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COMMON FIXED POINT THEOREMS UNDER GENERALIZED $(\psi - \phi)$-WEAK CONTRACTIONS IN S-METRIC SPACES WITH APPLICATIONS

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Abstract. The aim of this paper is to establish common fixed point theorems under generalized $(\psi - \phi)$-weak contractions in the setting of complete S-metric spaces and we support our result by some examples. Also an application of our results, we obtain some fixed point theorems of integral type. Our results extend Theorem 2.1 and 2.2 of Doric [5], Theorem 2.1 of Dutta and Choudhury [6], and many other several results from the existing literature.

1. Introduction

The classical Banach's contraction principle is one of the most useful results in fixed point theory. It is a very popular tool for solving existence problems in many different fields of mathematics. Banach contraction principle has been generalized in various ways either by using contractive conditions or by imposing some additional conditions on the ambient space. In a metric space setting it can be briefly stated as follows.

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Theorem 1.1. ([3]) *Let (X, d) be a complete metric space and $T\colon X \to X$ be a map satisfying*

$$d(T(x), T(y)) \leq q\, d(x, y), \quad \textit{for all } x, y \in X, \tag{1.1}$$

where $0 < q < 1$ is a constant. Then

(C1) *T has a unique fixed point u in X;*
(C2) *The Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by*

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, \ldots \tag{1.2}$$

converges to u, for any $x_0 \in X$.

Remark 1.2. (i) A map satisfying $(C1)$ and $(C2)$ is said to be a Picard operator (see, [13, 14]).
(ii) Inequality (1.1) also implies the continuity of T.

In 1997, Alber and Delabrieer in [2] introduced the concept of ϕ-weak contraction as follows.

Definition 1.3. ([2]) A mapping $T\colon X \to X$ is called a ϕ-weak contraction, if for each $x, y \in X$, there exists a function $\phi\colon [0, \infty) \to [0, \infty)$ such that ϕ is positive on $(0, +\infty)$ and $\phi(0) = 0$, and

$$d(T(x), T(y)) \leq d(x, y) - \phi(d(x, y)). \tag{1.3}$$

The authors defined such mappings for single-valued maps on Hilbert spaces and proved a novel fixed point result for weak contraction in the given space.

In 2001, Rhoades [12] has shown that the result which Alber and Delabrieer have proved in [2] is also valid in complete metric spaces. The result of Rhoades is as follows.

Theorem 1.4. *Let (X, d) be a nonempty complete metric space and let $T\colon X \to X$ be a ϕ-weak contraction on X. If ϕ is a continuous and nondecreasing function with $\phi(t) > 0$ for all $t > 0$ and $\phi(0) = 0$, then T has a unique fixed point.*

Remark 1.5. If we take $\phi(t) = qt$ where $0 < q < 1$, then (1.3) reduces to (1.1).

Dutta and Choudhury [6] in 2008, have introduced a new generalization of contraction principle and proved the following theorem.

Theorem 1.6. ([6]) *Let (X, d) be a complete metric space and let $T\colon X \to X$ be a self-mapping satisfying the inequality*

$$\psi(d(T(x), T(y))) \leq \psi(d(x, y)) - \phi(d(x, y)) \tag{1.4}$$

where $x, y \in X$, $\psi, \phi\colon [0,\infty) \to [0,\infty)$ are both continuous and monotone non-decreasing functions with $\psi(t) = 0 = \phi(t)$ if and only if $t = 0$. Then T has a unique fixed point.

Remark 1.7. (i) If we take $\psi(t) = t$ for all $t \geq 0$, then (1.4) reduces to (1.3).

(ii) If we take $\psi(t) = t$ for all $t \geq 0$ and $\phi(t) = (1-q)\psi(t)$ where $0 < q < 1$, then (1.4) reduces to (1.1).

In 2009, Doric [5] generalized Theorem 1.6 for a pair of maps as follows.

Theorem 1.8. ([5]) *Let (X, d) be a complete metric space and let $S, T\colon X \to X$ be two self-mappings satisfying the inequality*

$$\psi(d(T(x), S(y))) \leq \psi(M(x,y)) - \phi(M(x,y)) \tag{1.5}$$

for any $x, y \in X$, where $M(x,y)$ is given by

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Sy), \frac{1}{2}[d(x,Sy) + d(y,Tx)]\right\}$$

and

(1) *$\psi\colon [0,\infty) \to [0,\infty)$ is a continuous monotone nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$,*

(2) *$\phi\colon [0,\infty) \to [0,\infty)$ is a lower semi-continuous function with $\phi(t) = 0$ if and only if $t = 0$.*

Then there exists the unique point $u \in X$ such that $u = Tu = Su$.

In 2010, Abbas and Doric [1] proved similar results on fixed point in complete metric spaces involving four mappings while Murthy et al. [9] obtained fixed point results in complete metric spaces under (ψ, φ)-generalized weak contractive condition.

In 2012, Sedghi et al. [15] have introduced the notion of S-metric space which is a generalization of a G-metric space and D^*-metric space. In [15] the authors proved some properties of S-metric spaces. Also, they obtained some fixed point theorems in S-metric space for a self-map (see. [7],[8],[10],[11])

The main purpose of the present work is to generalize above few results in the setting of S-metric spaces. For this, we need the notion of S-metric space and its basic properties. So, first we recall the notion and basic properties of S-metric space.

2. Definitions and lemmas

We need the following definitions and lemmas in the sequel.

Definition 2.1. ([15]) Let X be a nonempty set and $S\colon X^3 \to [0,\infty)$ be a function satisfying the following conditions for all $x,\ y,\ z,\ t \in X$:

(SM1) $S(x,y,z) = 0$ if and only if $x = y = z$;
(SM2) $S(x,y,z) \le S(x,x,t) + S(y,y,t) + S(z,z,t)$.

Then the function S is called an S-metric on X and the pair (X,S) is called an S-metric space or simply SMS.

Example 2.2. ([15]) Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X. Then $S(x,y,z) = \|y+z-2x\| + \|y-z\|$ is an S-metric on X.

Example 2.3. ([15]) Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X. Then $S(x,y,z) = \|x-z\| + \|y-z\|$ is an S-metric on X.

Example 2.4. ([16]) Let $X = \mathbb{R}$ be the real line. Then $S(x,y,z) = |x-z| + |y-z|$ for all $x,y,z \in \mathbb{R}$ is an S-metric on X. This S-metric on X is called the usual S-metric on X.

Lemma 2.5. ([15], Lemma 2.5) *In an S-metric space, we have $S(x,x,y) = S(y,y,x)$ for all $x,y \in X$.*

Lemma 2.6. ([15], Lemma 2.12) *Let (X,S) be an S-metric space. If $x_n \to x$ and $y_n \to y$ as $n \to \infty$ then $S(x_n,x_n,y_n) \to S(x,x,y)$ as $n \to \infty$.*

Definition 2.7. ([15]) Let (X,S) be an S-metric space.

(1) A sequence $\{x_n\}$ in X converges to $x \in X$ if $S(x_n,x_n,x) \to 0$ as $n \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, we have $S(x_n,x_n,x) < \varepsilon$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
(2) A sequence $\{x_n\}$ in X is called Cauchy if $S(x_n,x_n,x_m) \to 0$ as $n,m \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n,m \ge n_0$, we have $S(x_n,x_n,x_m) < \varepsilon$.
(3) The S-metric space (X,S) is called complete if every Cauchy sequence in X is convergent in X.

Definition 2.8. Let T be a self mapping on an S-metric space (X,S). Then T is said to be continuous at $x \in X$ if for any sequence $\{x_n\}$ in X with $x_n \to x$ implies that $Tx_n \to Tx$ as $n \to \infty$.

Definition 2.9. ([15]) Let (X,S) be an S-metric space. A mapping $T\colon X \to X$ is said to be a contraction if there exists a constant $0 \le L < 1$ such that

$$S(Tx,Tx,Ty) \le L\,S(x,x,y) \tag{2.1}$$

for all $x, y \in X$.

If the S-metric space (X, S) is complete then the mapping defined as above has a unique fixed point.

Now, we generalize the definitions of ϕ-weak contractions and $(\psi - \phi)$-weak contractions in S-metric space as follows.

Definition 2.10. (Weak Contraction Mapping) Let (X, S) be an S-metric space. A mapping $T \colon X \to X$ is said to be ϕ-weak contraction if

$$S(Tx, Tx, Ty) \leq S(x, x, y) - \phi(S(x, x, y)), \tag{2.2}$$

where $x, y \in X$, $\phi \colon [0, \infty) \to [0, \infty)$ is continuous and nondecreasing, $\phi(t) = 0$ if and only if $t = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$.

Remark 2.11. If we take $\phi(t) = Lt$, where $0 < L < 1$ then (2.2) reduces to (2.1).

Definition 2.12. Let (X, S) be an S-metric space. A mapping $T \colon X \to X$ is said to be generalized $(\psi - \phi)$-weak contraction if for all $x, y \in X$

$$\psi(S(Tx, Tx, Ty)) \leq \psi(S(x, x, y)) - \phi(S(x, x, y)), \tag{2.3}$$

where $\psi, \phi \colon [0, \infty) \to [0, \infty)$ are both continuous and monotone nondecreasing functions with $\psi(t) = 0 = \phi(t)$ if and only if $t = 0$.

Remark 2.13.
(i) If we take $\psi(t) = t$ for all $t \geq 0$ and $\phi(t) = (1-L)\psi(t)$ where $0 < L < 1$, then (2.3) reduces to (2.1).
(ii) If we take $\psi(t) = t$ for all $t \geq 0$, then (2.3) reduces to (2.2).

3. Main results

In this section, we shall establish unique common fixed point theorems in a complete S-metric space for generalized $(\psi - \phi)$-weak contractions.

Theorem 3.1. *Let (X, S) be a complete S-metric space and $F, G \colon X \to X$ be two self mappings satisfying the inequality*

$$\psi(S(Fx, Fy, Gz)) \leq \psi(N(x, y, z)) - \phi(N(x, y, z)) \tag{3.1}$$

for all $x, y, z \in X$, where $N(x, y, z)$ is given by

$$\begin{aligned} N(x, y, z) \quad = \quad \max\Big\{ & S(x, y, z), \frac{1}{2}[S(x, x, Fx) + S(z, z, Gz)], \\ & \frac{1}{2}[S(x, x, Gz) + S(z, z, Fx)]\Big\} \end{aligned}$$

and

(A1) $\psi\colon [0,\infty) \to [0,\infty)$ *is a continuous monotone nondecreasing function with* $\psi(t) = 0$ *if and only if* $t = 0$,

(A2) $\phi\colon [0,\infty) \to [0,\infty)$ *is a lower semi-continuous function with* $\phi(t) = 0$ *if and only if* $t = 0$.

Then there exists the unique point $v \in X$ *such that* $v = Fv = Gv$.

Proof. For any $x_0 \in X$, we construct the sequence $\{x_n\}$ for $n \geq 0$ recursively as

$$x_{2n+1} = Gx_{2n}, \quad x_{2n} = Fx_{2n+1} \tag{3.2}$$

and prove that

$$S(x_{n+1}, x_{n+1}, x_n) \to 0 \text{ as } n \to \infty. \tag{3.3}$$

Suppose now that n is an odd number. Putting $x = y = x_n$ and $z = x_{n-1}$ in inequality (3.1), we get

$$\begin{aligned} \psi(S(x_{n+1}, x_{n+1}, x_n)) &= \psi(S(Fx_n, Fx_n, Gx_{n-1})) \\ &\leq \psi(N(x_n, x_n, x_{n-1})) \\ &\quad -\phi(N(x_n, x_n, x_{n-1})), \end{aligned} \tag{3.4}$$

which implies

$$\psi(S(x_{n+1}, x_{n+1}, x_n)) \leq \psi(N(x_n, x_n, x_{n-1})). \tag{3.5}$$

Using the properties of ψ and ϕ functions in the above inequality, we obtain

$$S(x_{n+1}, x_{n+1}, x_n) \leq N(x_n, x_n, x_{n-1}). \tag{3.6}$$

Now using condition (SM_2) and Lemma 2.5, we have

$$\begin{aligned} &N(x_n, x_n, x_{n-1}) \\ &= \max\Big\{S(x_n, x_n, x_{n-1}), \frac{1}{2}[S(x_n, x_n, Fx_n) + S(x_{n-1}, x_{n-1}, Gx_{n-1})], \\ &\qquad \frac{1}{2}[S(x_n, x_n, Gx_{n-1}) + S(x_{n-1}, x_{n-1}, Fx_n)]\Big\} \\ &= \max\Big\{S(x_n, x_n, x_{n-1}), \frac{1}{2}[S(x_n, x_n, x_{n+1}) + S(x_{n-1}, x_{n-1}, x_n)], \\ &\qquad \frac{1}{2}[S(x_n, x_n, x_n) + S(x_{n-1}, x_{n-1}, x_{n+1})]\Big\} \\ &= \max\Big\{S(x_n, x_n, x_{n-1}), \frac{1}{2}[S(x_{n+1}, x_{n+1}, x_n) + S(x_n, x_n, x_{n-1})], \\ &\qquad \frac{1}{2}[S(x_{n+1}, x_{n+1}, x_{n-1})]\Big\} \end{aligned}$$

$$\leq \max\Big\{S(x_n, x_n, x_{n-1}), \frac{1}{2}[S(x_{n+1}, x_{n+1} + x_n), S(x_n, x_n, x_{n-1})],$$
$$\frac{1}{2}[2S(x_{n+1}, x_{n+1}, x_n) + S(x_{n-1}, x_{n-1}, x_n)]\Big\}$$
$$= \max\Big\{S(x_n, x_n, x_{n-1}), \frac{1}{2}[S(x_{n+1}, x_{n+1}, x_n) + S(x_n, x_n, x_{n-1})],$$
$$\frac{1}{2}[2S(x_{n+1}, x_{n+1}, x_n) + S(x_n, x_n, x_{n-1})]\Big\}.$$

If $S(x_{n+1}, x_{n+1}, x_n) > S(x_n, x_n, x_{n-1})$, then

$$N(x_n, x_n, x_{n-1}) = S(x_{n+1}, x_{n+1}, x_n) > 0.$$

It implies that

$$\psi(S(x_{n+1}, x_{n+1}, x_n)) \leq \psi(S(x_{n+1}, x_{n+1}, x_n)) - \phi(S(x_{n+1}, x_{n+1}, x_n)) \quad (3.7)$$

which is a contradiction. So, we have

$$S(x_{n+1}, x_{n+1}, x_n) \leq N(x_n, x_n, x_{n-1}) \leq S(x_n, x_n, x_{n-1}). \quad (3.8)$$

Similarly, we can obtain the same inequality as above in the case when n is an even number. Therefore the sequence $\{S(x_{n+1}, x_{n+1}, x_n)\}$ is monotone decreasing and bounded. So there exists $c \geq 0$ such that

$$\lim_{n\to\infty} S(x_{n+1}, x_{n+1}, x_n) = \lim_{n\to\infty} N(x_n, x_n, x_{n-1}) = c \geq 0. \quad (3.9)$$

Letting $n \to \infty$ in inequality (3.4), we obtain

$$\psi(c) \leq \psi(c) - \phi(c), \quad (3.10)$$

which is a contradiction unless $c = 0$. Hence,

$$\lim_{n\to\infty} S(x_{n+1}, x_{n+1}, x_n) = 0. \quad (3.11)$$

Next we prove that $\{x_n\}$ is a Cauchy sequence. Because of (3.11) it is sufficient to show that $\{x_{2n}\}$ is a Cauchy sequence. If otherwise, then there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{2m(k)}\}$ and $\{x_{2n(k)}\}$ of $\{x_{2n}\}$ and increasing sequences of integers $\{2m(k)\}$ and $\{2n(k)\}$ such that $n(k)$ is smallest index for which,

$$n(k) > m(k) > k, \quad (3.12)$$

$$S(x_{2m(k)}, x_{2m(k)}, x_{2n(k)}) \geq \varepsilon. \quad (3.13)$$

Further corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (3.12). Then

$$S(x_{2m(k)}, x_{2m(k)}, x_{2n(k)-1}) < \varepsilon. \quad (3.14)$$

Now using (3.13), (3.14), $(SM2)$ and Lemma 2.5, we have

$$\begin{aligned} \varepsilon &\leq S(x_{2m(k)}, x_{2m(k)}, x_{2n(k)}) \\ &= S(x_{2n(k)}, x_{2n(k)}, x_{2m(k)}) \\ &\leq 2S(x_{2n(k)}, x_{2n(k)}, x_{2n(k)-1}) \\ &\quad +S(x_{2m(k)}, x_{2m(k)}, x_{2n(k)-1}) \\ &\leq \varepsilon + 2S(x_{2n(k)}, x_{2n(k)}, x_{2n(k)-1}). \end{aligned} \tag{3.15}$$

Letting $k \to \infty$ in equation (3.15) and using (3.11), we get

$$\lim_{k\to\infty} S(x_{2m(k)}, x_{2m(k)}, x_{2n(k)}) = \varepsilon. \tag{3.16}$$

Again, with the help of $(SM2)$ and Lemma 2.5, we have

$$\begin{aligned} S(x_{2m(k)}, x_{2m(k)}, x_{2n(k)}) &\leq 2S(x_{2m(k)}, x_{2m(k)}, x_{2m(k)-1}) \\ &\quad +S(x_{2n(k)}, x_{2n(k)}, x_{2m(k)-1}) \\ &= 2S(x_{2m(k)}, x_{2m(k)}, x_{2m(k)-1}) \\ &\quad +S(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)}). \end{aligned} \tag{3.17}$$

Also, with the help of $(SM2)$ and Lemma 2.5, we have

$$\begin{aligned} S(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)}) &\leq 2S(x_{2m(k)-1}, x_{2m(k)-1}, x_{2m(k)}) \\ &\quad +S(x_{2n(k)}, x_{2n(k)}, x_{2m(k)}) \\ &= 2S(x_{2m(k)}, x_{2m(k)}, x_{2m(k)-1}) \\ &\quad +S(x_{2m(k)}, x_{2m(k)}, x_{2n(k)}). \end{aligned} \tag{3.18}$$

Letting $k \to \infty$ in equation (3.18) and using (3.11), (3.14), (3.16) and (3.17), we get

$$\lim_{k\to\infty} S(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)}) = \varepsilon. \tag{3.19}$$

Again, note that with the help of $(SM2)$ and Lemma 2.5, we have

$$\begin{aligned} S(x_{2m(k)}, x_{2m(k)}, x_{2n(k)+1}) &\leq 2S(x_{2m(k)}, x_{2m(k)}, x_{2m(k)-1}) \\ &\quad +S(x_{2n(k)+1}, x_{2n(k)+1}, x_{2m(k)-1}) \\ &\leq 2S(x_{2m(k)}, x_{2m(k)}, x_{2m(k)-1}) \\ &\quad +2S(x_{2n(k)+1}, x_{2n(k)+1}, x_{2n(k)}) \\ &\quad +S(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)}). \end{aligned} \tag{3.20}$$

Also, with the help of $(SM2)$ and Lemma 2.5, we have

$$\begin{aligned} S(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)}) &= S(x_{2n(k)}, x_{2n(k)}, x_{2m(k)-1}) \\ &\leq 2S(x_{2n(k)}, x_{2n(k)}, x_{2n(k)+1}) \\ &\quad +S(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)+1}) \\ &= 2S(x_{2n(k)+1}, x_{2n(k)+1}, x_{2n(k)}) \\ &\quad +S(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)+1}) \\ &\leq 2S(x_{2n(k)+1}, x_{2n(k)+1}, x_{2n(k)}) \\ &\quad +2S(x_{2m(k)-1}, x_{2m(k)-1}, x_{2m(k)}) \\ &\quad +S(x_{2n(k)+1}, x_{2n(k)+1}, x_{2m(k)}) \\ &= 2S(x_{2n(k)+1}, x_{2n(k)+1}, x_{2n(k)}) \\ &\quad +2S(x_{2m(k)}, x_{2m(k)}, x_{2m(k)-1}) \\ &\quad +S(x_{2m(k)}, x_{2m(k)}, x_{2n(k)+1}). \end{aligned} \tag{3.21}$$

Letting $k \to \infty$ in equation (3.21) and using (3.11), (3.19) and (3.20), we get

$$\lim_{k\to\infty} S(x_{2m(k)}, x_{2m(k)}, x_{2n(k)+1}) = \varepsilon. \tag{3.22}$$

Now consider inequality (3.1) and putting $x = y = x_{2m(k)-1}$, $z = x_{2n(k)}$, we obtain

$$\begin{aligned} \psi(S(x_{2m(k)}, x_{2m(k)}, x_{2n(k)+1})) &= \psi(S(Fx_{2m(k)-1}, Fx_{2m(k)-1}, Gx_{2n(k)})) \\ &\leq \psi(S(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)})) \\ &\quad -\phi(S(x_{2m(k)-1}, x_{2m(k)-1}, x_{2n(k)})). \end{aligned} \tag{3.23}$$

Letting $k \to \infty$ in equation (3.23) and using (3.19) and (3.22), we get

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon) < \psi(\varepsilon),$$

which is a contradiction. This shows that $\{x_n\}$ is a Cauchy sequence and therefore is convergent in the complete S-metric space (X, S). So, suppose $x_n \to v$ as $n \to \infty$.

Now we prove that $v = Fv = Gv$. Indeed, suppose $v \neq Gv$, then for $S(v, v, Gv) > 0$, there exist $N_1, N_2 \in \mathbb{N}$ such that

$$S(x_{2n}, x_{2n}, v) < \frac{1}{4} S(v, v, Gv), \ \ \forall\, n \geq N_1, \tag{3.24}$$

$$S(x_{2n-1}, x_{2n-1}, v) < \frac{1}{4} S(v, v, Gv), \ \ \forall\, n \geq N_1 \tag{3.25}$$

and

$$S(x_{2n}, x_{2n}, x_{2n-1}) < S(v, v, Gv), \ \ \forall\, n \geq N_2. \tag{3.26}$$

Let $N_3 = \max\{N_1, N_2\}$, then for any $n \geq N_3$, we have

$$S(x_{2n}, x_{2n}, v) < \frac{1}{4}S(v, v, Gv), \tag{3.27}$$

$$S(x_{2n-1}, x_{2n-1}, v) < \frac{1}{4}S(v, v, Gv) \tag{3.28}$$

and

$$S(x_{2n}, x_{2n}, x_{2n-1}) < S(v, v, Gv). \tag{3.29}$$

Now, putting $x = x_{2n-1}$ and $y = v$ in equation (3.1), we obtain

$$\begin{aligned} \psi(S(x_{2n}, x_{2n}, Gv)) &= \psi(S(Fx_{2n-1}, Fx_{2n-1}, Gv)) \\ &\leq \psi(N(x_{2n-1}, x_{2n-1}, v)) \\ &\quad -\phi(N(x_{2n-1}, x_{2n-1}, v)), \end{aligned} \tag{3.30}$$

where

$$\begin{aligned} & N(x_{2n-1}, x_{2n-1}, v) \\ &= \max\Big\{S(x_{2n-1}, x_{2n-1}, v), \frac{1}{2}[S(x_{2n-1}, x_{2n-1}, Fx_{2n-1}) + S(v, v, Gv)], \\ &\qquad \frac{1}{2}[S(x_{2n-1}, x_{2n-1}, Gv) + S(v, v, Fx_{2n-1})]\Big\} \\ &= \max\Big\{S(x_{2n-1}, x_{2n-1}, v), \frac{1}{2}[S(x_{2n-1}, x_{2n-1}, x_{2n}) + S(v, v, Gv)], \\ &\qquad \frac{1}{2}[S(x_{2n-1}, x_{2n-1}, Gv) + S(v, v, x_{2n})]\Big\} \\ &= \max\Big\{S(x_{2n-1}, x_{2n-1}, v), \frac{1}{2}[S(x_{2n}, x_{2n}, x_{2n-1}) + S(v, v, Gv)], \\ &\qquad \frac{1}{2}[S(x_{2n-1}, x_{2n-1}, Gv) + S(x_{2n}, x_{2n}, v)]\Big\} \\ &= \max\Big\{S(x_{2n-1}, x_{2n-1}, v), \frac{1}{2}[S(x_{2n}, x_{2n}, x_{2n-1}) + S(v, v, Gv)], \\ &\qquad \frac{1}{2}[2S(x_{2n-1}, x_{2n-1}, v) + S(v, v, Gv) + S(x_{2n}, x_{2n}, v)]\Big\}. \end{aligned} \tag{3.31}$$

Using equation (3.27), (3.28) and (3.29) in (3.31), we obtain

$$\begin{aligned} N(x_{2n-1}, x_{2n-1}, v) &\leq \max\Big\{\frac{1}{4}S(v, v, Gv), \frac{1}{2}[S(v, v, Gv) + S(v, v, Gv)], \\ &\qquad \frac{1}{2}[2.\frac{1}{4}S(v, v, Gv) + S(v, v, Gv) + \frac{1}{4}S(v, v, Gv)]\Big\} \\ &= \max\Big\{\frac{1}{4}S(v, v, Gv), S(v, v, Gv), \frac{7}{8}S(v, v, Gv)\Big\}, \end{aligned}$$

that is,

$$N(x_{2n-1}, x_{2n-1}, v) \leq S(v, v, Gv). \tag{3.32}$$

Now using equation (3.32) in (3.30), we obtain

$$\psi(S(x_{2n}, x_{2n}, Gv)) \leq \psi(S(v, v, Gv)) - \phi(S(v, v, Gv)). \tag{3.33}$$

On letting $n \to \infty$ in inequality (3.33), we obtain

$$\psi(S(v, v, Gv)) \leq \psi(S(v, v, Gv)) - \phi(S(v, v, Gv)), \tag{3.34}$$

which is a contradiction unless $S(v, v, Gv) = 0$. Hence, we conclude that $v = Gv$. This shows that v is a fixed point of G. As

$$\begin{aligned} S(v, v, Fv) &\leq 2S(v, v, Gv) + S(Fv, Fv, Gv) \\ &= 2S(v, v, v) + S(Gv, Gv, Fv) \text{ (by Lemma 2.5)} \\ &< S(v, v, Fv), \end{aligned}$$

which is a contradiction. Hence $S(v, v, Fv) = 0$, that is, $v = Fv$. Thus v is a common fixed point of F and G.

Now to show that the common fixed point of F and G is unique. For this, suppose v' is another common fixed point of F and G such that $z = Fz = Gz$ with $z \neq v$. From (3.1), we have

$$\begin{aligned} \psi(S(v, v, v')) &= \psi(S(Fv, Fv, Gv')) \\ &\leq \psi(N(v, v, v')) - \phi(N(v, v, v')) \\ &\leq \psi(S(v, v, v')) - \phi(S(v, v, v')), \end{aligned}$$

which is a contradiction unless $S(v, v, v') = 0$. Thus we conclude that $v = v'$. This shows that the common fixed point of F and G is unique. This completes the proof. $\square$

If we take $\max\Big\{S(x, y, z), \frac{1}{2}[S(x, x, Fx) + S(z, z, Gz)], \frac{1}{2}[S(x, x, Gz)$ $+S(z, z, Fx)]\Big\} = S(x, y, z)$ in Theorem 3.1, then we obtain the following result as corollary.

Corollary 3.2. *Let (X, S) be a complete S-metric space and $F, G \colon X \to X$ be two self mappings satisfying the inequality*

$$\psi(S(Fx, Fy, Gz)) \leq \psi(S(x, y, z)) - \phi(S(x, y, z)), \tag{3.35}$$

for all $x, y, z \in X$, and where ψ and ϕ are functions defined as in Theorem 3.1*. Then there exists the unique point $v \in X$ such that $v = Fv = Gv$.*

Remark 3.3. Corollary 3.2 extends Theorem 2.1 of Dutta and Choudhury [6] for a pair of maps from complete metric space to that in the setting of complete S-metric space considered in this paper.

If we take $\max\Big\{S(x,y,z), \frac{1}{2}[S(x,x,Fx)+S(z,z,Gz)], \frac{1}{2}[S(x,x,Gz)$ $+S(z,z,Fx)]\Big\} = S(x,y,z)$ and $F=G=T$ in Theorem 3.1, then we obtain the following result as corollary.

Corollary 3.4. *Let (X,S) be a complete S-metric space and $T\colon X\to X$ be a self mapping satisfying the inequality*

$$\psi(S(Tx,Ty,Tz)) \le \psi(S(x,y,z)) - \phi(S(x,y,z)), \tag{3.36}$$

for all $x,y,z\in X$, and where ψ and ϕ are continuous monotone and nondecreasing functions defined on $[0,\infty)$ with $\psi(t)=\phi(t)=0$ if and only if $t=0$. Then T has a unique fixed point in X.

Remark 3.5. Corollary 3.4 extends Theorem 2.1 of Dutta and Choudhury [6] from complete metric space to that in the setting of complete S-metric space considered in this paper.

Also as a corollary, we have the following result.

Theorem 3.6. *Let (X,S) be a complete S-metric space and $T\colon X\to X$ be a self mapping satisfying the inequality*

$$\psi(S(Tx,Ty,Tz)) \le \psi(N(x,y,z)) - \phi(N(x,y,z)), \tag{3.37}$$

for all $x,y,z\in X$, where $N(x,y,z)$ is given by

$$\begin{aligned} N(x,y,z) &= \max\Big\{S(x,y,z), \frac{1}{2}[S(x,x,Tx)+S(z,z,Tz)], \\ &\qquad \frac{1}{2}[S(x,x,Tz)+S(z,z,Tx)]\Big\} \end{aligned}$$

and where ψ, ϕ are functions defined as in Theorem 3.1. Then T has a unique fixed point in X.

Proof. Follows from Theorem 3.1 by taking $F=G=T$. □

Remark 3.7. Theorem 3.6 extends Theorem 2.2 of Doric [5] from complete metric space to that in the setting of complete S-metric space considered in this paper.

The following results are direct consequences of Theorem 3.6.

Corollary 3.8. *Let (X,S) be a complete S-metric space and $T\colon X\to X$ be a mapping. Suppose there exists $\alpha\in[0,1)$ such that*

$$\begin{aligned} S(Tx,Ty,Tz) &\le \alpha\max\Big\{S(x,y,z), \frac{1}{2}\big[S(x,x,Tx)+S(z,z,Tz)\big], \\ &\qquad \frac{1}{2}\big[S(x,x,Tz)+S(z,z,Tx)\big]\Big\}, \end{aligned} \tag{3.38}$$

for all $x,y,z\in X$. Then T has a unique fixed point.

Corollary 3.9. *Let (X,S) be a complete S-metric space and $T\colon X\to X$ be a mapping. Suppose there exist nonnegative real numbers α,β and γ with $\alpha+\beta+\gamma<1$ such that*

$$\begin{aligned} S(Tx,Ty,Tz) \;\le\; & \alpha\, S(x,y,z)+\frac{\beta}{2}\big[S(x,x,Tx)+S(z,z,Tz)\big] \\ & +\frac{\gamma}{2}\big[S(x,x,Tz)+S(z,z,Tx)\big], \end{aligned} \tag{3.39}$$

for all $x,y,z\in X$. Then T has a unique fixed point.

Proof. Follows from Corollary 3.8, by noting that

$$\begin{aligned} & \alpha\, S(x,y,z)+\frac{\beta}{2}\big[S(x,x,Tx)+S(z,z,Tz)\big]+\frac{\gamma}{2}\big[S(x,x,Tz)+S(z,z,Tx)\big] \\ & \le(\alpha+\beta+\gamma)\max\Big\{S(x,y,z),\frac{1}{2}\big[S(x,x,Tx)+S(z,z,Tz)\big], \\ & \qquad \frac{1}{2}\big[S(x,x,Tz)+S(z,z,Tx)\big]\Big\}. \end{aligned} \tag{3.40}$$

□

4. Illustrations

Now we give some examples in support of our results.

Example 4.1. Let $X=[0,1]\cup\{2,3,4,\dots\}$ and

$$S(x,y,z)=\begin{cases} |x-y-z| & \text{if } x,y,z\in[0,1],\ x\ne y\ne z,\\ x+y+z & \text{if at least one of } x \text{ or } y \text{ or } z\notin[0,1] \text{ and } x\ne y\ne z,\\ 0 & \text{if } x=y=z, \end{cases}$$

for all $x,y,z\in X$. Then (X,S) is a complete S-metric space.

Let $\psi\colon[0,\infty)\to[0,\infty)$ be defined as

$$\psi(t)=\begin{cases} t & \text{if } 0\le t\le 1,\\ t^2 & \text{if } t>1, \end{cases}$$

and let $\phi\colon[0,\infty)\to[0,\infty)$ be defined as

$$\phi(t)=\begin{cases} t^2 & \text{if } 0\le t\le 1,\\ \frac{1}{2} & \text{if } t>1. \end{cases}$$

Let $T\colon X\to X$ be defined as

$$T(x)=\begin{cases} x-x^2 & \text{if } 0\le x\le 1,\\ x-1 & \text{if } x\in\{2,3,4,\dots\}. \end{cases}$$

Without loss of generality, we assume that $x>y>z$ with $x=\frac{3}{4}$, $y=\frac{1}{2}$ and $z=\frac{1}{4}$ and discuss the following cases.

Case I: Let $x \in [0, 1]$. Then

$$\begin{aligned}
\psi(S(Tx,Ty,Tz)) &= S(Tx,Ty,Tz) \\
&= S\Big(x - x^2, y - y^2, z - z^2\Big) \\
&= \Big[\big(x - x^2\big) - \big(y - y^2\big) - \big(z - z^2\big)\Big] \\
&= \Big[(x - y - z) - (x^2 - y^2 - z^2)\Big] \\
&\leq \Big[(x - y - z) - (x - y - z)^2\Big] \\
&= S(x,y,z) - (S(x,y,z))^2 \\
&= \psi(S(x,y,z)) - \phi(S(x,y,z)).
\end{aligned}$$

Case II: Let $x \in \{2, 3, 4, \dots\}$. Then

$$S(Tx,Ty,Tz) = S(x-1, y-1, z-z^2) \ \text{ if } z \in [0,1]$$

or

$$\begin{aligned}
S(Tx,Ty,Tz) &= (x-1) + (y-1) + (z-z^2) \\
&= x + y + z - z^2 - 2 \\
&\leq x + y + z - 2,
\end{aligned}$$

and

$$S(Tx,Ty,Tz) = S(x-1, y-1, z-1) \ \text{ if } y, z \in \{2, 3, 4, \dots\}$$

or

$$\begin{aligned}
S(Tx,Ty,Tz) &= x - 1 + y - 1 + z - 1 = x + y + z - 3 \\
&\leq x + y + z - 2.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\psi(S(Tx,Ty,Tz)) &= S(Tx,Ty,Tz)^2 \\
&\leq (x + y + z - 2)^2 \\
&< (x + y + z - 1)(x + y + z + 1) \\
&= (x + y + z)^2 - 1 < (x + y + z)^2 - \frac{1}{2} \\
&= (S(x,y,z))^2 - \phi(S(x,y,z)) \\
&= \psi(S(x,y,z)) - \phi(S(x,y,z)).
\end{aligned}$$

Case III: Let $x = 2$. Then $y, z \in [0, 1]$, $T(x) = 1$ and $S(Tx,Ty,Tz) = 1 + (y - y^2) + (z - z^2) \leq 2$. So, we have $\psi(S(Tx,Ty,Tz)) \leq \psi(2) = 4$. Again

$S(x,y,z) = 2 + y + z$. So,

$$\begin{aligned}
\psi(S(x,y,z)) - \phi(S(x,y,z)) &= (2+y+z)^2 - \phi((2+y+z)^2) \\
&= (2+y+z)^2 - \frac{1}{2} \\
&= \frac{7}{2} + y^2 + z^2 + 2yz + 4y + 4z > 4 \\
&= \psi(S(Tx,Ty,Tz)).
\end{aligned}$$

Considering all the above cases, we conclude that the inequality used in Corollary 3.4 remains valid for ψ, ϕ and T constructed in above example and consequently by an application of Corollary 3.4, T has a unique fixed point which is 0.

Example 4.2. Let $X = [0,1]$. We define $S\colon X^3 \to \mathbb{R}_+$ by

$$S(x,y,z) = \begin{cases} 0 & \text{if } x = y = z, \\ \max\{x,y,z\} & \text{if otherwise,} \end{cases}$$

for all $x, y, z \in X$. Then (X,S) is a complete S-metric space. We define $F, G\colon X \to X$ and ψ, ϕ on $\mathbb{R}_+$ by $F(x) = \frac{x}{2}$, $G(x) = 0$, $\psi(t) = 2t^2$ and $\phi(t) = t^2$ for all $x \in X$ and $t \in \mathbb{R}_+$.

Without loss of generality we assume that $x > y > z$. Then

$$S(Fx,Fy,Gz) = \max\left\{\frac{x}{2}, \frac{y}{2}, 0\right\} = \frac{x}{2},$$

and

$$S(x,y,z) = \max\{x,y,z\} = x.$$

Now, we consider

$$\psi(S(Fx,Fy,Gz)) = 2.\frac{x^2}{4} = \frac{x^2}{2},$$

$$\psi(S(x,y,z)) = 2x^2 \text{ and } \phi(S(x,y,z)) = x^2.$$

Therefore, we have

$$\psi(S(Fx,Fy,Gz)) = \frac{x^2}{2} \leq 2x^2 - x^2 = x^2 = \psi(S(x,y,z)) - \phi(S(x,y,z)).$$

Thus the inequality (3.35) of Corollary 3.2 holds. Hence F and G satisfy all the hypothesis of Corollary 3.2 and 0 is the unique common fixed point of F and G.

Example 4.3. Let $X = [0,1]$. We define $S\colon X^3 \to \mathbb{R}_+$ by

$$S(x,y,z) = \begin{cases} 0 & \text{if } x = y = z, \\ \max\{x,y,z\} & \text{if otherwise,} \end{cases}$$

for all $x, y, z \in X$. Then (X, S) is a complete S-metric space. We define $T\colon X \to X$ and ψ, ϕ on $\mathbb{R}_+$ by $T(x) = \frac{2x}{3}$, $\psi(t) = \frac{9}{8}t^2$ and $\phi(t) = \frac{1}{8}t^2$ for all $x \in X$ and $t \in \mathbb{R}_+$.

Without loss of generality we assume that $x > y > z$. Then, we have

$$S(Tx, Ty, Tz) = \max\left\{\frac{2x}{3}, \frac{2y}{3}, \frac{2z}{3}\right\} = \frac{2x}{3},$$
$$S(x, y, z) = \max\{x, y, z\} = x,$$
$$S(x, x, Tx) = \max\{x, x, \frac{2x}{3}\} = x,$$
$$S(z, z, Tz) = \max\{z, z, \frac{2z}{3}\} = z,$$
$$S(x, x, Tz) = \max\{x, x, \frac{2z}{3}\} = x,$$
$$S(z, z, Tx) = \max\{z, z, \frac{2x}{3}\} = \frac{2x}{3}$$

and

$$N(x, y, z) = \max\left\{x, \frac{1}{2}(x + z), \frac{1}{2}(x + \frac{2x}{3})\right\} = x.$$

Now, we consider

$$\begin{aligned} \psi(S(Tx, Ty, Tz)) &= \frac{x^2}{2} \leq x^2 = [N(x, y, z)]^2 \\ &= \frac{9}{8}[N(x, y, z)]^2 - \frac{1}{8}[N(x, y, z)]^2 \\ &= \psi(N(x, y, z)) - \phi(N(x, y, z)), \end{aligned}$$

that is,

$$\psi(S(Tx, Ty, Tz)) \leq \psi(N(x, y, z)) - \phi(N(x, y, z)).$$

Thus the inequality (3.37) of Theorem 3.6 holds. Hence T satisfies all the hypothesis of Theorem 3.6 and 0 is the unique fixed point of T.

Example 4.4. Let $X = [0, 1]$. We define $S\colon X^3 \to \mathbb{R}_+$ by

$$S(x, y, z) = \begin{cases} 0 & \text{if } x = y = z, \\ \max\{x, y, z\} & \text{if otherwise,} \end{cases}$$

for all $x, y, z \in X$. Then (X, S) is a complete S-metric space. We define $F, G\colon X \to X$ and ψ, ϕ on $\mathbb{R}_+$ by $F(x) = \frac{x}{2}$, $G(x) = \frac{x}{3}$, $\psi(t) = \frac{4}{3}t^2$ and $\phi(t) = \frac{1}{3}t^2$ for all $x \in X$ and $t \in \mathbb{R}_+$.

Without loss of generality we assume that $x > y > z$. Then, we have

$$S(Fx, Fy, Gz) = \max\left\{\frac{x}{2}, \frac{y}{2}, \frac{z}{3}\right\} = \frac{x}{2},$$
$$S(x, y, z) = \max\{x, y, z\} = x,$$

$$S(x,x,Fx) = \max\{x,x,\frac{x}{2}\} = x,$$
$$S(z,z,Gz) = \max\{z,z,\frac{z}{3}\} = z,$$
$$S(x,x,Gz) = \max\{x,x,\frac{z}{3}\} = x,$$
$$S(z,z,Fx) = \max\{z,z,\frac{x}{2}\} = \frac{x}{2}$$

and

$$N(x,y,z) = \max\left\{x, \frac{1}{2}[x+z], \frac{1}{2}[x+\frac{x}{2}]\right\} = x.$$

Now, we consider

$$\begin{aligned}
\psi(S(Fx,Fy,Gz)) &= \frac{x^2}{3} \leq x^2 = [N(x,y,z)]^2 \\
&= \frac{4}{3}[N(x,y,z)]^2 - \frac{1}{3}[N(x,y,z)]^2 \\
&= \psi(N(x,y,z)) - \phi(N(x,y,z)),
\end{aligned}$$

that is,

$$\psi(S(Fx,Fy,Gz)) \leq \psi(N(x,y,z)) - \phi(N(x,y,z)).$$

Thus the inequality (3.1) of Theorem 3.1 holds. Hence F and G satisfy all the hypothesis of Theorem 3.1 and 0 is the unique common fixed point of F and G.

Example 4.5. Let $X = [0,1]$ and let S be the usual S-metric, that is, $S(x,y,z) = |x-z| + |y-z|$ for all $x,y,z \in X$ be the S-metric on X. Then (X,S) is a complete S-metric space. We define $F,G\colon X \to X$ and ψ, ϕ on $\mathbb{R}_+$ by $F(x) = x - x^2$, $G(x) = x$, $\psi(t) = t$ and $\phi(t) = \frac{t^2}{6}$ for all $x \in X$ and $t \in \mathbb{R}_+$. Without loss of generality, we assume that $x > y > z$. Then, we have

$$\begin{aligned}
\psi(S(Fx,Fy,Gz)) &= S(Fx,Fy,Gz) \\
&= |Fx - Gz| + |Fy - Gz| \\
&= |(x-x^2) - z| + |(y-y^2) - z| \\
&= [(x-x^2) - z] + [(y-y^2) - z] \\
&= [(x-z) + (y-z)] - (x^2+y^2) \\
&\leq [(x-z) + (y-z)] - (x-y)^2 \\
&\leq [(x-z) + (y-z)] - \frac{1}{6}[(x-z) + (y-z)]^2 \\
&= S(x,y,z) - \phi(S(x,y,z)) \\
&= \psi(S(x,y,z)) - \phi(S(x,y,z)),
\end{aligned}$$

that is,

$$\psi(S(Fx, Fy, Gz)) \leq \psi(S(x, y, z)) - \phi(S(x, y, z)).$$

Thus the inequality (3.35) of Corollary 3.2 holds. Hence F and G satisfy all the hypothesis of Corollary 3.2 and 0 is the unique common fixed point of F and G.

5. Applications

As an application of our results, we introduce some fixed point theorems of integral type.

Denote Φ the set of functions $\phi\colon [0, +\infty) \to [0, +\infty)$ satisfying the following hypothesis:

(H1) ϕ is a Lebesgue-integrable mapping on each compact subset of $[0, +\infty)$;
(H2) for any $\varepsilon > 0$ we have $\int_0^\varepsilon \phi(s)ds > 0$.

It is an easy matter, to see that the mapping $\psi\colon [0, +\infty) \to [0, +\infty)$ defined by

$$\psi(t) = \int_0^t \phi(s)ds \tag{5.1}$$

an altering distance function. Now, we have the following result.

Corollary 5.1. *Let (X, S) be a complete S-metric space. Let $F, G\colon X \to X$ be two self mappings and $\phi, \mu \in \Phi$, we have*

$$\begin{aligned} &\int_0^{S(Fx,Fy,Gz)} \phi(s)ds \qquad (5.2)\\ &\leq \int_0^{\max\left\{S(x,y,z), \frac{1}{2}[S(x,x,Fx)+S(z,z,Gz)], \frac{1}{2}[S(x,x,Gz)+S(z,z,Fx)]\right\}} \phi(s)ds \\ &\quad - \int_0^{\max\left\{S(x,y,z), \frac{1}{2}[S(x,x,Fx)+S(z,z,Gz)], \frac{1}{2}[S(x,x,Gz)+S(z,z,Fx)]\right\}} \mu(s)ds, \end{aligned}$$

for all $x, y, z \in X$. Then F and G have a unique common fixed point.

Proof. Follows from Theorem 3.1 by taking

$$\psi(t) = \int_0^t \phi(s)ds, \quad \phi(t) = \int_0^t \mu(s)ds. \tag{5.3}$$

□

If we take $F = G = T$ in Corollary 5.1, then we obtain the following result.

Corollary 5.2. *Let (X,S) be a complete S-metric space and $T\colon X\to X$ be a mapping, and for $\phi,\mu\in\Phi$, we have*

$$\begin{aligned}&\int_0^{S(Tx,Ty,Tz)}\phi(s)ds \qquad (5.4)\\ &\leq \int_0^{\max\left\{S(x,y,z),\frac{1}{2}[S(x,x,Tx)+S(z,z,Tz)],\frac{1}{2}[S(x,x,Tz)+S(z,z,Tx)]\right\}}\phi(s)ds\\ &\quad-\int_0^{\max\left\{S(x,y,z),\frac{1}{2}[S(x,x,Tx)+S(z,z,Tz)],\frac{1}{2}[S(x,x,Tz)+S(z,z,Tx)]\right\}}\mu(s)ds\end{aligned}$$

for all $x,y,z\in X$. Then T has a unique fixed point.

Proof. Follows from Theorem 3.1 by taking $F=G=T$ and

$$\psi(t)=\int_0^t\phi(s)ds,\quad \phi(t)=\int_0^t\mu(s)ds. \qquad (5.5)$$

□

If we take $\max\Big\{S(x,y,z),\frac{1}{2}[S(x,x,Fx)+S(z,z,Gz)],\frac{1}{2}[S(x,x,Gz)$ $+S(z,z,Fx)]\Big\}=S(x,y,z)$ in Corollary 5.1, then we obtain the following result.

Corollary 5.3. *Let (X,S) be a complete S-metric space. Let $F,G\colon X\to X$ be two self mappings and $\phi,\mu\in\Phi$, we have*

$$\int_0^{S(Fx,Fy,Gz)}\phi(s)ds \;\leq\; \int_0^{S(x,y,z)}\phi(s)ds-\int_0^{S(x,y,z)}\mu(s)ds \qquad (5.6)$$

for all $x,y,z\in X$. Then F and G have a unique common fixed point.

If we take $F=G=T$ in Corollary 5.3, then we obtain the following result.

Corollary 5.4. *Let (X,S) be a complete S-metric space. Let $T\colon X\to X$ be a mapping and $\phi,\mu\in\Phi$, we have*

$$\int_0^{S(Tx,Ty,Tz)}\phi(s)ds \;\leq\; \int_0^{S(x,y,z)}\phi(s)ds-\int_0^{S(x,y,z)}\mu(s)ds \qquad (5.7)$$

for all $x,y,z\in X$. Then T has a unique fixed point.

The following result is a special case of Corollary 5.4.

Corollary 5.5. *Let (X,S) be a complete S-metric space. Let $T\colon X\to X$ be a mapping. Suppose that there exists $k\in[0,1)$ such that for $\phi\in\Phi$, we have*

$$\int_0^{S(Tx,Ty,Tz)}\phi(s)ds\leq k\int_0^{S(x,y,z)}\phi(s)ds \qquad (5.8)$$

for all $x, y, z \in X$. Then T has a unique fixed point.

Proof. Follows from Corollary 5.4 by taking $\mu(s) = (1-k)\phi(s)$ where $0 \leq k < 1$. $\square$

Remark 5.6. Corollary 5.5 extends Theorem 2.1 of Branciari [4] from complete metric space to the setting of complete S-metric space.

6. Conclusion

In this paper, we define generalized $(\psi - \phi)$-weak contractions in S-metric space and establish some unique common fixed point theorems in the framework of complete S-metric spaces and we give some examples in support of our results. Also an application of our results, we obtained some fixed point theorems of integral type. Especially, Theorem 3.1 and Theorem 3.6 respectively extend and generalize Theorem 2.1 and 2.2 of Doric [5] and Corollary 3.2 and 3.4 extend and generalize Theorem 2.1 of Dutta and Choudhury [6] from complete metric space to that in the setting of complete S-metric space.

Our results also extend and generalize several results from the existing literature regarding S-metric space.

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