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HYERS-ULAM STABILITY OF THE JENSEN FUNCTIONAL EQUATION IN QUASI-BANACH SPACES

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Abstract. In this paper, we prove the Hyers-Ulam-Rassias stability property for the Jensen functional equations

$$f(x+y)+f(x+\sigma(y))=2f(x);\ \ f(x+y)-f(x+\sigma(y))=2f(y)\quad x,y\in E_1$$

for mappings from a normed space E_1 into a quasi-Banach space E_2.

1. Introduction and preliminaries

The stability problem of functional equations was posed for the first time by S. M. Ulam [31] in the year 1940. Ulam stated the problem as follows:

Given a group G_1, a metric group (G_2, d), a number $\epsilon > 0$ and a mapping f: $G_1 \longrightarrow G_2$ which satisfies $d(f(xy), f(x)f(y)) < \epsilon$ for all $x, y \in G_1$, does there exist an homomorphism g: $G_1 \longrightarrow G_2$ and a constant $k > 0$, depending only on G_1 and G_2 such that

$$d(f(x), g(x)) < k\epsilon$$

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for all $x \in G_1$?

In 1941, D. H. Hyers [7] considered the case of approximately additive mappings $f\colon\ E \longrightarrow E'$, where E and E' are Banach spaces and f satisfies the following inequality

$$\|f(x+y)-f(x)-f(y)\| \leq \epsilon$$

for all $x, y \in E$. It was shown that the limit $T(x)=\lim_{n\longrightarrow+\infty}\frac{f(2^n x)}{2^n}$ exits for all $x \in E$ and that $T\colon\ E \longrightarrow E'$ is the unique additive mapping satisfying

$$\|f(x)-T(x)\| \leq \epsilon.$$

In 1978, Th. M. Rassias [24] provided a generalization of Hyers's stability theorem which allows the Cauchy difference to be unbounded, as follows:

Theorem 1.1. [24] *Let $f\colon V \longrightarrow X$ be a mapping between Banach spaces and let $p<1$ be fixed. If f satisfies the inequality*

$$\|f(x+y)-f(x)-f(y)\| \leq \theta(\|x\|^p+\|y\|^p) \tag{1.1}$$

for some $\theta \geq 0$ and for all $x, y \in V$ ($x, y \in V\setminus\{0\}$ if $p<0$). Then there exists a unique additive mapping $T\colon V \longrightarrow X$ such that

$$\|f(x)-T(x)\| \leq \frac{2\theta}{2^p-2}\|x\|^p \tag{1.2}$$

for all $x \in V$ ($x \in V\setminus\{0\}$ if $p<0$).
If, in addition, $f(tx)$ is continuous in t for each fixed $x \in V$, then T is linear.

In 1990, during the 27th International Symposium of functional equations, Th. M. Rassias asked the question whether such a theorem can also be proved for values of p greater or equal to one [22]. Z. Gajda [5] following the same approach as in [24] provided an affirmative solution to Rassias' question for p strictly greater than one. However, it was shown independently by Z. Gajda [5] and Th. M. Rassias and P. Šemrl [27] that a similar result for the case of value of p equal to one can not be obtained.

The concept of the linear mapping, that was introduced for the first time in 1978 by Th. M. Rassias and followed later by several other mathematicians is known today as Hyers-Ulam-Rassias stability. During the last decades several stability problems of functional equations have been investigated by a number of mathematicians, the reader can be referred for example to the monographs [9, 12] and [2]-[28],[30].

We consider some basic concepts concerning quasi-β-normed spaces and some preliminaries results.

Definition 1.2. *Let X be a linear space. A quasi-norm is a real-valued function on X satisfying the following:*

(1) $\|x\| \geq 0$ *for all* $x \in X$ *and* $\|x\| = 0$ *if and only if* $x = 0$.
(2) $\|\lambda x\| = |\lambda|\|x\|$ *for all* $\lambda \in \mathbb{R}$ *and all* $x \in X$.
(3) *There is a constant* $K \geq 1$ *such that*
$$\|x + y\| \leq K(\|x\| + \|x\|) \tag{1.3}$$
for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a quasi-normed space if $\|\cdot\|$ is a quasi-norm on X. The smallest possible K is called the modulus of concavity of $\|\cdot\|$.

A quasi-Banach space is a complete quasi-normed space.

A quasi-norm $\|\cdot\|$ is called a β-norm $(0 < \beta \leq 1)$ if
$$\|x + y\|^\beta \leq \|x\|^\beta + \|y\|^\beta \tag{1.4}$$
for all $x, y \in X$. In this case a quasi-Banach space is called a β-Banach space. We refer to [1, 29] for the concept of quasi-normed spaces and quasi-Banach spaces. Given a β-norm, the formula $d(x, y) = \|x - y\|^\beta$ gives us a translation invariant metric on X. By the Aoki-Rolewicz Theorem [29] (see also [1]), each quasi norm is equivalent to some β-norm. Since it is much easier to work with β-norm than quasi-norm, hence we restrict our attention mainly to β-norm.

In [18] C. Park generalized the concept of quasi-normed spaces as follows

Definition 1.3. *Let X be a linear space. A function $\|\cdot\|$: $X \longrightarrow [0, \infty)$ is called a generalized quasi-norm if and only if it satisfies the following properties:*

(1) $\|x\| \geq 0$ *for all* $x \in X$ *and* $\|x\| = 0$ *if and only if* $x = 0$.
(2) $\|\lambda x\| = |\lambda|\|x\|$ *for all* $\lambda \in \mathbb{R}$ *and all* $x \in X$.
(3) *There is a constant* $K \geq 1$ *such that*
$$\|\sum_{j=1}^{\infty} x_j\| \leq K \sum_{j=1}^{\infty} \|x_j\| \tag{1.5}$$
for all $x_1, x_2, \ldots \in X$ *with* $\sum_{j=1}^{\infty} x_j \in X$.

The pair $(X, \|\cdot\|)$ is called a generalized quasi-normed space if $\|\cdot\|$ is a generalized quasi-norm on X. The smallest possible K is called the modulus of concavity of $\|\cdot\|$. A generalized quasi-Banach space is a complete generalized quasi-normed space.

In this paper we consider the Jensen functional equations
$$f(x + y) + f(x + \sigma(y)) = 2f(x), \qquad x, y \in E_1, \tag{1.6}$$

$$f(x+y) - f(x+\sigma(y)) = 2f(y), \qquad x, y \in E_1 \tag{1.7}$$

where σ: $E_1 \longrightarrow E_1$ is an involution of E_1, i.e., $\sigma(x+y) = \sigma(x) + \sigma(y)$ and $\sigma(\sigma(x)) = x$ for all $x, y \in E_1$.

Recently G.H. Kim [16] have improved the stability of equation (1.6) over an abelian group under the condition

$$\|f(x+y) + f(x+\sigma(y)) - 2f(x)\| \le \varphi(x) \text{ or } \varphi(y).$$

In [17] G.H. Kim proved the Hyers-Ulam-Rassias stability of equation (1.7) in normed space, with $\sigma(x) = -x$ and $0 \le p < 1$. In [2] the stability of other generalized Jensen functional equations have been investigated.

The stability problems of several functional equations in quasi-Banach spaces have been extensively investigated by a number of authors, we refer for example to [19], [21] and [30].

Our main goal in this paper is to investigate the Hyers-Ulam stability problem for the equations (1.6) and (1.7) in generalized quasi-Banach spaces and in quasi-β-Banach spaces.

2. Hyers-Ulam stability of (1.6) with $p < 1$ and $p > 1$

In this section we investigate the Hyers-Ulam stability for the equation (1.6).

Theorem 2.1. *Let E_1 be a normed space, E_2 a generalized quasi-Banach space and f: $E_1 \longrightarrow E_2$ a mapping which satisfies the inequality*

$$\|f(x+y) + f(x+\sigma(y)) - 2f(x)\|_{E_2} \le \theta(\|x\|^p + \|y\|^p) \tag{2.1}$$

for some $\theta \ge 0$, $p > 1$ and for all $x, y \in E_1$. Then there exits a unique mapping $J : E_1 \longrightarrow E_2$, defined by

$$J(x) = \lim_{n\to+\infty} 2^n \{f(\frac{x}{2^n}) + (\frac{1}{2^n} - 1)f(\frac{x}{2^{n+1}} + \frac{\sigma(x)}{2^{n+1}})\} \tag{2.2}$$

that is a solution of the Jensen functional equation (1.6) such that

$$\|f(x) - J(x)\|_{E_2} \le \frac{2K\theta}{2^p - 2}\|x\|^p + \frac{K\theta}{(2^p-1)(2^p-2)}\|x + \sigma(x)\|^p] \tag{2.3}$$

for all $x \in E_1$.

Proof. Suppose that f satisfies the inequality (2.1). Replacing x, y by $\frac{x}{2^n}$, (resp. by $\frac{x}{2^{n+1}} + \frac{\sigma(x)}{2^{n+1}}$), we obtain

$$\|f(\frac{x}{2^{n-1}}) + f(\frac{x}{2^n} + \frac{\sigma(x)}{2^n}) - 2f(\frac{x}{2^n})\|_{E_2} \le \frac{2\theta}{2^{np}}\|x\|^p, \tag{2.4}$$

respectively

$$\|2f(\frac{x}{2^n}+\frac{\sigma(x)}{2^n})-2f(\frac{x}{2^{n+1}}+\frac{\sigma(x)}{2^{n+1}})\|_{E_2}\le\frac{2\theta}{2^{(n+1)p}}\|x+\sigma(x)\|^p. \tag{2.5}$$

Make the induction assumption:

$$\begin{aligned}&\|f(x)-2^n\{f(\frac{x}{2^n})+(\frac{1}{2^n}-1)f(\frac{x}{2^{n+1}}+\frac{\sigma(x)}{2^{n+1}})\}\|_{E_2}\\ \le\ &\frac{2\theta K}{2^p}\|x\|^p[1+2^{1-p}+\ldots+2^{(n-1)(1-p)}]\\ &+\frac{2\theta K}{2^{2p}}\|x+\sigma(x)\|^p[(1-\frac{1}{2})+(1-\frac{1}{2^2})2^{1-p}+\ldots\\ &+(1-\frac{1}{2^n})2^{(n-1)(1-p)}].\end{aligned} \tag{2.6}$$

For $n=1$, by using the triangle inequality (1.5), we get

$$\begin{aligned}&\|f(x)-2\{f(\frac{x}{2})+(\frac{1}{2}-1)f(\frac{x}{2^2}+\frac{\sigma(x)}{2^2})\}\|_{E_2}\\ \le\ &K\|f(x)+f(\frac{x}{2}+\frac{\sigma(x)}{2})-2f(\frac{x}{2})\|_{E_2}\\ &+K\|f(\frac{x}{4}+\frac{\sigma(x)}{4})-f(\frac{x}{2}+\frac{\sigma(x)}{2})\|_{E_2}\\ \le\ &K\frac{2\theta}{2^p}\|x\|^p+K\frac{\theta}{2^{2p}}\|x+\sigma(x)\|^p=\frac{2\theta K}{2^p}\|x\|^p+\frac{2\theta K}{2^{2p}}\|x+\sigma(x)\|^p(1-\frac{1}{2}).\end{aligned}$$

So (2.6) is true for $n=1$. We will show that the induction assumption (2.6) is true with n replaced by $n+1$.

$$\begin{aligned}&\|f(x)-2^{n+1}\{f(\frac{x}{2^{n+1}})+(\frac{1}{2^{n+1}}-1)f(\frac{x}{2^{n+2}}+\frac{\sigma(x)}{2^{n+2}})\}\|_{E_2}\\ =\ &\|f(x)-2^n\{f(\frac{x}{2^n})+(\frac{1}{2^n}-1)f(\frac{x}{2^{n+1}}+\frac{\sigma(x)}{2^{n+1}})\}\\ &+2^n\{f(\frac{x}{2^n})+f(\frac{x}{2^{n+1}}+\frac{\sigma(x)}{2^{n+1}})-2f(\frac{x}{2^{n+1}})\}\\ &+2^{n+1}(\frac{1}{2^{n+1}}-1)\{f(\frac{x}{2^{n+1}}+\frac{\sigma(x)}{2^{n+1}})-f(\frac{x}{2^{n+2}}+\frac{\sigma(x)}{2^{n+2}})\}\|_{E_2}\\ =\ &\|f(x)-2^{n-1}\{f(\frac{x}{2^{n-1}})+(\frac{1}{2^{n-1}}-1)f(\frac{x}{2^n}+\frac{\sigma(x)}{2^n})\}\\ &+2^{n-1}\{f(\frac{x}{2^{n-1}})+f(\frac{x}{2^n}+\frac{\sigma(x)}{2^n})-2f(\frac{x}{2^n})\}\end{aligned}$$

$$
\begin{aligned}
&+2^n(\frac{1}{2^n}-1)\{f(\frac{x}{2^n}+\frac{\sigma(x)}{2^n})-f(\frac{x}{2^{n+1}}+\frac{\sigma(x)}{2^{n+1}})\}\\
&+2^n\{f(\frac{x}{2^n})+f(\frac{x}{2^{n+1}}+\frac{\sigma(x)}{2^{n+1}})-2f(\frac{x}{2^{n+1}})\}\\
&+2^{n+1}(\frac{1}{2^{n+1}}-1)\{f(\frac{x}{2^{n+1}}+\frac{\sigma(x)}{2^{n+1}})-f(\frac{x}{2^{n+2}}+\frac{\sigma(x)}{2^{n+2}})\}\|_{E_2}\\
= \ & \|\{f(x)-2\{f(\frac{x}{2})+(\frac{1}{2}-1)f(\frac{x}{2^2}+\frac{\sigma(x)}{2^2})\}\\
&+2\{f(\frac{x}{2})+f(\frac{x}{2^2}+\frac{\sigma(x)}{2^2})-2f(\frac{x}{2^2})\}\\
&+2^2(\frac{1}{2^2}-1)\{f(\frac{x}{2^2}+\frac{\sigma(x)}{2^2})-f(\frac{x}{2^3}+\frac{\sigma(x)}{2^3})\}\\
&+2^2\{f(\frac{x}{2^2})+f(\frac{x}{2^3}+\frac{\sigma(x)}{2^3})-2f(\frac{x}{2^3})\}\\
&+2^3(\frac{1}{2^3}-1)\{f(\frac{x}{2^3}+\frac{\sigma(x)}{2^3})-f(\frac{x}{2^4}+\frac{\sigma(x)}{2^4})\}+\cdots\\
&+2^{n-1}\{f(\frac{x}{2^{n-1}})+f(\frac{x}{2^n}+\frac{\sigma(x)}{2^n})-2f(\frac{x}{2^n})\}\\
&+2^n(\frac{1}{2^n}-1)\{f(\frac{x}{2^n}+\frac{\sigma(x)}{2^n})-f(\frac{x}{2^{n+1}}+\frac{\sigma(x)}{2^{n+1}})\}\\
&+2^n\{f(\frac{x}{2^n})+f(\frac{x}{2^{n+1}}+\frac{\sigma(x)}{2^{n+1}})-2f(\frac{x}{2^{n+1}})\}\\
&+2^{n+1}(\frac{1}{2^{n+1}}-1)\{f(\frac{x}{2^{n+1}}+\frac{\sigma(x)}{2^{n+1}})-f(\frac{x}{2^{n+2}}+\frac{\sigma(x)}{2^{n+2}})\}\|_{E_2}\\
\leq \ & \frac{2\theta K}{2^p}\|x\|^p[1+2^{1-p}+\ldots+2^{n(1-p)}]\\
&+\frac{2\theta K}{2^{2p}}\|x+\sigma(x)\|^p[(1-\frac{1}{2})+(1-\frac{1}{2^2})2^{1-p}+\cdots+(1-\frac{1}{2^{n+1}})2^{n(1-p)}].
\end{aligned}
$$

Now, the inequality (2.6) is proved for all n.

Next, we will show that the sequence functions

$$
J_n(x)=2^n\{f(\frac{x}{2^n})+(\frac{1}{2^n}-1)f(\frac{x}{2^{n+1}}+\frac{\sigma(x)}{2^{n+1}})\} \tag{2.7}
$$

is a Cauchy sequence for every $x\in E_1$. Combine (2.4) and (2.5) by use of the triangle inequality (1.5) to show that

$$
\begin{aligned}
& \|J_{n+1}(x) - J_n(x)\|_{E_2} \\
= \ & \|2^{n+1}\{f(\frac{x}{2^{n+1}}) + (\frac{1}{2^{n+1}} - 1)f(\frac{x}{2^{n+2}} + \frac{\sigma(x)}{2^{n+2}})\} \\
& -2^n\{f(\frac{x}{2^n}) + (\frac{1}{2^n} - 1)f(\frac{x}{2^{n+1}} + \frac{\sigma(x)}{2^{n+1}})\}\|_{E_2} \\
\leq \ & 2^n K\|f(\frac{x}{2^n}) + f(\frac{x}{2^{n+1}} + \frac{\sigma(x)}{2^{n+1}}) - 2f(\frac{x}{2^{n+1}})\|_{E_2} \\
& +2^{n+1}K(1 - \frac{1}{2^{n+1}})\|f(\frac{x}{2^{n+1}} + \frac{\sigma(x)}{2^{n+1}}) - f(\frac{x}{2^{n+2}} + \frac{\sigma(x)}{2^{n+2}})\|_{E_2} \\
\leq \ & 2^n K\frac{2\theta}{2^{(n+1)p}}\|x\|^p + 2^n K(1 - \frac{1}{2^{n+1}})\frac{2\theta}{2^{(n+2)p}}\|x + \sigma(x)\|^p \\
\leq \ & 2^{n(1-p)}\frac{2\theta K}{2^p}[\|x\|^p + \frac{1}{2^p}\|x + \sigma(x)\|^p].
\end{aligned}
$$

Since $2^{1-p} < 1$, the desired conclusion follows. However, E_2 is a generalized quasi-Banach space, it follows that the functions $J_n(x)$ form a sequence which converges to some function $J(x)$ for all x in E_1.

Let us show now that J is a solution of Jensen functional equation (1.6). Indeed,

$$
\begin{aligned}
& \|J_n(x+y) + J_n(x + \sigma(y)) - 2J_n(x)\|_{E_2} \\
= \ & \|2^n\{f(\frac{x+y}{2^n}) + (\frac{1}{2^n} - 1)f(\frac{x+y}{2^{n+1}} + \frac{\sigma(x) + \sigma(y)}{2^{n+1}})\} \\
& +2^n\{f(\frac{x + \sigma(y)}{2^n}) + (\frac{1}{2^n} - 1)f(\frac{x + \sigma(x)}{2^{n+1}} + \frac{y + \sigma(y)}{2^{n+1}})\} \\
& -2^{n+1}\{f(\frac{x}{2^n}) + (\frac{1}{2^n} - 1)f(\frac{x}{2^{n+1}} + \frac{\sigma(x)}{2^{n+1}})\}\|_{E_2} \\
\leq \ & 2^n K\|f(\frac{x+y}{2^n}) + f(\frac{x + \sigma(y)}{2^n}) - 2f(\frac{x}{2^n})\|_{E_2} \\
& +2^n K(1 - \frac{1}{2^n})\|f(\frac{x + \sigma(x)}{2^{n+1}} + \frac{y + \sigma(y)}{2^{n+1}}) \\
& +f(\frac{x + \sigma(x)}{2^{n+1}} + \frac{y + \sigma(y)}{2^{n+1}}) - 2f(\frac{x}{2^{n+1}} + \frac{\sigma(x)}{2^{n+1}})\}\|_{E_2} \\
\leq \ & 2^{n(1-p)}K\theta[\|x\|^p + \|y\|^p + \frac{1}{2^p}\|x + \sigma(x)\|^p + \frac{1}{2^p}\|y + \sigma(y)\|^p].
\end{aligned}
$$

Here $2^{1-p} < 1$, then by letting $n \to +\infty$, we get that J is a solution of Jensen functional equation (1.6).

Assume now that there exist two functions $J_i : E_1 \longrightarrow E_2$ $(i = 1, 2)$ that are solutions of equation (1.6) with inequality (2.3). First, we will prove by mathematical induction that

$$J_i(\frac{x}{2^n})+(\frac{1}{2^n}-1)J_i(\frac{x}{2^{n+1}}+\frac{\sigma(x)}{2^{n+1}})=\frac{1}{2^n}J_i(x),\ (i=1,2). \tag{2.8}$$

From equation (1.6) it follows that if we replace x and y by $\frac{x}{2^{n+2}}+\frac{\sigma(x)}{2^{n+2}}$, we get

$$J_i(\frac{x}{2^{n+1}}+\frac{\sigma(x)}{2^{n+1}})=J_i(\frac{x}{2^{n+2}}+\frac{\sigma(x)}{2^{n+2}}),$$

for all $n\in\mathbb{N}$. Hence we have for $n=1$

$$J_i(\frac{x}{2})-\frac{1}{2}J_i(\frac{x}{4}+\frac{\sigma(x)}{4})=J_i(\frac{x}{2})-\frac{1}{2}J_i(\frac{x}{2}+\frac{\sigma(x)}{2})=\frac{1}{2}J_i(x).$$

This proves (2.8) for $n=1$. The inductive step must now be demonstrated to hold true for the integer $n+1$, that is,

$$\begin{aligned}
& J_i(\frac{x}{2^{n+1}})+(\frac{1}{2^{n+1}}-1)J_i(\frac{x}{2^{n+2}}+\frac{\sigma(x)}{2^{n+2}}) \\
= & \frac{1}{2}[J_i(\frac{x}{2^n})+(\frac{1}{2^n}-1)J_i(\frac{x}{2^{n+1}}+\frac{\sigma(x)}{2^{n+1}})] \\
& +J_i(\frac{x}{2^{n+1}})-\frac{1}{2}J_i(\frac{x}{2^n})-\frac{1}{2}J_i(\frac{x}{2^{n+1}}+\frac{\sigma(x)}{2^{n+1}}) \\
= & \frac{1}{2}[\frac{1}{2^n}J_i(x)]+0 \\
= & \frac{1}{2^{n+1}}J_i(x).
\end{aligned}$$

Therefore, the relation (2.8) is true for any naturel number n. Now, we are able to prove the uniqueness of the mapping J. For all $x\in E_1$ and all $n\in\mathbb{N}$, we have

$$\begin{aligned}
& \|J_1(x)-J_2(x)\|_{E_2} \\
= & 2^n\|J_1(\frac{x}{2^n})+(\frac{1}{2^n}-1)J_1(\frac{x}{2^{n+1}}+\frac{\sigma(x)}{2^{n+1}}) \\
& -J_2(\frac{x}{2^n})-(\frac{1}{2^n}-1)J_2(\frac{x}{2^{n+1}}+\frac{\sigma(x)}{2^{n+1}})\|_{E_2} \\
\leq & 2^nK\|J_1(\frac{x}{2^n})-f(\frac{x}{2^n})\|_{E_2}+2^nK\|J_2(\frac{x}{2^n})-f(\frac{x}{2^n})\|_{E_2} \\
& +2^nK(1-\frac{1}{2^n})\|J_1(\frac{x}{2^{n+1}}+\frac{\sigma(x)}{2^{n+1}})-f(\frac{x}{2^{n+1}}+\frac{\sigma(x)}{2^{n+1}})\|_{E_2}
\end{aligned}$$

$$+2^nK(1-\frac{1}{2^n})\|J_2(\frac{x}{2^{n+1}}+\frac{\sigma(x)}{2^{n+1}})-f(\frac{x}{2^{n+1}}+\frac{\sigma(x)}{2^{n+1}})\|_{E_2}$$
$$\leq\ 2^{n(1-p)}\frac{4K^2\theta}{2^p-2}[(2^p+1)\|x\|^p+\frac{1}{2^p-1}\|x+\sigma(x)\|^p].$$

Finally, by letting $n \to +\infty$, we obtain $J_1(x) = J_2(x)$ for all $x \in E_1$. This completes the proof of Theorem 2.1. □

In the following theorem, we shall prove a result about Hyers-Ulam stability of equation (1.6) for the case $p < 1$.

Theorem 2.2. *Let E_1 be a normed space and E_2 a generalized quasi-Banach space. If a function $f\colon E_1 \longrightarrow E_2$ satisfies the inequality*

$$\|f(x+y)+f(x+\sigma(y))-2f(x)\|_{E_2}\leq\theta(\|x\|^p+\|y\|^p) \tag{2.9}$$

for some $\theta \geq 0$, $p < 1$ and for all $x, y \in E_1$. Then there exits a unique mapping $j\colon E_1 \longrightarrow E_2$, that is a solution of the Jensen functional equation (1.6) such that $j(e) = f(e)$ and

$$\|f(x)-j(x)-f(e)\|_{E_2}\leq K\theta[\|x\|^p+\frac{1}{2-2^p}\|x-\sigma(x)\|^p],\quad x\in E_1. \tag{2.10}$$

If $p < 0$, then inequality (2.9) holds for $x, y \neq 0$ and (2.10) for $x \neq 0$ and $x - \sigma(x) \neq 0$.

Proof. Let f: $E_1 \longrightarrow E_2$ satisfies the inequality (2.9). Then also $f - f(e)$ satisfies (2.9). Without loss of generality, we assume that $f(e) = 0$.

Letting $y = -x$ in (2.9) yields

$$\|f(x)-\frac{1}{2}f(x-\sigma(x))\|_{E_2}\leq\theta\|x\|^p. \tag{2.11}$$

Now, by replacing x and y by $2^{n-1}x - 2^{n-1}\sigma(x)$ in (2.9), we get

$$\|f(2^nx-2^n\sigma(x))-2f(2^{n-1}x-2^{n-1}\sigma(x))\|_{E_2}\leq 2\theta 2^{(n-1)p}\|x-\sigma(x))\|^p. \tag{2.12}$$

By applying the inductive argument, we obtain

$$\|f(x)-\frac{1}{2^n}f(2^{n-1}x-2^{n-1}\sigma(x))\|_{E_2} \tag{2.13}$$
$$\leq\ K\theta\|x\|^p+K\frac{\theta}{2}\|x-\sigma(x))\|^p[1+2^{p-1}+2^{2(p-1)}+\ldots+2^{(n-2)(p-1)}]$$

For n+1, we have

$$\begin{aligned}
&\|f(x)-\frac{1}{2^{n+1}}f(2^nx-2^n\sigma(x))\|_{E_2}\\
=&\ \|\{f(x)-\frac{1}{2^n}f(2^{n-1}x-2^{n-1}\sigma(x))\}\\
&+\frac{1}{2^{n+1}}\{2f(2^{n-1}x-2^{n-1}\sigma(x))-f(2^nx-2^n\sigma(x))\}\|_{E_2}\\
=&\ \|\{f(x)-\frac{1}{2^{n-1}}f(2^{n-2}x-2^{n-2}\sigma(x))\}\\
&+\frac{1}{2^n}\{2f(2^{n-2}x-2^{n-2}\sigma(x))-f(2^{n-1}x-2^{n-1}\sigma(x))\}\\
&+\frac{1}{2^{n+1}}\{2f(2^{n-1}x-2^{n-1}\sigma(x))-f(2^nx-2^n\sigma(x))\}\|_{E_2}\\
=&\ \|\{f(x)-\frac{1}{2}f(x-\sigma(x))\}+\frac{1}{2^2}\{2f(x-\sigma(x))-f(2x-2\sigma(x))\}\\
&+\cdots\\
&+\frac{1}{2^n}\{2f(2^{n-2}x-2^{n-2}\sigma(x))-f(2^{n-1}x-2^{n-1}\sigma(x))\}\\
&+\frac{1}{2^{n+1}}\{2f(2^{n-1}x-2^{n-1}\sigma(x))-f(2^nx-2^n\sigma(x))\}\|_{E_2}\\
\le&\ K[\theta\|x\|^p+\frac{2\theta}{2^2}\|x-\sigma(x)\|^p+\cdots+\frac{2^{(n-1)p}}{2^{n+1}}2\theta\|x-\sigma(x)\|^p]\\
=&\ K\theta\|x\|^p+\frac{\theta K}{2}\|x-\sigma(x))\|^p[1+2^{(p-1)}+\ldots+2^{(n-1)(p-1)}].
\end{aligned}$$

Which proves the validity of inequality (2.13).
Put for $n\in\mathbb{N}$, $(n\ge 1)$

$$j_n(x)=\frac{1}{2^n}f(2^{n-1}x-2^{n-1}\sigma(x)),\quad x\in E_1. \tag{2.14}$$

From (2.12) we have for $n\in\mathbb{N}$ and $x\in E_1$

$$\begin{aligned}
&\|j_{n+1}(x)-j_n(x)\|_{E_2}\\
=&\ \|\frac{1}{2^{n+1}}f(2^nx-2^n\sigma(x))-\frac{1}{2^n}f(2^{n-1}x-2^{n-1}\sigma(x))\|_{E_2}\\
=&\ \frac{1}{2^{n+1}}\|f(2^nx-2^n\sigma(x))-2f(2^{n-1}x-2^{n-1}\sigma(x))\|_{E_2}\\
\le&\ 2^{n(p-1)}\frac{\theta}{2^p}\|x-\sigma(x)\|^p.
\end{aligned}$$

Since $2^{p-1}<1$, hence $\{j_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence for every $x\in E_1$. However, E_2 is a generalized quasi-Banach space. Therefore, define $j(x)=$

$\lim_{n\to+\infty} j_n(x)$ for any $x \in E_1$. Now, we can verify that j is a solution of the Jensen functional equation (1.6). For all $x, y \in E_1$, we have

$$\begin{aligned}
& \|j_n(x+y) + j_n(x+\sigma(y)) - 2j_n(x)\|_{E_2} \\
= & \frac{1}{2^n}\|f(2^{n-1}x + 2^{n-1}y - 2^{n-1}\sigma(x) - 2^{n-1}\sigma(y)) \\
& + f(2^{n-1}x + 2^{n-1}\sigma(y) - 2^{n-1}\sigma(x) - 2^{n-1}y) \\
& - 2f(2^{n-1}x - 2^{n-1}\sigma(x))\|_{E_2} \\
= & \frac{1}{2^n}\|f(2^{n-1}(x-\sigma(x)) + 2^{n-1}(y-\sigma(y))) \\
& + f(2^{n-1}(x-\sigma(x)) + 2^{n-1}\sigma(y-\sigma(y))) \\
& - 2f(2^{n-1}(x-\sigma(x)))\|_{E_2} \\
\leq & \frac{\theta}{2^p}2^{n(p-1)}[\|x-\sigma(x)\|^p + \|y-\sigma(y)\|^p].
\end{aligned}$$

Since $2^{p-1} < 1$, it follows that j is a solution of Jensen functional equation (1.6). It remains to prove that there is only one solutions of Jensen functional equation (1.6) which satisfies (2.10). For the contrary, suppose that there are two such mappings, say j_1 and j_2 such that $j_1(e) = j_2(e) = 0$. First, we can verify by induction that

$$j_i(2^{n-1}x - 2^{n-1}\sigma(x)) = 2^n j_i(x). \tag{2.15}$$

For all $x \in E_1$ and all $n \in \mathbb{N}$, we have

$$\begin{aligned}
\|j_1(x) - j_2(x)\|_{E_2} &= \frac{1}{2^n}\|j_1(2^{n-1}x - 2^{n-1}\sigma(x)) - j_2(2^{n-1}x - 2^{n-1}\sigma(x))\|_{E_2} \\
&\leq \frac{K}{2^n}\|j_1(2^{n-1}x - 2^{n-1}\sigma(x)) - f(2^{n-1}x - 2^{n-1}\sigma(x))\|_{E_2} \\
&\quad + \frac{K}{2^n}\|j_2(2^{n-1}x - 2^{n-1}\sigma(x)) - f(2^{n-1}x - 2^{n-1}\sigma(x))\|_{E_2} \\
&\leq 2^{n(p-1)}\frac{4K^2\theta}{2^p(2-2^p)}\|x-\sigma(x)\|^p.
\end{aligned}$$

If we let $n \to +\infty$, we get $j_1(x) = j_2(x)$ for all $x \in E_1$. This completes the proof of Theorem 2.2. □

3. Hyers-Ulam stability of (1.7) with $p < 1$ and $p > 1$

In this section, we investigate the Hyers-Ulam stability for equation (1.7).

Theorem 3.1. *Let E_1 be a normed space and E_2 a generalized quasi-Banach space. If a function $f\colon E_1 \longrightarrow E_2$ satisfies the inequality*

$$\|f(x+y)-f(x+\sigma(y))-2f(y)\|_{E_2} \leq \theta(\|x\|^p+\|y\|^p) \tag{3.1}$$

for some $\theta \geq 0$, $p>1$ and for all $x,y \in E_1$. Then there exits a unique mapping $S: E_1 \longrightarrow E_2$, defined by

$$S(x)=\lim_{n\to+\infty} 2^n f(\frac{x}{2^n}) \tag{3.2}$$

that is a solution of the functional equation (1.7) such that

$$\|f(x)-S(x)\|_{E_2} \leq \frac{2\theta K^2}{2^p-2}\|x\|^p+\frac{K^2\theta}{2(2^p-2)}\|x+\sigma(x)\|^p, \quad x\in E_1. \tag{3.3}$$

Proof. Suppose that f satisfies inequality (3.1). Substituting $x=0$ and $y=\frac{x}{2}+\frac{\sigma(x)}{2}$ in (3.1) we obtain

$$\|f(\frac{x}{2}+\frac{\sigma(x)}{2})\|_{E_2} \leq \frac{\theta}{2^{p+1}}\|x+\sigma(x)\|^p. \tag{3.4}$$

By replacing x,y by $\frac{x}{2}$ in (3.1) and applying the triangle inequality (1.5) we get

$$\|f(x)-2f(\frac{x}{2})\|_{E_2} \leq \frac{2\theta K}{2^p}\|x\|^p+\frac{\theta K}{2^{p+1}}\|x+\sigma(x)\|^p. \tag{3.5}$$

Now, we will show by induction on n that

$$\begin{aligned}\|f(x)-2^n f(\frac{x}{2^n})\|_{E_2} \quad &\leq \quad \frac{2\theta K^2}{2^p}\|x\|^p[1+2^{1-p}+\ldots+2^{(n-1)(1-p)}] \\ &+\frac{\theta K^2}{2^{p+1}}\|x+\sigma(x)\|^p[1+2^{1-p}+\cdots+2^{(n-1)(1-p)}].\end{aligned} \tag{3.6}$$

From (3.5) it follows that (3.6) is true for $n=1$. Now, we will prove the validity of the inequality (3.6) for $n\in\mathbb{N}$.

$$
\begin{aligned}
& \|f(x) - 2^n f(\frac{x}{2^n})\|_{E_2} \\
= \ & \|[f(x) - 2^{n-1} f(\frac{x}{2^{n-1}})] + 2^{n-1}[f(\frac{x}{2^{n-1}}) - 2f(\frac{x}{2^n})]\|_{E_2} \\
= \ & \|[f(x) - 2^{n-2} f(\frac{x}{2^{n-2}})] + 2^{n-2}[f(\frac{x}{2^{n-2}}) - 2f(\frac{x}{2^{n-1}})] \\
& +2^{n-1}[f(\frac{x}{2^{n-1}}) - 2f(\frac{x}{2^n})]\|_{E_2} \\
= \ & \|f(x) - 2f(\frac{x}{2})] + 2[f(\frac{x}{2}) - 2f(\frac{x}{4})] \\
& + \cdots + 2^{n-2}[f(\frac{x}{2^{n-2}}) - 2f(\frac{x}{2^{n-1}})] + 2^{n-1}[f(\frac{x}{2^{n-1}}) - 2f(\frac{x}{2^n})]\|_{E_2} \\
\leq \ & K[\frac{2\theta K}{2^p}\|x\|^p + \frac{\theta K}{2^{p+1}}\|x+\sigma(x)\|^p + \frac{4\theta K}{2^{2p}}\|x\|^p \\
& + \frac{2\theta K}{2^{2p+1}}\|x+\sigma(x)\|^p + \cdots + 2^{n-2}\frac{2\theta K}{2^p}\frac{1}{2^{(n-2)p}}\|x\|^p \\
& +2^{n-2}\frac{\theta K}{2^{p+1}}\frac{1}{2^{(n-2)p}}\|x+\sigma(x)\|^p + 2^{n-1}\frac{2\theta K}{2^p}\frac{1}{2^{(n-1)p}}\|x\|^p \\
& +2^{n-1}\frac{\theta K}{2^{p+1}}\frac{1}{2^{(n-1)p}}\|x+\sigma(x)\|^p] \\
= \ & \frac{2\theta K^2}{2^p}\|x\|^p[1 + 2^{1-p} + \cdots + 2^{(n-1)(1-p)}] \\
& + \frac{\theta K^2}{2^{p+1}}\|x+\sigma(x)\|^p[1 + 2^{1-p} + \cdots + 2^{(n-1)(1-p)}].
\end{aligned}
$$

This proves (3.6) for all n. Let us now define the sequence functions

$$
S_n(x) = 2^n f(\frac{x}{2^n}), \quad x \in E_1, \quad n \in \mathbb{N}. \tag{3.7}
$$

We shall verify that $\{S_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence for every $x \in E_1$. Indeed, in view of (3.5), we get for all natural number n that

$$
\begin{aligned}
\|S_{n+1}(x) - S_n(x)\|_{E_2} &= \|2^{n+1}f(\frac{x}{2^{n+1}}) - 2^n f(\frac{x}{2^n})\|_{E_2} \\
&\leq 2^{n(1-p)}K[\frac{2\theta}{2^p}\|x\|^p + \frac{\theta}{2^{p+1}}\|x+\sigma(x)\|^p].
\end{aligned}
$$

Since $2^{1-p} < 1$, we have proved our statement. However, E_2 is a generalized quasi-Banach space, thus we can define $S(x) = \lim_{n\to+\infty} S_n(x)$ for $x \in E_1$. The function S satisfies (1.7). Indeed, by using (3.7) and (3.1), we get

$$\begin{aligned}
&\|S_n(x+y) - S_n(x+\sigma(y)) - 2S_n(y)\|_{E_2}\\
= \;& 2^n\|f(\frac{x}{2^n}+\frac{y}{2^n}) - f(\frac{x}{2^n}+\frac{\sigma(y)}{2^n}) - 2f(\frac{y}{2^n})\|_{E_2}\\
\leq \;& 2^{n(1-p)}\theta[\|x\|^p + \|y\|^p].
\end{aligned}$$

Hence, from $2^{1-p} < 1$, we get that S is a solution of equation (1.7).

Assume now that there exist two functions $S_i : E_1 \longrightarrow E_2$ $(i = 1, 2)$ that are solutions of (1.7) which satisfies the inequality (3.3). First, we will prove by mathematical induction on n that

$$S_i(\frac{x}{2^n}) = \frac{1}{2^n}S_i(x). \tag{3.8}$$

By letting $x = y = \frac{x}{2^n} + \frac{\sigma(x)}{2^n}$ in (1.7), we get $S_i(\frac{x}{2^n} + \frac{\sigma(x)}{2^n}) = 0$ for all $n \in \mathbb{N}$, so for $n = 1$, we have

$$\begin{aligned}
S_i(\frac{x}{2}) &= \frac{1}{2}[2S_i(\frac{x}{2}) + S_i(\frac{x}{2}+\frac{\sigma(x)}{2}) - S_i(x)] + \frac{1}{2}S_i(x)\\
&= 0 + \frac{1}{2}S_i(x)\\
&= \frac{1}{2}S_i(x).
\end{aligned}$$

This proves (3.8) for n = 1. The inductive step must now be demonstrated to hold true for the integer n + 1, that is,

$$\begin{aligned}
S_i(\frac{x}{2^{n+1}}) &= \frac{1}{2}[2S_i(\frac{x}{2^{n+1}}) + S_i(\frac{x}{2^{n+1}}+\frac{\sigma(x)}{2^{n+1}}) - S_i(\frac{x}{2^n})] + \frac{1}{2}S_i(\frac{x}{2^n})\\
&= 0 + \frac{1}{2^{n+1}}S_i(x)\\
&= \frac{1}{2^{n+1}}S_i(x).
\end{aligned}$$

Which proves (3.8) for $n+1$. Now, we will prove the uniqueness of the mapping S. For all $x \in E_1$ and all $n \in \mathbb{N}$, we have

$$\begin{aligned}
\|S_1(x) - S_2(x)\|_{E_2} &= 2^n\|S_1(\frac{x}{2^n}) - S_2(\frac{x}{2^n})\|_{E_2}\\
&\leq 2^nK\|S_1(\frac{x}{2^n}) - f(\frac{x}{2^n})\|_{E_2} + 2^nK\|S_2(\frac{x}{2^n}) - f(\frac{x}{2^n})\|_{E_2}\\
&\leq 2^{n(1-p)}\frac{K^3\theta}{2^p-2}[4\|x\|^p + \|x+\sigma(x)\|^p].
\end{aligned}$$

Finally, by letting $n \to +\infty$, we obtain $S_1(x) = S_2(x)$ for all $x \in E_1$. This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let E_1 be a normed space and E_2 a generalized quasi-Banach space. If a function $f\colon E_1 \longrightarrow E_2$ satisfies the inequality*

$$\|f(x+y) - f(x+\sigma(y)) - 2f(y)\|_{E_2} \leq \theta(\|x\|^p + \|y\|^p) \tag{3.9}$$

for some $\theta \geq 0$, $p < 1$ and for all $x, y \in E_1$. Then there exits a unique mapping $h\colon E_1 \longrightarrow E_2$, that is a solution of equation (1.7) such that

$$\|f(x) - h(x)\|_{E_2} \leq \frac{2K^2\theta}{2-2^p}[\|x\|^p + \frac{1}{2}\|x+\sigma(x)\|^p], \quad x \in E_1. \tag{3.10}$$

Proof. If we replace x and y by $x + \sigma(x)$, we get

$$\|f(x+\sigma(x))\|_{E_2} \leq \theta\|x+\sigma(x)\|^p. \tag{3.11}$$

Substituting $x = y = x$ in (3.9) and using the triangle inequality (1.5) we get

$$\|f(x) - \frac{1}{2}f(2x)\|_{E_2} \leq K\theta[\|x\|^p + \frac{1}{2}\|x+\sigma(x)\|^p]. \tag{3.12}$$

Now, we shall verify that for every $n \in \mathbb{N}$, we have

$$\begin{aligned} \|f(x) - \frac{1}{2^n}f(2^n x)\|_{E_2} \leq \; & K^2\theta\|x\|^p[1 + 2^{p-1} + \ldots + 2^{(n-1)(p-1)}] \\ & + \frac{K^2\theta}{2}\|x+\sigma(x))\|^p[1 + 2^{p-1} + \ldots + 2^{(n-1)(p-1)}]. \end{aligned} \tag{3.13}$$

The inequality (3.12) means that (3.13) is satisfied for $n = 1$. Now we have for $n + 1$,

$$\begin{aligned}
&\|f(x)-\frac{1}{2^{n+1}}f(2^{n+1}x)\|_{E_2}\\
=\;& \|[f(x)-\frac{1}{2^n}f(2^nx)]+\frac{1}{2^n}[f(2^nx)-\frac{1}{2}f(2^{n+1}x)]\|_{E_2}\\
=\;& \|[f(x)-\frac{1}{2^{n-1}}f(2^{n-1}x)]+\frac{1}{2^{n-1}}[f(2^{n-1}x)-\frac{1}{2}f(2^nx)]\\
&+\frac{1}{2^n}[f(2^nx)-\frac{1}{2}f(2^{n+1}x)]\|_{E_2}\\
=\;& \|[f(x)-\frac{1}{2}f(2x)]+\frac{1}{2}[f(2x)-\frac{1}{2}f(4x))]\\
&+\ldots+\frac{1}{2^{n-1}}[f(2^{n-1}x)-\frac{1}{2}f(2^nx)]+\frac{1}{2^n}[f(2^nx)-\frac{1}{2}f(2^{n+1}x)]\|_{E_2}\\
\le\;& K\|f(x)-\frac{1}{2}f(2x)\|_{E_2}+\frac{K}{2}\|f(2x)-\frac{1}{2}f(4x))\|_{E_2}\\
&+\cdots+\frac{K}{2^{n-1}}\|f(2^{n-1}x)-\frac{1}{2}f(2^nx)\|_{E_2}+\frac{K}{2^n}\|f(2^nx)-\frac{1}{2}f(2^{n+1}x)\|_{E_2}\\
\le\;& K[\theta K\|x\|^p+\frac{\theta K}{2}\|x+\sigma(x)\|^p+\frac{2^p\theta K}{2}\|x\|^p+\frac{2^p\theta K}{4}\|x+\sigma(x)\|^p\\
&+\cdots+\frac{2^{(n-1)p}}{2^{n-1}}\theta K\|x\|^p+\frac{2^{(n-1)p}}{2^{n-1}}\frac{\theta K}{2}\|x+\sigma(x)\|^p\\
&+\frac{2^{np}}{2^n}\theta K\|x\|^p+\frac{2^{np}}{2^n}\frac{\theta K}{2}\|x+\sigma(x)\|^p]\\
=\;& K^2\theta\|x\|^p[1+2^{p-1}+\ldots+2^{n(p-1)}]\\
&+\frac{K^2\theta}{2}\|x+\sigma(x)\|^p[1+2^{p-1}+\ldots+2^{n(p-1)}].
\end{aligned}$$

Which proves the inequality (3.13). Now, put for $n\in\mathbb{N}$

$$h_n(x)=\frac{1}{2^n}f(2^nx),\quad x\in E_1. \tag{3.14}$$

From (3.12), we have for $n\in\mathbb{N}$ and $x\in E_1$

$$\begin{aligned}
\|h_{n+1}(x)-h_n(x)\|_{E_2} &= \|\frac{1}{2^{n+1}}f(2^{n+1}x)-\frac{1}{2^n}f(2^nx)\|_{E_2}\\
&= \frac{1}{2^n}\|f(2^nx)-\frac{1}{2}f(2^{n+1}x)\|_{E_2}\\
&\le 2^{n(p-1)}\theta K[\|x\|^p+\frac{1}{2}\|x+\sigma(x)\|^p.
\end{aligned}$$

Since $2^{p-1} < 1$, hence $\{h_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence for every $x \in E_1$. However, E_2 is a complete space, define, therefore

$$h(x) = \lim_{n\to+\infty} h_n(x), \quad x \in E_1.$$

Now, we will show that h is a solution of equation (1.7). Let x, y be two elements of E_1. From (3.9) it follows that

$$\begin{aligned} & \|h_n(x+y) - h_n(x+\sigma(y)) - 2h_n(y)\|_{E_2} \\ = \; & \frac{1}{2^n}\|f(2^n x + 2^n x) - f(2^n x + 2^n\sigma(y)) - 2f(2^n y)\|_{E_2} \\ \leq \; & 2^{n(p-1)}\theta[\|x\|^p + \|y\|^p]. \end{aligned}$$

By letting $n \to +\infty$, we get the desired result that

$$h(x+y) - h(x+\sigma(y)) = 2h(y),$$

for all $x, y \in E_1$.

Assume now that there exist two functions $h_i : E_1 \to E_2$ $(i = 1, 2)$ that are solutions of equation (1.7) with (3.10). First, we will prove by mathematical induction

$$2^n h_i(x) = h_i(2^n x). \tag{3.15}$$

By letting $x = y = 2^{n-1}x$ in (1.7), we get

$$h_i(2^n x) - h_i(2^{n-1}x + 2^{n-1}\sigma(x)) = 2h_i(2^{n-1}x), \tag{3.16}$$

because $h_i(2^{n-1}x + 2^{n-1}\sigma(x)) = 0$. For $n = 1$, we have

$$h_i(2x) - h_i(x + \sigma(x)) = h_i(2x) = 2h_i(x).$$

By using (3.16) the inductive step hold true for the integer $n + 1$. Therefore, the equality (3.15) is true for any naturel number n. Now, we able to prove the uniqueness of the mapping h. For all $x \in E_1$ and all $n \in \mathbb{N}$, we have

$$\begin{aligned} \|h_1(x) - h_2(x)\|_{E_2} &= \frac{1}{2^n}\|h_1(2^n x) - h_2(2^n x)\|_{E_2} \\ &\leq K\frac{1}{2^n}[\|h_1(2^n x) - f(2^n x)\|_{E_2} + \|h_2(2^n x) - f(2^n x)\|_{E_2}] \\ &\leq \frac{4K^3\theta}{2 - 2^p}2^{n(p-1)}[\|x\|^p + \frac{1}{2}\|x + \sigma(x)\|^p]. \end{aligned}$$

If we let $n \to +\infty$, we get $h_1(x) = h_2(x)$ for all $x \in E_1$. This completes the proof. □

4. Hyers-Ulam stability of equation (1.6) and (1.7) in β-Banach spaces

Theorem 4.1. *Let E_1 be a normed space, E_2 a β-Banach space and $f\colon E_1 \longrightarrow E_2$ a mapping which satisfies the inequality*

$$\|f(x+y)+f(x+\sigma(y))-2f(x)\|_{E_2} \le \theta(\|x\|^p+\|y\|^p) \tag{4.1}$$

for some $\theta \ge 0$, $p>1$ and for all $x,y \in E_1$. Then there exits a unique mapping $J : E_1 \longrightarrow E_2$, defined by

$$J(x) = \lim_{n\to+\infty} 2^n f(\frac{x}{2^n}) \tag{4.2}$$

that is a solution of the Jensen functional equation (1.6) such that

$$\|f(x)-J(x)\|_{E_2} \le \frac{2K2^p\theta}{(2^{\beta p}-2^{\beta})^{\frac{1}{\beta}}}\|x\|^p + \frac{\theta K 2^p}{2(2^{\beta p}-2^{\beta})^{\frac{1}{\beta}}}\|x+\sigma(x)\|^p \tag{4.3}$$

Proof. Replacing $x=0$ and $y=\frac{x}{2}+\frac{\sigma(x)}{2}$ in (4.1), we find

$$\|f(\frac{x}{2}+\frac{\sigma(x)}{2})\|_{E_2} \le \frac{\theta}{2^{p+1}}\|x+\sigma(x)\|^p, \tag{4.4}$$

By replacing x,y by $\frac{x}{2}$ in (4.1) and using the triangle inequality (1.3) we get

$$\|f(x)-2f(\frac{x}{2})\|_{E_2} \le \frac{2K\theta}{2^p}\|x\|^p + \frac{K\theta}{2^{p+1}}\|x+\sigma(x)\|^p, \tag{4.5}$$

for all $x \in E_1$. Replacing x by $\frac{x}{2^n}$ in (4.5) and multiply both sides of (4.5) to 2^n, we get

$$\|2^n f(\frac{x}{2^n}) - 2^{n+1} f(\frac{x}{2^{n+1}})\|_{E_2} \le 2^{n(1-p)}K\theta[2\|x\|^p + \frac{\|x+\sigma(x)\|^p}{2}] \tag{4.6}$$

for all $x \in E_1$ and all nonnegative integers n. Using (4.6) and the inequality (1.4), we have

$$\|2^m f(\frac{x}{2^m}) - 2^n f(\frac{x}{2^n})\|_{E_2}^{\beta} \le \sum_{k=m}^{n-1} 2^{k(1-p)\beta}K^{\beta}\theta^{\beta}[2\|x\|^p + \frac{\|x+\sigma(x)\|^p}{2}]^{\beta} \tag{4.7}$$

for all $x \in E_1$ and all nonnegative integers n and m with $m<n$. This show that $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in E_1$. Consequently, we can define $J : E_1 \longrightarrow E_2$ by

$$J(x) = \lim_{n\longrightarrow+\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in E_1$. Putting $m=0$ in (4.7) and taking the limit as $n \longrightarrow +\infty$, we obtain (4.3). Let us now show that J is a solution of Jensen functional equation (1.6). Indeed,

$$\begin{aligned} & \|J_n(x+y)+J_n(x+\sigma(y))-2J_n(x)\|_{E_2} \\ = \;& 2^n\|f(\frac{x}{2^n}+\frac{y}{2^n})+f(\frac{x}{2^n}+\frac{\sigma(y)}{2^n})-2f(\frac{x}{2^n})\| \\ \leq \;& 2^{n(1-p)}\theta[\|x\|^p+\|y\|^p]. \end{aligned}$$

Here $2^{1-p} < 1$, then by letting $n \to +\infty$, we get that J is a solution of equation (1.6). The uniqueness of the mapping J can be proved by using some computations similar to the ones of the proof of Theorem 2.1. This ends the proof of Theorem 4.1. □

Theorem 4.2. *Let E_1 be a normed space, E_2 a β-Banach space and $f\colon E_1 \longrightarrow E_2$ a mapping which satisfies the inequality*

$$\|f(x+y)+f(x+\sigma(y))-2f(x)\|_{E_2} \leq \theta(\|x\|^p+\|y\|^p) \tag{4.8}$$

for some $\theta \geq 0$, $p \in]0,1[$ and for all $x, y \in E_1$. Then there exits a unique mapping $j : E_1 \longrightarrow E_2$, that is a solution of the Jensen functional equation (1.6) such that

$$\|f(x)-j(x)\|_{E_2} \leq \frac{2K\theta}{(2^\beta-2^{\beta p})^{\frac{1}{\beta}}}\|x\|^p + \frac{K\theta}{2(2^\beta-2^{\beta p})^{\frac{1}{\beta}}}\|x+\sigma(x)\|^p, \quad x \in E_1. \tag{4.9}$$

Proof. Letting $x = 0$ and $y = x + \sigma(x)$ in (4.8), we get

$$\|f(x+\sigma(x))\|_{E_2} \leq \frac{\theta}{2}\|x+\sigma(x)\|^p. \tag{4.10}$$

Replacing x and y by x in (4.8) and using the triangle inequality (1.3), we get

$$\|f(x)-\frac{1}{2}f(2x)\|_{E_2} \leq K\theta\|x\|^p + \frac{\theta K}{4}\|x+\sigma(x)\|^p, \tag{4.11}$$

for all $x \in E_1$. If we replace x in (4.11) by $2^n x$ and multiply both sides of (4.11) to $\frac{1}{2^n}$, then we have

$$\|\frac{f(2^n x)}{2^n} - \frac{f(2^{n+1}x)}{2^{n+1}}\|_{E_2} \leq 2^{(p-1)n}[K\theta\|x\|^p + \frac{\theta K}{4}\|x+\sigma(x)\|^p], \tag{4.12}$$

for all $x \in E_1$ and all nonnegative integers n. Since E_2 is a β-Banach space, we have

$$\|\frac{f(2^n x)}{2^n} - \frac{f(2^m x)}{2^m}\|_{E_2}^\beta \leq \Sigma_{k=m}^{n-1} 2^{k\beta(p-1)}[K\theta\|x\|^p + \frac{\theta K}{4}\|x+\sigma(x)\|^p]^\beta, \tag{4.13}$$

for all $x \in E_1$ and all nonnegative integers n and m with $m < n$. Therefore we conclude that the sequence $\{\frac{f(2^n x)}{2^n}\}$ is a Cauchy sequence in E_2 for all $x \in E_1$.

Since E_2 is complete, so we can define j: $E_1 \longrightarrow E_2$ by

$$j(x) = \lim_{n \longrightarrow +\infty} \frac{f(2^n x)}{2^n}$$

for all $x \in E_1$. Letting $m = 0$ and passing the limit $n \longrightarrow +\infty$ in (4.13), we get (4.9). The rest of the proof can be derived by using some computations of the proof of Theorem 2.1 and Theorem 4.1. □

Theorem 4.3. *Let E_1 be a normed space, E_2 a β-Banach space and f: $E_1 \longrightarrow E_2$ a mapping which satisfies the inequality*

$$\|f(x+y) + f(x+\sigma(y)) - 2f(x)\|_{E_2} \leq \theta(\|x\|^p + \|y\|^p) \tag{4.14}$$

for some $\theta \geq 0$, $p \leq 0$ and for all $x, y \in E_1$. Then there exits a unique mapping $h : E_1 \longrightarrow E_2$, defined by

$$j(x) = \lim_{n \to +\infty} \frac{1}{2^n} f(2^{n-1}x - 2^{n-1}\sigma(x)) \tag{4.15}$$

that is a solution of the Jensen functional equation (1.6) such that

$$\|f(x) - j(x)\|_{E_2} \leq \theta[\|x\|^{\beta p} + \frac{\|x+\sigma(x)\|^{\beta p}}{2^\beta - 2^{\beta p}}]^{\frac{1}{\beta}}, \quad x \in E_1. \tag{4.16}$$

Proof. Let f: $E_1 \longrightarrow E_2$ satisfies the inequality (4.14), then also $f - f(e)$ satisfies (4.14). Without loss of generality we assume that $f(e) = 0$. By letting $y = -x$ in (4.14), we obtain

$$\|f(x) - \frac{1}{2} f(x - \sigma(x))\|_{E_2} \leq \theta \|x\|^p. \tag{4.17}$$

Now, if we replace x and y in (4.14) by $2^{n-1}x - 2^{n-1}\sigma(x)$, we get

$$\|f(2^n x - 2^n \sigma(x)) - 2f(2^{n-1}x - 2^{n-1}\sigma(x))\|_{E_2} \leq 2\theta 2^{(n-1)p} \|x - \sigma(x))\|^p. \tag{4.18}$$

By applying the inductive argument, we obtain

$$\begin{aligned}
& \|f(x) - \frac{1}{2^n} f(2^{n-1}x - 2^{n-1}\sigma(x))\|_{E_2}^\beta \qquad (4.19)\\
= \quad & \|\{f(x) - \frac{1}{2} f(x - \sigma(x))\} + \frac{1}{2^2}\{2f(x - \sigma(x)) - f(2x - 2\sigma(x))\}\\
& + \cdots + \frac{1}{2^n}\{2f(2^{n-2}x - 2^{n-2}\sigma(x)) - f(2^{n-1}x - 2^{n-1}\sigma(x))\}\|_{E_2}^\beta\\
\leq \quad & \|f(x) - \frac{1}{2} f(x - \sigma(x))\|_{E_2}^\beta + \frac{1}{2^{2\beta}}\|2f(x - \sigma(x)) - f(2x - 2\sigma(x))\|_{E_2}^\beta\\
& + \cdots + \frac{1}{2^{n\beta}}\|\{2f(2^{n-2}x - 2^{n-2}\sigma(x)) - f(2^{n-1}x - 2^{n-1}\sigma(x))\|_{E_2}^\beta\\
\leq \quad & \theta^\beta \|x\|^{p\beta} + \frac{\theta^\beta}{2^\beta}\|x - \sigma(x))\|^{p\beta}[1 + 2^{\beta(p-1)} + \cdots + 2^{(n-2)\beta(p-1)}],
\end{aligned}$$

for all $x \in E_1$. Put for $n \in \mathbb{N}$, $(n \geq 1)$

$$j_n(x) = \frac{1}{2^n} f(2^{n-1}x - 2^{n-1}\sigma(x)), \quad x \in E_1. \tag{4.20}$$

Using (4.18) we get

$$\begin{aligned} \|j_{n+1}(x) - j_n(x)\|_{E_2}^{\beta} &= \|\tfrac{1}{2^{n+1}} f(2^n x - 2^n \sigma(x)) - \tfrac{1}{2^n} f(2^{n-1}x - 2^{n-1}\sigma(x))\|_{E_2}^{\beta} \\ &\leq 2^{\beta n(p-1)} \tfrac{\theta}{2^p} \|x - \sigma(x)\|^{\beta p}. \end{aligned}$$

From $2^{\beta(p-1)} < 1$, we deduce that $\{j_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence for every $x \in E_1$. Since, E_2 is a β-Banach space, we can define the following limit

$$j(x) = \lim_{n\to+\infty} j_n(x)$$

for any $x \in E_1$ and we can easily verify that j is a solution of Jensen functional equation (1.6). The rest of the proof is similar to the proof of Theorem 2.2.

The following results follows by using some ideas of the proof of Theorem 3.1 and Theorem 3.2. □

Theorem 4.4. *Let E_1 be a normed space and E_2 a β-Banach space. If a function $f\colon E_1 \longrightarrow E_2$ satisfies the inequality*

$$\|f(x+y) - f(x+\sigma(y)) - 2f(y)\|_{E_2} \leq \theta(\|x\|^p + \|y\|^p) \tag{4.21}$$

for some $\theta \geq 0$, $p > 1$ and for all $x, y \in E_1$. Then there exits a unique mapping $S : E_1 \longrightarrow E_2$, defined by

$$S(x) = \lim_{n\to+\infty} 2^n f(\frac{x}{2^n}) \tag{4.22}$$

that is a solution of the functional equation (1.7) such that

$$\|f(x) - S(x)\|_{E_2} \leq \frac{2K\theta}{(2^{\beta p} - 2^{\beta})^{\frac{1}{\beta}}} [\|x\|^p + \frac{\|x + \sigma(x)\|^p}{4}], \quad x \in E_1. \tag{4.23}$$

Theorem 4.5. *Let E_1 be a normed space and E_2 a β-Banach space. If a function $f\colon E_1 \longrightarrow E_2$ satisfies the inequality*

$$\|f(x+y) - f(x+\sigma(y)) - 2f(y)\|_{E_2} \leq \theta(\|x\|^p + \|y\|^p) \tag{4.24}$$

for some $\theta \geq 0$, $p < 1$ and for all $x, y \in E_1$. Then there exits a unique mapping $g : E_1 \longrightarrow E_2$, defined by

$$g(x) = \lim_{n\to+\infty} \frac{1}{2^n} f(2^n x) \tag{4.25}$$

that is a solution of Jensen functional equation (1.7) such that

$$\|f(x)-g(x)\|_{E_2} \leq \frac{2K\theta}{(2^{\beta}-2^{\beta p})^{\frac{1}{\beta}}}[\|x\|^p + \frac{1}{2}\|x+\sigma(x)\|^p], \quad x \in E_1. \tag{4.26}$$

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