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ON ASYMPTOTICALLY PSEUDOCONTRACTIVE MAPPINGS

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Abstract. Let K be a nonempty closed convex subset of a real Banach space E, $T : K \to K$ a uniformly continuous asymptotically pseudocontractive mapping having $T(K)$ bounded with sequence $\{k_n\}_{n\geq 0} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ such that $p \in F(T) = \{x \in K : Tx = x\}$. Let $\{\alpha_n\}_{n\geq 0} \in [0, 1]$ be such that $\sum_{n\geq 0} \alpha_n^2 = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$. For arbitrary $x_0 \in K$ let $\{x_n\}_{n\geq 0}$ be iteratively defined by

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T^n x_n, \ n \geq 0.$$

Then $\{x_n\}_{n\geq 0}$ converges strongly to $p \in F(T)$.

1. Introduction

Let E be a real Banach space and K be a nonempty convex subset of E. Let J denote the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = ||x||^2 \ \text{ and } \ ||f^*|| = ||x||\},$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. We shall denote the single-valued duality map by j.

Let $T : D(T) \subset E \to E$ be a mapping with domain $D(T)$ in E.

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Definition 1.1. *The mapping T is said to be uniformly L-Lipschitzian if there exists $L>0$ such that for all $x, y \in D(T)$*

$$\|T^n x - T^n y\| \leq L \|x - y\|.$$

Definition 1.2. *T is said to be nonexpansive if for all x, $y \in D(T)$, the following inequality holds:*

$$\|Tx - Ty\| \leq \|x - y\|.$$

Definition 1.3. *T is said to be asymptotically nonexpansive [2], if there exists a sequence $\{k_n\}_{n\geq 0} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that*

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all $x, y \in D(T), n \geq 1$.

Definition 1.4. *T is said to be asymptotically pseudocontractive if there exists a sequence $\{k_n\}_{n\geq 0} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ and there exists $j(x-y) \in J(x-y)$ such that*

$$\langle T^n x - T^n y, j(x-y)\rangle \leq k_n \|x - y\|^2$$

for all $x, y \in D(T), n \geq 1$.

Remark 1.5. 1. It is easy to see that every asymptotically nonexpansive mapping is uniformly L-Lipschitzian.

2. If T is asymptotically nonexpansive mapping then for all x, $y \in D(T)$ there exists $j(x-y) \in J(x-y)$ such that

$$\begin{aligned} \langle T^n x - T^n y, j(x-y)\rangle &\leq \|T^n x - T^n y\| \|x - y\| \\ &\leq k_n \|x - y\|^2, \ n \geq 1. \end{aligned}$$

Hence every asymptotically nonexpansive mapping is asymptotically pseudocontractive.

3. Rhoades in [6] showed that the class of asymptotically pseudocontractive mappings properly contains the class of asymptotically nonexpansive mappings.

The asymptotically pseudocontractive mappings were introduced by Schu [7] who proved the following theorem:

Theorem 1.6. *Let K be a nonempty bounded closed convex subset of a Hilbert space H, $T: K \to K$ a completely continuous, uniformly L-Lipschitzian and asymptotically pseudocontractive with sequence $\{k_n\} \subset [1,\infty)$; $q_n = 2k_n - 1$,*

$\forall n \in N$; $\sum(q_n^2 - 1) < \infty$; $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$; $\epsilon < \alpha_n < \beta_n \leq b$, $\forall n \in N$, $\epsilon > 0$ *and* $b \in (0, L^{-2}[(1+L^2)^{\frac{1}{2}} - 1])$; $x_1 \in K$ *for all* $n \in N$, *define*

$$x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n T^n x_n.$$

Then $\{x_n\}$ *converges to some fixed point of* T.

The recursion formula of Theorem 1.6 is a modification of the well-known Mann iteration process (see [4]).

Also among the most recent results about the same topic, following are due to Ofoedu [5].

Theorem 1.7. *[5] Let* K *be a nonempty closed convex subset of a real Banach space* E, $T : K \to K$ *a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with sequence* $\{k_n\}_{n\geq 0} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ *such that* $x^* \in F(T) = \{x \in K : Tx = x\}$. *Let* $\{\alpha_n\}_{n\geq 0} \subset [0,1]$ *be such that* $\sum_{n\geq 0} \alpha_n = \infty$, $\sum_{n\geq 0} \alpha_n^2 < \infty$ *and* $\sum_{n\geq 0} \alpha_n(k_n - 1) < \infty$. *For arbitrary* $x_0 \in K$ *let* $\{x_n\}_{n\geq 0}$ *be iteratively defined by*

$$x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n T^n x_n, \ n \geq 0.$$

Suppose there exists a strictly increasing function $\psi : [0,\infty) \to [0,\infty)$, $\psi(0) = 0$ *such that*

$$\langle T^n x - x^*, j(x - x^*)\rangle \leq k_n ||x - x^*||^2 - \psi(||x - x^*||), \ \forall x \in K. \qquad \text{(O)}$$

Then $\{x_n\}_{n\geq 0}$ *is bounded.*

Theorem 1.8. *[5] Let* K *be a nonempty closed convex subset of a real Banach space* E, $T : K \to K$ *a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with sequence* $\{k_n\}_{n\geq 0} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ *such that* $x^* \in F(T) = \{x \in K : Tx = x\}$. *Let* $\{\alpha_n\}_{n\geq 0} \subset [0,1]$ *be such that* $\sum_{n\geq 0} \alpha_n = \infty$, $\sum_{n\geq 0} \alpha_n^2 < \infty$ *and* $\sum_{n\geq 0} \alpha_n(k_n - 1) < \infty$. *For arbitrary* $x_0 \in K$ *let* $\{x_n\}_{n\geq 0}$ *be iteratively defined by*

$$x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n T^n x_n, \ n \geq 0.$$

Suppose there exists a strictly increasing function $\psi : [0,\infty) \to [0,\infty)$, $\psi(0) = 0$ *such that*

$$\langle T^n x - x^*, j(x - x^*)\rangle \leq k_n ||x - x^*||^2 - \psi(||x - x^*||), \ \forall x \in K.$$

Then $\{x_n\}_{n\geq 0}$ *converges strongly to* $x^* \in F(T)$.

Remark 1.9. One can easily see that if we take in Theorems 1.7 and 1.8, $\alpha_n = \frac{1}{n^\sigma}$; $0 < \sigma < \frac{1}{2}$, then $\sum \alpha_n = \infty$, but also $\sum \alpha_n^2 = \infty$. Hence the conclusions of Theorems 1.7 1nd 1.8 can be improved. The same argument can be applied on the results of Chidume and Chidume in [1].

In this paper, we establish the strong convergence for a modified Mann iterative scheme associated with asymptotically pseudocontractive mappings in real Banach spaces. Moreover, our technique of proofs is of independent interest. We also generalize the results of Schu [7] from Hilbert spaces to more general Banach spaces and improve the results of Ofoedu [5].

2. Main Results

We will need the following results.

Lemma 2.1. *[9] Let $J : E \to 2^E$ be the normalized duality mapping. Then for any $x, y \in E$, we have*

$$||x+y||^2 \leq ||x||^2 + 2\langle y, j(x+y)\rangle, \quad \forall j(x+y) \in J(x+y).$$

Lemma 2.2. *[8] If there exists a positive integer N such that for all $n \geq N, n \in \mathbb{N}$,*

$$\rho_{n+1} \leq (1-\theta_n)\rho_n + b_n,$$

then

$$\lim_{n\to\infty} \rho_n = 0,$$

where $\theta_n \in [0,1)$, $\sum_{n=0}^{\infty} \theta_n = \infty$, and $b_n = o(\theta_n)$.

Lemma 2.3. *If there exists a positive integer N such that for all $n \geq N$, $n \in \mathbb{N}$,*

$$\rho_{n+1} \leq (1-\delta_n^l)\rho_n + b_n; \; l \geq 1,$$

then

$$\lim_{n\to\infty} \rho_n = 0,$$

where $\delta_n \in [0,1)$, $\sum_{n=0}^{\infty} \delta_n^l = \infty$, and $b_n = o(\delta_n)$.

Proof. Since $b_n = o(\delta_n)$, let $b_n = \varepsilon_n \delta_n$, and $\varepsilon_n \to 0$. By a straightforward induction, one obtains

$$0 \leq \rho_{n+1} \leq \prod_{j=k}^{n}(1-\delta_n^l)\rho_k + \sum_{j=k}^{n}\left[\delta_j \prod_{i=j+1}^{n}(1-\delta_n^l)\right]\varepsilon_j. \tag{W}$$

We have

$$\prod_{j=k}^{n}(1-\delta_n^l) \leq e^{-\sum_{j=k}^{n}\delta_n^l} \to 0,$$

and

$$\sum_{j=k}^{n}\delta_j \prod_{i=j+1}^{n}(1-\delta_n^l) \leq 1, \text{ for all } n, \; k.$$

Given $\varepsilon > 0$, pick k such that $\varepsilon_j \leq \varepsilon$ for all $j \geq k$, from (W) we have

$$0 \leq \lim_{n\to\infty} \inf \rho_n \leq \lim_{n\to\infty} \sup \rho_n \leq \varepsilon.$$

Letting $\varepsilon \to 0$, we obtain $\lim_{n\to\infty} \rho_n = 0$. This completes the proof. □

Theorem 2.4. *Let K be a nonempty closed convex subset of a real Banach space E, $T : K \to K$ a uniformly continuous asymptotically pseudocontractive mapping having $T(K)$ bounded with sequence $\{k_n\}_{n\geq 0} \subset [1,\infty)$, $\lim_{n\to\infty} k_n = 1$ such that $p \in F(T) = \{x \in K : Tx = x\}$. Let $\{\alpha_n\}_{n\geq 0} \in [0,1]$ be such that $\sum_{n\geq 0} \alpha_n^2 = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$. For arbitrary $x_0 \in K$ let $\{x_n\}_{n\geq 0}$ be iteratively defined by*

$$x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n T^n x_n,\ n \geq 0. \tag{2.1}$$

Then $\{x_n\}_{n\geq 0}$ converges strongly to $p \in F(T)$.

Proof. Because p is a fixed point of T, then the set of fixed points $F(T)$ of T is nonempty.

Since T has bounded range, we set

$$M_1 = ||x_0 - p|| + \sup_{n\geq 0} ||T^n x_n - p||.$$

Obviously $M_1 < \infty$.

It is clear that $||x_0 - p|| \leq M_1$. Let $||x_n - p|| \leq M_1$. Next we will prove that $||x_{n+1} - p|| \leq M_1$.

Consider

$$\begin{aligned} ||x_{n+1} - p|| &= ||(1-\alpha_n)x_n + \alpha_n T^n x_n - p|| \\ &= ||(1-\alpha_n)(x_n - p) + \alpha_n (T^n x_n - p)|| \\ &\leq (1-\alpha_n)||x_n - p|| + \alpha_n ||T^n x_n - p|| \\ &\leq (1-\alpha_n)M_1 + M_1\alpha_n \\ &= M_1. \end{aligned}$$

So, from the above discussion, we can conclude that the sequence $\{x_n - p\}_{n\geq 0}$ is bounded. Let $M_2 = \sup_{n\geq 0} ||x_n - p||$ and $M = M_1 + M_2$. Then, $M < \infty$.

Now from Lemma 2.1 for all $n \geq 0$, we obtain

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= ||(1-\alpha_n)x_n+\alpha_n T^n x_n - p||^2 \\
&= ||(1-\alpha_n)(x_n-p)+\alpha_n(T^n x_n - p)||^2 \\
&\leq (1-\alpha_n)^2||x_n-p||^2+2\alpha_n\langle T^n x_n - p, j(x_{n+1}-p)\rangle \\
&= (1-\alpha_n)^2||x_n-p||^2+2\alpha_n\langle T^n x_{n+1} - p,\ j(x_{n+1}-p)\rangle \\
&\quad +2\alpha_n\langle T^n x_n - T^n x_{n+1}, j(x_{n+1}-p)\rangle \\
&\leq (1-\alpha_n)^2||x_n-p||^2+2\alpha_n k_n||x_{n+1}-p||^2 \qquad (2.2)\\
&\quad +2\alpha_n \|T^n x_n - T^n x_{n+1}\| \, \|x_{n+1}-p\| \\
&\leq (1-\alpha_n)^2||x_n-p||^2+2\alpha_n k_n||x_{n+1}-p||^2 \\
&\quad +2\alpha_n\lambda_n,
\end{aligned}$$

where

$$\lambda_n = M\left\|T^n x_n - T^n x_{n+1}\right\|. \tag{2.3}$$

Using (2.1) we have

$$\begin{aligned}
\|x_n - x_{n+1}\| &= \alpha_n\left\|x_n - T^n x_n\right\| \\
&\leq \alpha_n\left(\|x_n-p\|+\|T^n x_n - p\|\right) \\
&\leq 2M\alpha_n. \qquad (2.4)
\end{aligned}$$

From the condition $\lim_{n\to\infty}\alpha_n = 0$ and (2.4), we obtain

$$\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0,$$

and the uniform continuity of T leads to

$$\lim_{n\to\infty}\|T^n x_n - T^n x_{n+1}\| = 0,$$

thus, we have

$$\lim_{n\to\infty}\lambda_n = 0. \tag{2.5}$$

The real function $f:[0,\,\infty)\to[0,\,\infty)$, defined by $f(t)=t^2$ is increasing and convex. For all $\lambda\in[0,1]$ and $t_1,\,t_2>0$ we have

$$((1-\lambda)t_1+\lambda t_2)^2 \leq (1-\lambda)t_1^2+\lambda t_2^2. \tag{2.6}$$

Consider

$$\begin{aligned}
||x_{n+1}-p||^2 &= ||(1-\alpha_n)x_n+\alpha_n T^n x_n - p||^2 \\
&= ||(1-\alpha_n)(x_n-p)+\alpha_n(T^n x_n - p)||^2 \\
&\leq \left[(1-\alpha_n)\|x_n-p\|+\alpha_n\|T^n x_n - p\|\right]^2 \\
&\leq (1-\alpha_n)\|x_n-p\|^2+\alpha_n\|T^n x_n - p\|^2 \\
&\leq (1-\alpha_n)\|x_n-p\|^2+M^2\alpha_n. \qquad (2.7)
\end{aligned}$$

Substituting (2.7) in (2.2), we get

$$\begin{aligned}\|x_{n+1}-p\|^2 \leq\ & [(1-\alpha_n)^2+2\alpha_n(1-\alpha_n)k_n]||x_n-p||^2 \\ & +2\alpha_n\left(M^2k_n\alpha_n+\lambda_n\right). \end{aligned} \tag{2.8}$$

Consider

$$\begin{aligned}(1-\alpha_n)^2+2\alpha_n(1-\alpha_n)k_n =\ & (1-\alpha_n)^2+2\alpha_n(1-\alpha_n) \\ & +2\alpha_n(1-\alpha_n)(k_n-1) \\ \leq\ & 1-\alpha_n^2+2\alpha_n(k_n-1),\end{aligned}$$

and consequently from (2.8), we obtain

$$\begin{aligned}||x_{n+1}-p||^2 \leq\ & [1-\alpha_n^2+2\alpha_n(k_n-1)||x_n-p||^2 \\ & +2\alpha_n\left(M^2k_n\alpha_n+\lambda_n\right) \\ \leq\ & \left(1-\alpha_n^2\right)||x_n-p||^2 \\ & +2[M^2k_n\alpha_n+\lambda_n+M^2(k_n-1)]\alpha_n \\ =\ & \left(1-\alpha_n^2\right)||x_n-p||^2+\varepsilon_n\alpha_n,\end{aligned} \tag{2.9}$$

where $\varepsilon_n = 2\left[M^2k_n\alpha_n+\lambda_n+M^2(k_n-1)\right]$. Now with the help of $\sum_{n\geq 0}\alpha_n^2 = \infty$, $\lim\limits_{n\to\infty}\alpha_n = 0$, (2.5) and Lemma 2.3, we obtain from (2.9) that

$$\lim_{n\to\infty}||x_n-p|| = 0.$$

This completes the proof. □

Corollary 2.5. *Let K be a nonempty closed convex subset of a real Banach space E, $T : K \to K$ a uniformly L-Lipschitzian asymptotically pseudo-contractive mapping having $T(K)$ bounded with sequence $\{k_n\}_{n\geq 0} \subset [1,\infty)$, $\lim\limits_{n\to\infty}k_n = 1$ such that $p \in F(T) = \{x \in K : Tx = x\}$. Let $\{\alpha_n\}_{n\geq 0} \in [0,1]$ be such that $\sum_{n\geq 0}\alpha_n^2 = \infty$ and $\lim\limits_{n\to\infty}\alpha_n = 0$. For arbitrary $x_0 \in K$ let $\{x_n\}_{n\geq 0}$ be iteratively defined by*

$$x_{n+1} = (1-\alpha_n)\,x_n+\alpha_nT^nx_n,\ n\geq 0.$$

Then $\{x_n\}_{n\geq 0}$ converges strongly to $p \in F(T)$.

Remark 2.6. 1. We try to remove the conditions like (O) form the existing literature.

2. It is worth to mention that, our results are new and do not exist in the literature.

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