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CONVERGENCE THEOREMS OF PROXIMAL TYPE ALGORITHM FOR A CONVEX FUNCTION AND MULTIVALUED MAPPINGS IN HILBERT SPACES

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Abstract. In this paper we study the weak and strong convergence to minimizers of convex function of proximal point algorithm SP-iteration of three multivalued nonexpansive mappings in a Hilbert space.

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1. Introduction

Let H be a Hilbert space and $f : H \to (-\infty, \infty]$ be a proper and convex function. One of the major problem for optimization is to find a point $x \in H$ such that

$$f(x) = \min_{y \in H} f(y).$$

We denote the set of all minimizers of f on H by $\arg\min_{y\in H} f(y)$.

The proximal point algorithm (PPA) is an important tool in solving optimization problem which was initiated by Martinet [10] in 1970. Later Rockafallar [14] studied the converging of PPA for finding a solution of the unconstrained convex minimization problem in H as follows:

Let f be a proper convex and lower semi-continuous function on H. The PPA is defined by $x_1 \in H$ and

$$x_{n+1} = \arg\min_{y \in H} \Big\{ f(y) + \frac{1}{2\lambda_n} ||y - x_n||^2 \Big\},$$

where $\lambda_n > 0$ for all $n \in \mathbb{N}$. It was shown that if f has a minimizers and $\sum_{n=1}^{\infty} \lambda_n = \infty$, then the sequence $\{x_n\}$ converges weakly to a minimizers of f. However, PPA does not necessarily converge strongly in general. Recently, several authors proposed modification of Rockafellar's PPA to have strong convergence for example [7, 8].

In the recent years, the problem of finding a common element of the set of solutions of various convex minimization problem and the set of fixed point for multivalued mapping in the framework of Hilbert spaces has been intensely studied by many authors, see for instance [3, 4, 9, 12].

In 2011, Phuengrattana and Suantai [13] introduced SP iteration as follows:

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n T x_n, \\ y_n = (1-\beta_n)z_n + \beta_n T z_n, \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n T y_n \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences of positive numbers in [0,1].

In this paper, we present PPA-SP-iteration for three multivalued nonexpansive mappings. We prove strong and weak convergence theorems of the proposed iteration process for proper convex and lower semi-continuous function and common fixed point of three multivalued nonexpansive mappings in a Hilbert space under some standard conditions.

2. Preliminaries

In this section, we collect some well-known concepts and relevant results which will be used frequently.

Let H be a real Hilbert space with inner product and C be a nonempty subset of H. Let $CB(C)$ and $K(C)$ denote the families of nonempty closed bounded subset and nonempty compact subset of C, respectively. The Pompeiu-Hausdorff metric on $CB(C)$ is defined by

$$\mathcal{H}(A,B) = \max\{\sup \operatorname{dist}_{x\in A}(x,B), \sup \operatorname{dist}_{x\in B}(y,A)\}$$

for $A, B \in CB(C)$, where $\operatorname{dist}(x, C) = \inf\{||x-y|| : y \in C\}$.

An element $x \in C$ is called a fixed point of a multivalued mapping $T : C \to CB(C)$ if $x \in Tx$. The set of fixed point of T is denoted by $F(T)$.

Recall that a multivalued mapping $T : C \to CB(C)$ is said to be nonexpansive if

$$\mathcal{H}(Tx,Ty) \leq ||x-y||$$

for all $x, y \in C$.

Let $f : H \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. For any $\lambda > 0$, define the Moreau-Yosida resolvent of f in a real Hilbert space H as follows:

$$J_\lambda x = \arg\min_{y\in H}\Big\{f(y) + \frac{1}{2\lambda}||y-x||^2\Big\},$$

for all $x \in H$. It was shown in [5] that the set of fixed point of the resolvent associated with f coincides with the set of minimizers of f. Also, the resolvent J_λ of f is nonexpansive for all $\lambda > 0$ (See [6]).

It is known in [11] that a Hilbert space H satisfies Opial's condition that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\limsup_{n\to\infty} ||x_n - x|| < \limsup_{n\to\infty} ||x_n - y||$$

hold for every $y \in H$ with $y \neq x$.

Lemma 2.1. ([6]) *Let H be a real Hilbert space and $f : H \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Then, for each $x \in H$ and $\lambda > \mu > 0$, the following resolvent identity holds:*

$$J_\lambda x = J_\mu\Big(\frac{\lambda-\mu}{\lambda}J_\lambda x + \frac{\mu}{\lambda}x\Big).$$

Lemma 2.2. ([1]) *Let H be a real Hilbert space and $f : H \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Then, for all $x, y \in H$ and $\lambda > 0$, the following inequality holds:*

$$\frac{1}{2\lambda}||J_\lambda x - y||^2 - \frac{1}{2\lambda}||x-y||^2 + \frac{1}{2\lambda}||x - J_\lambda x||^2 \leq f(y) - f(J_\lambda x).$$

Lemma 2.3. ([2]) *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a nonexpansive single valued mapping. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ with $x_n - Tx_n \to 0$, then $x = Tx$.*

Lemma 2.4. ([15]) *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to K(C)$ be a multivalued nonexpansive mapping. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $y_n \in Tx_n$ with $x_n - y_n \to 0$, then $x \in Tx$.*

Lemma 2.5. *Let H be a Hilbert space. Let $R \in [0, \infty)$ be such that $\limsup_{n\to\infty} ||x_n|| \le R$, $\limsup_{n\to\infty} ||y_n|| \le R$ and $\lim_{n\to\infty} ||\alpha_n x_n + (1-\alpha_n)y_n|| = R$ where $\alpha_n \in [a, b]$, with $0 < a \le b < 1$. Then, we have $\lim_{n\to\infty} ||x_n - y_n|| = 0$.*

3. Main results

Lemma 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2, T_3 : C \to K(C)$ be multivalued nonexpansive mappings and $f : C \to (-\infty, \infty]$ be a proper, convex and lower semi-continuous function. Suppose that*

$$\mathcal{F} = F(T_1) \cap F(T_2) \cap F(T_3) \cap \operatorname*{arg\,min}_{y\in H} f(y)$$

is nonempty and $p \in \mathcal{F}$. For $x_1 \in C$, let the PPA-SP-iteration process $\{x_n\}$ for multivalued mapping be defined by:

$$\begin{cases} w_n = \operatorname*{arg\,min}_{y\in H}[f(y) + \frac{1}{2\lambda_n}||y - x_n||^2], \\ z_n = (1-\gamma_n)w_n + \gamma_n r_n, \quad r_n \in T_1 w_n, \\ y_n = (1-\beta_n)z_n + \beta_n q_n, \quad q_n \in T_2 z_n, \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n p_n, \quad p_n \in T_3 y_n \end{cases} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequence in $[0, 1]$ such that $0 < a \le \alpha_n, \beta_n, \gamma_n < 1$ for all $n \in \mathbb{N}$ and $\{\lambda_n\}$ is a sequence such that $\lambda_n \ge \lambda > 0$ for all $n \in \mathbb{N}$ and some λ. Then, the following statements hold:

(i) $\lim_{n\to\infty} ||x_n - p||$ *exist for all $p \in \mathcal{F}$,*
(ii) $\lim_{n\to\infty} ||x_n - w_n|| = 0$,
(iii) $\lim_{n\to\infty} ||r_n - w_n|| = \lim_{n\to\infty} ||q_n - z_n|| = \lim_{n\to\infty} ||p_n - y_n|| = 0$.

Proof. Let $p \in \mathcal{F}$, Then, $p = T_1 p = T_2 p = T_3 p$ and $f(p) \le f(y)$ for all $y \in C$. Hence, we have

$$f(p) + \frac{1}{2\lambda_n}||p - p||^2 \le f(y) + \frac{1}{2\lambda_n}||y - p||$$

for all $y \in C$ and hence $p = J_{\lambda_n}(p)$ for each $n \in \mathbb{N}$.

(i) Note that $w_n = J_{\lambda_n} x_n$ and J_{λ_n} is nonexpansive map for each $n \in \mathbb{N}$. So we have

$$||w_n - p|| = ||J_{\lambda_n}(x_n) - J_{\lambda_n}(p)|| \le ||x_n - p||. \tag{3.2}$$

Now, from (3.1), we have

$$\begin{aligned} ||z_n - p|| &\leq (1-\gamma_n)||w_n - p|| + \gamma_n||p_n - p|| \\ &\leq (1-\gamma_n)||w_n - p|| + \gamma_n\mathcal{H}(T_1 w_n, T_1 p) \\ &\leq (1-\gamma_n)||w_n - p|| + \gamma_n||w_n - p|| \\ &= ||w_n - p||, \end{aligned} \tag{3.3}$$

$$\begin{aligned} ||y_n - p|| &\leq (1-\beta_n)||z_n - p|| + \beta_n\mathcal{H}(T_2 z_n, T_2 p) \\ &\leq (1-\beta_n)||z_n - p|| + \beta_n||z_n - p|| \\ &\leq ||z_n - p|| \\ &\leq ||w_n - p|| \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} ||x_{n+1} - p|| &\leq (1-\alpha_n)||y_n - p|| + \alpha_n\mathcal{H}(T_3 y_n, T_3 p) \\ &\leq (1-\alpha_n)||y_n - p|| + \alpha_n||y_n - p|| \\ &= ||y_n - p|| \\ &\leq ||z_n - p|| \\ &\leq ||w_n - p|| \\ &\leq ||x_n - p||. \end{aligned} \tag{3.5}$$

This gives $\lim_{n\to\infty} ||x_n - p||$ exists. Let us assume $\lim_{n\to\infty} ||x_n - p|| = c$.

(ii) Next we show that $\lim_{n\to\infty} ||x_n - w_n|| = 0$. By Lemma 2.2, we get

$$\frac{1}{2\lambda_n}||w_n - p||^2 - \frac{1}{2\lambda_n}||x_n - p||^2 + \frac{1}{2\lambda_n}||x_n - w_n||^2 \leq f(p) - f(w_n).$$

Since $f(p) \leq f(w_n)$ for each $n \in \mathbb{N}$, it follows that

$$||x_n - w_n||^2 \leq ||x_n - p||^2 - ||w_n - p||^2.$$

In order to show that $\lim_{n\to\infty} ||x_n - w_n|| = 0$ it is sufficient to show that

$$\lim_{n\to\infty} ||w_n - p|| = c.$$

Now, from (3.5), we have

$$||x_{n+1} - p|| \leq ||w_n - p||.$$

Hence, we get

$$c \leq \liminf_{n\to\infty} ||w_n - p||,$$

Also, from (3.2), we see that

$$\limsup_{n\to\infty} ||w_n - p|| \leq c.$$

Therefore, we get

$$\lim_{n\to\infty} ||w_n - p|| = c.$$

This gives

$$\begin{aligned}\lim_{n\to\infty} ||x_n - w_n||^2 &\leq \lim_{n\to\infty} ||x_n - p||^2 - \lim_{n\to\infty} ||w_n - p||^2 \\ &= 0.\end{aligned}$$

Hence $\lim_{n\to\infty} ||x_n - w_n|| = 0$.

(iii) From (3.2), we have $\lim_{n\to\infty} ||w_n - p|| = c$. It follows from (3.1), we have

$$\begin{aligned}||r_n - p|| &\leq \mathcal{H}(T_1 w_n, T_1 p) \\ &\leq ||w_n - p||.\end{aligned}$$

Hence $\limsup_{n\to\infty} ||r_n - p|| \leq c$. From (3.1)

$$\lim_{n\to\infty} ||z_n - p|| = \lim_{n\to\infty} ||(1-\gamma_n)w_n + \gamma_n r_n - p|| = c.$$

By using Lemma 2.5, $\lim_{n\to\infty} ||w_n - r_n|| = 0$.

Now from (3.2) and (3.3), we get

$$\lim_{n\to\infty} ||z_n - p|| = c$$

and

$$\begin{aligned}||q_n - p|| &\leq \mathcal{H}(T_2 z_n, T_2 p) \\ &\leq ||z_n - p||.\end{aligned}$$

Hence $\limsup_{n\to\infty} ||q_n - p|| \leq c$.

From (3.1), we have

$$\lim_{n\to\infty} ||y_n - p|| = \lim_{n\to\infty} ||(1-\beta_n)z_n + \beta_n q_n - p|| = c.$$

By using Lemma 2.5, we get $\lim_{n\to\infty} ||q_n - z_n|| = 0$.
From (3.2) and (3.4), we have

$$\lim_{n\to\infty} ||y_n - p|| = c$$

and

$$\begin{aligned}||p_n - p|| &\leq \mathcal{H}(T_3 y_n, T_3 p) \\ &\leq ||y_n - p||.\end{aligned}$$

Hence $\limsup_{n\to\infty} ||p_n - p|| \leq c$.

From (3.1), we obtain

$$\lim_{n\to\infty} ||x_{n+1} - p|| = \lim_{n\to\infty} ||(1-\beta_n)y_n + \beta_n p_n - p|| = c.$$

By using Lemma 2.5, we obtain

$$\lim_{n\to\infty} ||p_n - y_n|| = 0.$$

This completes the proof. □

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2, T_3 : C \to K(C)$ be three multivalued nonexpansive mappings and $f : C \to (-\infty, \infty]$ be a proper, convex and lower semi-continuous function. Suppose that*

$$\mathcal{F} = F(T_1) \cap F(T_2) \cap F(T_3) \cap \arg\min_{y\in H} f(y)$$

is nonempty and $p \in \mathcal{F}$. For $x_1 \in C$, let the PPA-SP-iteration process $\{x_n\}$ for multivalued mapping be defined by (3.1)*. Then the sequence $\{x_n\}$ converges weakly to an element of $\mathcal{F}$.*

Proof. By using Lemma 2.1, we have

$$\begin{aligned}
||J_\lambda x_n - x_n|| &\leq ||J_\lambda x_n - w_n|| + ||w_n - x_n|| \\
&= ||J_\lambda x_n - w_n|| + ||w_n - x_n|| \\
&= ||J_\lambda x_n - J_\lambda\Big(\frac{\lambda_n - \lambda}{\lambda_n} J_{\lambda_n} x_n + \frac{\lambda}{\lambda_n} x_n\Big)|| + ||w_n - x_n|| \\
&\leq ||x_n - \Big(\frac{\lambda_n - \lambda}{\lambda_n} J_{\lambda_n} x_n + \frac{\lambda}{\lambda_n} x_n\Big)|| + ||w_n - x_n|| \\
&\leq (1 - \frac{\lambda}{\lambda_n})||x_n - J_{\lambda_n} x_n|| + \frac{\lambda}{\lambda_n}||x_n - w_n|| + ||w_n - x_n|| \\
&\leq (1 - \frac{\lambda}{\lambda_n})||x_n - w_n||+)||x_n - w_n|| \\
&\to 0 \ \text{ as } \ n \to \infty.
\end{aligned}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup q \in C$. By using Lemma 3.1 (ii) implies that $w_{n_i} \to q \in C$. This implies by Lemma 2.4 that $q \in \mathcal{F}$. Since J_λ is nonexpansive single valued mapping, from Lemma 2.4, we get

$$q \in F(J_\lambda) = \arg\min_{y\in H} f(y).$$

Hence, we have $q \in \mathcal{F}$. We will show that $x_n \to q$. To show this, suppose not so, there exist a subsequence $x_{n_j} \to q^{'} \in C$ and $q \neq q^{'}$. Again, as above, we can conclude that $q^{'} \in \mathcal{F}$. Since $\lim_{n\to\infty} ||x_n - p||$ exist for all $p \in \mathcal{F}$, by the

Opial's condition, we have

$$\begin{aligned}
\limsup_{i\to\infty} ||x_{n_i} - q|| &< \limsup_{i\to\infty} ||x_{n_i} - q^{'}|| \\
&= \lim_{n\to\infty} ||x_n - q^{'}|| \\
&= \limsup_{j\to\infty} ||x_{n_j} - q^{'}|| \\
&< \limsup_{j\to\infty} ||x_{n_j} - q|| \\
&= \lim_{n\to\infty} ||x_n - q|| \\
&= \limsup_{i\to\infty} ||x_{n_i} - q||.
\end{aligned}$$

This is a contradiction. Therefore, $q = q^{'}$ and so $\{x_n\}$ converges weakly to an element of $\mathcal{F}$. □

If $T = T_1 = T_2 = T_3$, then corollary can be obtain directly from the above theorem.

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to K(C)$ be a multivalued nonexpansive mapping and $f : C \to (-\infty, \infty]$ be a proper, convex and lower semi-continuous function. Suppose that*

$$\mathcal{F} = F(T) \cap \underset{y\in H}{\arg\min} f(y)$$

is nonempty and $p \in \mathcal{F}$. For $x_1 \in C$, let the PPA-SP-iteration process $\{x_n\}$ for multivalued mapping be defined by:

$$\begin{cases}
w_n = \underset{y\in H}{\arg\min}[f(y) + \frac{1}{2\lambda_n}||y - x_n||^2], \\
z_n = (1-\gamma_n)w_n + \gamma_n r_n, \quad r_n \in Tw_n, \\
y_n = (1-\beta_n)z_n + \beta_n q_n, \quad q_n \in Tz_n, \\
x_{n+1} = (1-\alpha_n)y_n + \alpha_n p_n, \quad p_n \in Ty_n
\end{cases} \tag{3.6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequence in $[0,1]$ such that $0 < a \leq \alpha_n, \beta_n, \gamma_n < 1$ for all $n \in \mathbb{N}$ and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \in \mathbb{N}$ and some λ. Then, the sequence $\{x_n\}$ converges weakly to an element of $\mathcal{F}$.

Next, we will prove strong convergence theorem under some standard conditions.

Theorem 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2, T_3 : C \to K(C)$ be three multivalued nonexpansive mappings and $f : C \to (-\infty, \infty]$ be a proper, convex and lower semi-continuous* *function. Suppose that*

$$\mathcal{F} = F(T_1) \cap F(T_2) \cap F(T_3) \cap \arg\min_{y \in H} f(y)$$

is nonempty and $p \in \mathcal{F}$. For $x_1 \in C$, let the PPA-SP-iteration process $\{x_n\}$ for multivalued mapping be defined by (3.1). Then the sequence $\{x_n\}$ converges strongly to an element of $\mathcal{F}$ if and only if $\liminf_{n\to\infty} dist(x_n, \mathcal{F}) = 0$.

Proof. Suppose that $\{x_n\}$ converges strongly to $p \in \mathcal{F}$, then $\lim_{n\to\infty} ||x_n - p|| = 0$. Since $0 \leq \text{dist}(x_n, \mathcal{F}) \leq ||x_n - p||$, it follows that

$$\lim_{n\to\infty} \text{dist}(x_n, \mathcal{F}) = 0.$$

Therefore, the $\liminf_{n\to\infty} \text{dist}(x_n, \mathcal{F}) = 0$.

Conversely, suppose that $\liminf_{n\to\infty} \text{dist}(x_n, \mathcal{F}) = 0$. We therefore have

$$\lim_{n\to\infty} \text{dist}(x_n, \mathcal{F}) = 0.$$

Suppose that $\{x_{n_k}\}$ is any arbitrary subsequence of $\{x_n\}$ and that $\{p_k\}$ is a sequence in $\mathcal{F}$ such that $||x_{n_k} - p_k|| \leq \frac{1}{2k}$ for all $n, k \in \mathbb{N}$.

From (3.5), we have

$$||x_{n_{k+1}} - p_k|| \leq ||x_{n_k} - p_k|| \leq \frac{1}{2k}.$$

Thus

$$\begin{aligned} ||p_{k+1} - p_k|| &\leq ||p_{k+1} - x_{n_{k+1}}|| + ||x_{n_{k+1}} - p_k|| \\ &\leq \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}}. \end{aligned}$$

This gives that $\{p_k\}$ is a Cauchy sequence in C. Suppose that $\lim_{k\to\infty} p_k = q$. Then,

$$\begin{aligned} ||q - T_1 q|| &= \lim_{n\to\infty} \text{dist}(p_k, T_1 q) \\ &\leq \lim_{n\to\infty} \mathcal{H}(T_1 p_k, T_1 q) \\ &\leq \lim_{n\to\infty} ||p_k - q|| \\ &= 0. \end{aligned}$$

This implies $||q - T_1 q|| = 0$.

Similarly, we can conclude that $||q - T_2 q|| = 0$ and $||q - T_3 q|| = 0$.
And also

$$\begin{aligned} ||q - J_\lambda q|| &= \lim_{k\to\infty} ||p_k - J_\lambda q|| \\ &\leq \lim_{k\to\infty} ||J_\lambda p_k - J_\lambda q|| \\ &\leq \lim_{k\to\infty} ||p_k - q|| \\ &= 0. \end{aligned}$$

Hence, $q \in \mathcal{F}$ and the sequence $\{x_n\}$ converges strongly to q. □

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