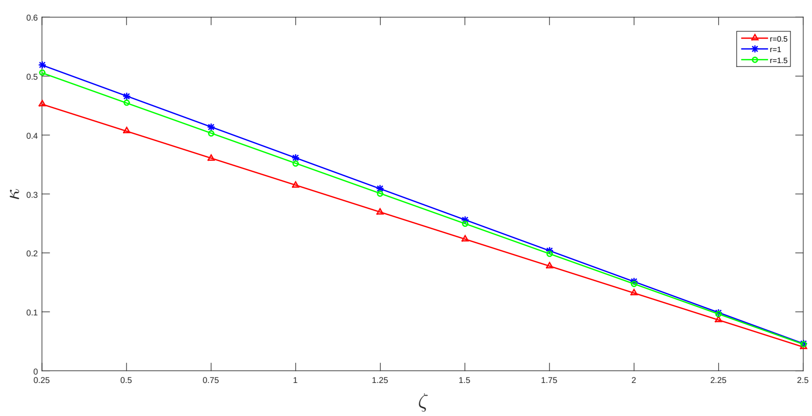

FIGURE 1. Graph of Showing the relation between r, ζ and κ, $(\delta \leq \kappa)$

In the similar argument, we can discuss about ζ and κ for $r = 2, 3, ..., n$.

5. Conclusion

In this paper, we defined and proved the fixed point theorems for $\xi_\mu-$admissible, ξ_μ-β and ξ_μ-β-ψ condensing operators. The Corollaries proved in the Section 3 are the existing results in the literature [2, 11, 19]. The nonlinear quadratic Volterra-Stieltjes integral equation is the generalization of the main results in [6, 14], and is derived by using the function of bounded variation and Riemann-Stieltjes integral. We investigated the solvability of nonlinear quadratic Volterra-Stieltjes integral equation by using ξ_μ-admissible and ξ_μ-β-condensing operator. Finally, we have provided the numerical values for which the illustrative example has a solution.

Acknowledgments: This research was partially supported by Valaya Alongkorn Rajabhat University under the Royal Patronage, Thailand.

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