



## GENERALIZED PSEUDO-DIFFERENTIAL OPERATORS INVOLVING FRACTIONAL FOURIER TRANSFORM

B. B. Waphare<sup>1</sup> and P. D. Pansare<sup>2</sup>

<sup>1</sup>MIT Arts, Commerce and Science College, Alandi (D), Pune, India  
e-mail: [balasahebwapahare@gmail.com](mailto:balasahebwapahare@gmail.com)

<sup>2</sup>Research Scholar, Savitribai Phule Pune University, Pune, India  
e-mail: [pradippansare@gmail.com](mailto:pradippansare@gmail.com)

**Abstract.** Generalized pseudo-differential operators (PDO) involving fractional Fourier transform associate with the symbol  $a(x, y)$  whose derivatives satisfy certain growth condition is defined. The product of two generalized pseudo-differential operators is shown to be a generalized pseudo-differential operator.

### 1. INTRODUCTION

If  $\phi \in L_1(\mathbb{R})$  then the Fourier transform of a function is defined by

$$\tilde{\phi}(y) = (\mathcal{F}(\phi))(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixy} \phi(x) dx, \quad y \in \mathbb{R} \quad (1.1)$$

and if  $\tilde{\phi} \in L_1(\mathbb{R})$  then the inverse Fourier transform is given by

$$\phi(x) = (\mathcal{F}^{-1}(\tilde{\phi}))(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ixy} \tilde{\phi}(y) dy, \quad x \in \mathbb{R}. \quad (1.2)$$

The one dimensional fractional Fourier transform ([5], [6]) with parameters  $\alpha, \beta$  of  $\phi(x)$  denoted by  $(\mathcal{F}_{\alpha, \beta}(\phi))(t) = \hat{\phi}_{\alpha, \beta}(t)$  is given by

$$(\mathcal{F}_{\alpha, \beta}(\phi))(t) = \hat{\phi}_{\alpha, \beta}(t) = \int_{\mathbb{R}} K_{\alpha, \beta}(x, t) \phi(x) dx, \quad (1.3)$$

---

<sup>0</sup>Received August 3, 2020. Revised October 4, 2020. Accepted February 5, 2021.

<sup>0</sup>2010 Mathematics Subject Classification: 47G30, 46F12.

<sup>0</sup>Keywords: Generalized pseudo-differential operator, Fourier transform, fractional Fourier transform.

<sup>0</sup>Corresponding author: B .B. Waphare([balasahebwapahare@gmail.com](mailto:balasahebwapahare@gmail.com)).

where  $K_{\alpha,\beta}(x,t) = \begin{cases} C_{\alpha,\beta} e^{i(x^2+t^2) \cot(\alpha-\beta) - ixt \csc(\alpha-\beta)} & \text{if } (\alpha - \beta) \neq n\pi, \\ \frac{1}{(2\pi)^{\alpha+\beta}} e^{-ixt} & \text{if } (\alpha - \beta) = \frac{\pi}{2}, \end{cases}$

for  $n$  is an integer and

$$C_{\alpha,\beta} = (2\pi i \sin(\alpha - \beta))^{-(\alpha+\beta)} e^{i(\alpha-\beta)(\alpha+\beta)} = \left( \frac{1 - i \cot(\alpha - \beta)}{2\pi} \right)^{\alpha+\beta}.$$

It is inverted by

$$\phi(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \overline{K_{\alpha,\beta}(x,t)} (\mathcal{F}_{\alpha,\beta}(\phi))(t) dt, \quad (1.4)$$

where

$$\overline{K_{\alpha,\beta}(x,t)} = C'_{\alpha,\beta} e^{-i(x^2+t^2) \cot(\alpha-\beta) + ixt \csc(\alpha - \beta)} \quad (1.5)$$

and

$$\begin{aligned} C'_{\alpha,\beta} &= \frac{(2\pi i \sin(\alpha - \beta))^{\alpha+\beta}}{\sin(\alpha - \beta)} e^{-i(\alpha-\beta)(\alpha+\beta)} \\ &= \frac{(2\pi i)^{\alpha+\beta} (\sin(\alpha - \beta))^{\alpha+\beta}}{\sin(\alpha - \beta)} (\cos(\alpha - \beta) - i \sin(\alpha - \beta))^{\alpha+\beta} \\ &= \frac{(2\pi i)^{\alpha+\beta} (\sin(\alpha - \beta))^{2(\alpha+\beta)}}{\sin(\alpha - \beta) (\sin(\alpha - \beta))^{\alpha+\beta}} (\cos(\alpha - \beta) - i \sin(\alpha - \beta))^{\alpha+\beta} \\ &= \frac{(2\pi i)^{\alpha+\beta} (\sin(\alpha - \beta))^{2(\alpha+\beta)}}{\sin(\alpha - \beta)} (\cot(\alpha - \beta) - i)^{\alpha+\beta} \\ &= \frac{(2\pi)^{\alpha+\beta} (\sin(\alpha - \beta))^{2(\alpha+\beta)}}{\sin(\alpha - \beta)} (1 + i \cot(\alpha - \beta))^{\alpha+\beta} \\ &= \frac{(\sin(\alpha - \beta))^{2(\alpha+\beta)}}{\sin(\alpha - \beta)} [2\pi(1 + i \cot(\alpha - \beta))]^{\alpha+\beta}. \end{aligned}$$

**Definition 1.1.** A tempered distribution  $\phi \in H^s(\mathbb{R})$  (the Sobolev space),  $s \in \mathbb{R}$ , if its fractional Fourier transform  $\mathcal{F}_{\alpha,\beta}\phi$  corresponding to a locally integrable function  $(\mathcal{F}_{\alpha,\beta}(\phi))(t)$  over  $\mathbb{R}$  such that

$$\|\phi\|_{H^s} = \left( \int_{\mathbb{R}} \left| (1 + |t|^2)^{s/2} \mathcal{F}_{\alpha,\beta}(\phi)(t) \right|^2 dt \right)^{1/2} < \infty. \quad (1.6)$$

This space is complete with respect to the norm  $\|\phi\|_{H^s}$ .

## 2. PROPERTIES OF FRACTIONAL FOURIER TRANSFORM

First we recall the definition of the Schwartz space  $S(\mathbb{R})$ .

**Definition 2.1.** The space  $S$ , the so called space of smooth functions of rapid descent is defined as follows:  $\phi$  is member of  $S$  if and only if it is complex valued  $C^\infty$  function on  $\mathbb{R}$  and for every choice of  $b$  and  $c$  of non negative integers, it satisfies

$$\Gamma_{b,c}(\phi) = \sup_{x \in \mathbb{R}} |x^b D^c \phi(x)| < \infty. \quad (2.1)$$

**Proposition 2.2.** Let  $K_{\alpha,\beta}(x, t)$  be the kernel of fractional Fourier transform and  $\Delta_x^r = \left(\frac{d}{dx} - ix \cot(\alpha - \beta)\right)^r$ . Then

$$\Delta_x^r K_{\alpha,\beta}(x, t) = (-it \csc(\alpha - \beta))^r K_{\alpha,\beta}(x, t),$$

for all  $r \in \mathbb{N}_0$ .

*Proof.*

$$\begin{aligned} \frac{d}{dx} [K_{\alpha,\beta}(x, y)] &= C_{\alpha,\beta} \frac{d}{dx} \left[ e^{i[(x^2+t^2) \cos(\alpha-\beta)]^{(\alpha+\beta)} - ixt \csc(\alpha-\beta)} \right] \\ &= K_{\alpha,\beta}(x, t) i(x \cot(\alpha - \beta) - t \csc(\alpha - \beta)). \end{aligned}$$

So that

$$\left(\frac{d}{dx} - ix \cot(\alpha - \beta)\right) K_{\alpha,\beta}(x, t) = (-it \csc(\alpha - \beta)) K_{\alpha,\beta}(x, t).$$

Continuing in this way, we get

$$\left(\frac{d}{dx} - ix \cot(\alpha - \beta)\right)^r K_{\alpha,\beta}(x, t) = (-it \csc(\alpha - \beta))^r K_{\alpha,\beta}(x, t).$$

□

**Proposition 2.3.** For all  $\phi \in S(\mathbb{R})$ , we have

$$\int_{\mathbb{R}} \Delta_x^r K_{\alpha,\beta}(x, t) \phi(x) dx = \int_{\mathbb{R}} K_{\alpha,\beta}(x, t) (\Delta'_x)^r \phi(x) dx,$$

for all  $r \in \mathbb{N}_0$ , where  $\Delta'_x = -\left(\frac{d}{dx} + ix \cot(\alpha - \beta)\right)$ .

*Proof.* First we prove

$$\int_{\mathbb{R}} \Delta_x K_{\alpha,\beta}(x, t) \phi(x) dx = \int_{\mathbb{R}} K_{\alpha,\beta}(x, t) (\Delta'_x) \phi(x) dx.$$

Using integration by parts, we have

$$\begin{aligned} &\int_{\mathbb{R}} \left(\frac{d}{dx} - ix \cot(\alpha - \beta)\right) K_{\alpha,\beta}(x, t) \phi(x) dx \\ &= - \int_{\mathbb{R}} K_{\alpha,\beta}(x, t) \left(\frac{d}{dx} + ix \cot(\alpha - \beta)\right) \phi(x) dx. \end{aligned}$$

Therefore,

$$\int_{\mathbb{R}} \Delta_x K_{\alpha,\beta}(x,t) \phi(x) dx = \int_{\mathbb{R}} K_{\alpha,\beta}(x,t) (\Delta'_x) \phi(x) dx.$$

In general, we have

$$\int_{\mathbb{R}} \Delta_x^r K_{\alpha,\beta}(x,t) \phi(x) dx = \int_{\mathbb{R}} K_{\alpha,\beta}(x,t) (\Delta'_x)^r \phi(x) dx.$$

□

**Proposition 2.4.** *Let  $\phi \in S$ . Then*

$$(\mathcal{F}_{\alpha,\beta}(\Delta'_x)^r \phi(x))(t) = (-it \csc(\alpha, \beta))^r (\mathcal{F}_{\alpha,\beta} \phi(x))(t)$$

for all  $r \in \mathbb{N}_0$ .

*Proof.* By using Proposition 2.2 and Proposition 2.3, we have

$$\begin{aligned} (\mathcal{F}_{\alpha,\beta}(\Delta'_x)^r \phi(x))(t) &= \int_{\mathbb{R}} K_{\alpha,\beta}(x,t) (\Delta'_x)^r \phi(x) dx \\ &= \int_{\mathbb{R}} \Delta_x^r K_{\alpha,\beta}(x,t) \phi(x) dx \\ &= (it \csc(\alpha - \beta))^r \int_{\mathbb{R}} K_{\alpha,\beta}(x,t) \phi(x) dx \\ &= (-it \csc(\alpha, \beta))^r (\mathcal{F}_{\alpha,\beta} \phi(x))(t). \end{aligned}$$

□

### 3. PRODUCT OF TWO GENERALIZED PSEUDO DIFFERENTIAL OPERATORS

Sobolev space  $H^s(\mathbb{R})$  is defined by (1.6). The variant of this, denoted by  $H^s(\mathbb{R} \times \mathbb{R})$ , where  $s \in \mathbb{R}$  defined by [2] as follows:

A tempered distribution  $\phi \in S'(\mathbb{R}^2)$  is said to belong to  $H^s(\mathbb{R} \times \mathbb{R})$ , if  $(\mathcal{F}_{\alpha,\beta} \phi)(t, \eta)$  is locally integrable on  $\mathbb{R}^2$  and

$$[(1 + |t|^2)(1 + |\eta|^2)]^{s/2} (\mathcal{F}_{\alpha,\beta} \phi)(t, \eta) \in L^2(\mathbb{R}^2).$$

A norm in this space is defined as

$$\begin{aligned} \|\phi\|_{H^s(\mathbb{R} \times \mathbb{R})} &= \left( \int_{\mathbb{R}} \int_{\mathbb{R}} |(1 + |t|^2)^{s/2} (1 + |\eta|^2)^{s/2} (\mathcal{F}_{\alpha,\beta} \phi)(t, \eta)|^2 dt d\eta \right)^{1/2} \\ &< \infty, \quad \phi \in S'(\mathbb{R} \times \mathbb{R}). \end{aligned} \quad (3.1)$$

**Definition 3.1.** The function  $a(x, y) : C^\infty(\mathbb{R} \times \mathbb{R}) \mapsto \mathbb{C}$  belongs to the class  $S^m$  if and only if for all  $q, v, b \in \mathbb{N}_0$ , there exist  $D_{b,v,m} > 0$  such that

$$(1 + |x|^q) |D_x^b D_y^v a(x, y)| \leq D_{b,v,m} (1 + |y|)^{m-v}. \quad (3.2)$$

**Lemma 3.2.** *For any symbol  $a \in S^m$ ,  $m \in \mathbb{R}$  and  $l > 1 \in \mathbb{N}$ , there exists a positive constant  $C_m$  such that*

$$|(\mathcal{F}_{\alpha,\beta}a)(t,\eta)| \leq C_m(1+|\eta|)^m(1+t^2 \csc^2(\alpha-\beta))^{-l/2}. \quad (3.3)$$

*Proof.* We know that

$$(\mathcal{F}_{\alpha,\beta}a)(t,\eta) = C_{\alpha,\beta} \int_{\mathbb{R}} e^{(i(x^2+t^2) \cot(\alpha-\beta))^{(\alpha+\beta)} - ixt \csc(\alpha-\beta)} a(x,\eta) dx.$$

So that

$$(1+it \csc(\alpha-\beta))^l (\mathcal{F}_{\alpha,\beta}a)(t,\eta) = \int_{\mathbb{R}} K_{\alpha,\beta}(x,t)(1-\Delta'_x)^l a(x,\eta) dx \quad (3.4)$$

for all  $\eta, t \in \mathbb{R}, l > 1 \in \mathbb{N}$ . Now

$$\begin{aligned} (1-\Delta'_x)^l a(x,\eta) &= \sum_{r=0}^l \binom{l}{r} (-1)^r (\Delta'_x)^r a(x,\eta) \\ &= \sum_{r=0}^l \binom{l}{r} (-1)^r \sum_{k=0}^r P(x,k) D_x^k a(x,\eta) \\ &= \sum_{r=0}^l \binom{l}{r} (-1)^r \sum_{k=0}^r \sum_{l_1=0}^k d_{l_1} x^{l_1} D_x^k a(x,\eta), \end{aligned}$$

where  $P(x,k)$  is a polynomial of maximum degree  $r$ . Using (3.2) in above equation, we have

$$\left| (1-\Delta'_x)^l a(x,\eta) \right| \leq \sum_{r=0}^l \binom{l}{r} \sum_{k=0}^r \sum_{l_1=0}^k |d_{l_1}| D_{k,m}(1+|\eta|)^m (1+|x|)^{-l}. \quad (3.5)$$

Therefore by (3.4) and (3.5), we get

$$\begin{aligned} &|(1+it \csc(\alpha-\beta))^l| |(\mathcal{F}_{\alpha,\beta}a)(t,\eta)| \\ &\leq |C_{\alpha,\beta}| \sum_{r=0}^l \binom{l}{r} \sum_{k=0}^r \sum_{l_1=0}^k |d_{l_1}| D_{k,m}(1+|\eta|)^m \times \int_{\mathbb{R}} (1+|x|)^{-l} dx. \end{aligned}$$

Since integral is convergent for large value of  $l$ , there exists a constant  $C_m > 0$  depending on  $l, \alpha, \beta, k, r, m$  such that

$$|(\mathcal{F}_{\alpha,\beta}a)(t,\eta)| \leq C_m(1+|\eta|)^m(1+t^2 \csc^2(\alpha-\beta))^{-l/2}.$$

□

A linear partial differential operator  $A(x, \Delta'_x)$  on  $\mathbb{R}$  is given by

$$A(x, \Delta'_x) = \sum_{r=0}^m a_r(x) (\Delta'_x)^r, \quad (3.6)$$

where the coefficient  $a_r(x)$  are functions defined on  $\mathbb{R}$  and  $\Delta'_x = -(\frac{d}{dx} + ix \cot(\alpha - \beta))$ . If we replace  $(\Delta'_x)^r$  in (3.6) by monomial  $(-it \csc(\alpha - \beta))^r$  in  $\mathbb{R}$ , then we obtain the so called symbol

$$A(x, t) = \sum_{r=0}^m a_r(x) (-it \csc(\alpha - \beta))^r. \quad (3.7)$$

In order to get another representation of the operator  $A(x, \Delta'_x)$ , let  $\phi \in S(\mathbb{R})$ , by (1.3), (1.4) and Proposition 2.4, we have

$$\begin{aligned} (A(x, \Delta'_x)\phi)(x) &= \sum_{r=0}^m a_r(x) \mathcal{F}_{\alpha, \beta}^{-1} \mathcal{F}_{\alpha, \beta} (\Delta'_x)^r \phi(x) \\ &= \sum_{r=0}^m a_r(x) \mathcal{F}_{\alpha, \beta}^{-1} (-it \csc(\alpha - \beta))^r (\mathcal{F}_{\alpha, \beta} \phi)(t) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \overline{K_{\alpha, \beta}(x, t)} A(x, t) (\mathcal{F}_{\alpha, \beta} \phi)(t) dt, \end{aligned}$$

where  $\overline{K_{\alpha, \beta}(x, t)}$  is defined by (1.5). If we replace the symbol  $A(x, t)$  by more general symbol  $a(x, t)$  which is no longer polynomial in  $t$ , we get the generalized pseudo-differential operator  $A_{a, \alpha, \beta}$  defined below.

**Definition 3.3.** Let  $a(x, y)$  be a complex valued function belonging to the space  $C^\infty(\mathbb{R} \times \mathbb{R})$ , and its derivatives satisfy certain growth conditions (3.2). Then the generalized pseudo-differential operator  $A_{a, \alpha, \beta}$  associated with the symbol  $a(x, y)$  is defined by

$$(A_{a, \alpha, \beta} \phi) = \frac{1}{2\pi} \int_{\mathbb{R}} \overline{K_{\alpha, \beta}(x, y)} a(x, y) (\mathcal{F}_{\alpha, \beta} \phi)(y) dy, \quad (3.8)$$

where  $(\mathcal{F}_{\alpha, \beta} \phi)(y)$  is defined in (1.3).

For pseudo-differential operator involving Fourier transform, we may refer to [3, 4].

**Theorem 3.4.** For any symbol  $a(x, y) \in S^m$ , the associated operator  $(A_{a, \alpha, \beta} \phi)(x)$  can be represented by

$$\begin{aligned} (A_{a, \alpha, \beta} \phi)(x) &= \left( \frac{C'_{\alpha, \beta}}{2\pi} \right)^2 \int_{\mathbb{R}} \int_{\mathbb{R}} \left( e^{-i(x^2 + t^2 + y^2 + \eta^2) \cot(\alpha - \beta)} \right)^{(\alpha + \beta)} \\ &\quad \times e^{i(xt + y\eta)} (\mathcal{F}_{\alpha, \beta} a)(t, \eta) (\mathcal{F}_{\alpha, \beta} \phi)(\eta) dt d\eta, \end{aligned} \quad (3.9)$$

where  $\phi \in S(\mathbb{R})$ .

*Proof.* Proof can be completed by making use of (1.3) and (1.4) in (3.8).  $\square$

**Corollary 3.5.** For any symbol  $a \in S^m$  and  $\phi \in S(\mathbb{R})$ ,

$$\mathcal{F}_{\alpha,\beta} [A_{a,\alpha,\beta}\phi] (t, \eta) = (\mathcal{F}_{\alpha,\beta} a) (t, \eta) (\mathcal{F}_{\alpha,\beta} \phi) (\eta), \quad (3.10)$$

where the fractional Fourier transform is taken with respect to all the variables  $x$  and  $y$ .

**Definition 3.6.** Let  $\sigma(x, t) \in S^{m_1}$  and  $\tau(y, t) \in S^{m_2}$ . Then the product of two generalized pseudo-differential operators  $B_{\tau,\alpha,\beta}$  and  $A_{\sigma,\alpha,\beta}$  associated with symbol  $\tau(y, t)$  and  $\sigma(x, t)$  respectively is defined by

$$(B_{\tau,\alpha,\beta} A_{\sigma,\alpha,\beta} \phi) (x, y) = \frac{1}{2\pi} \int_{\mathbb{R}} \overline{K_{\alpha,\beta}(x, t)} \tau(y, t) \mathcal{F}_{\alpha,\beta} [A_{\sigma,\alpha,\beta}] (t, y) dt, \quad (3.11)$$

if the integral is convergent.

**Theorem 3.7.** Let  $\sigma(x, t) \in S^{m_1}$  and  $\tau(y, t) \in S^{m_2}$ . Then the product of two pseudo-differential operators  $B_{\tau,\alpha,\beta}$  and  $A_{\sigma,\alpha,\beta}$  is again a generalized pseudo-differential operator whose symbol is in  $S^{m_1+m_2}$ .

*Proof.* Let  $\phi \in S(\mathbb{R})$ . Then from the Definition 3.6 and (3.10), we have

$$\begin{aligned} & (B_{\tau,\alpha,\beta} A_{\sigma,\alpha,\beta} \phi) (x, y) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \overline{K_{\alpha,\beta}(x, t)} \tau(y, t) (\mathcal{F}_{\alpha,\beta} \sigma)(y, t) (\mathcal{F}_{\alpha,\beta} \phi)(t) dt. \end{aligned} \quad (3.12)$$

This shows that  $\tau(y, t) (\mathcal{F}_{\alpha,\beta} \sigma)(y, t)$  is symbol of the product  $B_{\tau,\alpha,\beta} A_{\sigma,\alpha,\beta}$ .

Now, we have to show that this symbol is in  $S^{m_1+m_2}$ .

For  $a_1, b_1 \in \mathbb{N}_0$ , we have

$$\begin{aligned} & |(D_y)^{a_1} (D_t)^{b_1} \tau(y, t) (\mathcal{F}_{\alpha,\beta} \sigma)(y, t)| \\ & \leq \sum_{a_2=0}^{a_1} \binom{a_1}{a_2} \sum_{b_2=0}^{b_1} \binom{b_1}{b_2} |(D_y)^{a_1-a_2} (D_t)^{b_1-b_2} \tau(y, t)| \\ & \quad \times |(D_y)^{a_2} (D_t)^{b_2} (\mathcal{F}_{\alpha,\beta} \sigma)(y, t)|. \end{aligned} \quad (3.13)$$

Now,

$$\begin{aligned}
& (D_y)^{a_2} (D_t)^{b_2} (\mathcal{F}_{\alpha, \beta} \sigma)(y, t) \\
&= C_{\alpha, \beta} \int_{\mathbb{R}} D_y^{a_2} \left( e^{[i(x^2+y^2) \cot(\alpha-\beta)](\alpha+\beta) - ixy \csc(\alpha-\beta)} \right) D_t^{b_2} \sigma(y, t) dx \\
&= C_{\alpha, \beta} \int_{\mathbb{R}} \sum_{a_3=0}^{a_2} \binom{a_2}{a_3} D_y^{a_2-a_3} \left( e^{[i(x^2+y^2) \cot(\alpha-\beta)](\alpha+\beta)} \right) \\
&\quad \times D_y^{a_3} \left( e^{-ixy \csc(\alpha-\beta)} \right) D_t^{b_2} \sigma(y, t) dx \\
&= C_{\alpha, \beta} \int_{\mathbb{R}} \sum_{a_3=0}^{a_2} \binom{a_2}{a_3} e^{[i(x^2+y^2) \cot(\alpha-\beta)](\alpha+\beta)} \\
&\quad \times P_{a_2-a_3}(y, (i \cot(\alpha-\beta))(\alpha+\beta)) \\
&\quad \times (-ix \csc(\alpha-\beta))^{a_3} e^{-ix \csc(\alpha-\beta)} D_t^{b_2} \sigma(y, t) dx \\
&= C_{\alpha, \beta} \sum_{a_3=0}^{a_2} \binom{a_2}{a_3} \sum_{s=0}^{a_2-a_3} a_s \cot(\alpha-\beta) y^s (-i \csc(\alpha-\beta))^{a_3} \\
&\quad \times \int_{\mathbb{R}} e^{[i(x^2+y^2) \cot(\alpha-\beta)](\alpha+\beta) - ixy \csc(\alpha-\beta)} x^{a_3} D_t^{b_2} \sigma(y, t) dx.
\end{aligned}$$

Thus,

$$\begin{aligned}
& |(D_y)^{a_2} (D_t)^{b_2} (\mathcal{F}_{\alpha, \beta} \sigma)(y, t)| \\
&\leq |C_{\alpha, \beta}| \sum_{a_3=0}^{a_2} \binom{a_2}{a_3} \sum_{s=0}^{a_2-a_3} |a_s \cot(\alpha-\beta)| \\
&\quad \times |y|^s |\csc(\alpha-\beta)|^{a_3} \int_{\mathbb{R}} |x^{a_3}| |D_t^{b_2} \sigma(y, t)| dx \\
&\leq |C_{\alpha, \beta}| \sum_{a_3=0}^{a_2} \binom{a_2}{a_3} \sum_{s=0}^{a_2-a_3} |a_s \cot(\alpha-\beta)| \\
&\quad \times (1+|y|)^s |\csc(\alpha-\beta)|^{a_3} \int_{\mathbb{R}} (1+|x|)^{a_3} |D_t^{b_2} \sigma(y, t)| dx \\
&\leq D_{b_3, m} |C_{\alpha, \beta}| \sum_{a_3=0}^{a_2} \binom{a_2}{a_3} \sum_{s=0}^{a_2-a_3} |a_s \cot(\alpha-\beta)| \\
&\quad \times (1+|y|)^s |\csc(\alpha-\beta)|^{a_3} (1+|t|)^{m_1-b_2} \int_{\mathbb{R}} (1+|x|)^{a_3-q} dx.
\end{aligned}$$

The  $x$ -integral being convergent for  $q > a_3 + 1$ , therefore

$$|(D_y)^{a_2} (D_t)^{b_2} (\mathcal{F}_{\alpha, \beta} \sigma)(y, t)| \leq L(1+|y|)^s (1+|t|)^{m_1-b_2}, \quad (3.14)$$

where  $L$  is a positive constant depending on  $\alpha - \beta, a_2, a_3, s, b_2, m$  and  $q$ .



By (3.13), (3.14) and (3.2), we have

$$\begin{aligned} & |(D_y)^{a_2}(D_t)^{b_2}(\mathcal{F}_{\alpha,\beta}\sigma)(y,t)| \\ & \leq L \sum_{a_2=0}^{a_1} \binom{a_1}{a_2} \sum_{b_2=0}^{b_1} \binom{b_1}{b_2} C_{a_1-a_2,b_1-b_2,m_2,s} (1+|t|)^{m_1+m_2-b_1} \\ & \leq L_1(1+|t|)^{m_1+m_2-b_1}, \end{aligned}$$

where  $L_1$  is a positive constant depending on  $a_1, a_2, b_1, b_2, m_2, s$  and  $L$ . Therefore the symbol of the product  $B_{\tau,\alpha,\beta}A_{\sigma,\alpha,\beta}$  is in  $S^{m_1+m_2}$ .  $\square$

**Theorem 3.8.** *Let  $\sigma(x,t) \in S^{m_1}$  and  $\tau(y,t) \in S^{m_2}$ . Then for certain  $C_1 > 0, m_1, m_2 \in \mathbb{R}_+$ ,*

$$\| (B_{\tau,\alpha,\beta}A_{\sigma,\alpha,\beta}\phi)(x,y) \|_{H^s(\mathbb{R} \times \mathbb{R})}^2 \leq C_1 \| \phi \|_{H^{s+m_1+m_2}(\mathbb{R})}^2,$$

for all  $\phi \in S(\mathbb{R})$ .

*Proof.* From Definition 3.6 and (3.12), it follows that  $(B_{\tau,\alpha,\beta}A_{\sigma,\alpha,\beta}\phi)(x,y)$  has the fractional Fourier transform equal to  $\tau(y,t)(\mathcal{F}_{\alpha,\beta}\sigma)(y,t)$ . Therefore,

$$\begin{aligned} & \| (B_{\tau,\alpha,\beta}A_{\sigma,\alpha,\beta}\phi)(x,y) \|_{H^s(\mathbb{R} \times \mathbb{R})}^2 \\ & = \| (1+|t|^2)^{s/2}(1+|y|^2)^{s/2}\mathcal{F}_{\alpha,\beta}(B_{\tau,\alpha,\beta}A_{\sigma,\alpha,\beta}\phi)(t,y) \|_{L^2(\mathbb{R} \times \mathbb{R})}^2 \\ & = \int_{\mathbb{R}} \int_{\mathbb{R}} |(1+|t|^2)^{s/2}(1+|y|^2)^{s/2}\tau(y,t) \\ & \quad \times (\mathcal{F}_{\alpha,\beta}\sigma)(y,t)(\mathcal{F}_{\alpha,\beta}\phi)(t)|^2 dy dt. \end{aligned}$$

Since from (3.2)

$$|\tau(y,t)| \leq C_{m_2}(1+|t|)^{m_2}(1+|y|^2)^{-l},$$

and from (3.3)

$$|(\mathcal{F}_{\alpha,\beta}\sigma)(y,t)| \leq C_{m_1}(1+|t|)^{m_1}(1+y^2 \csc^2(\alpha-\beta))^{-l/2}.$$

We know that

$$(1+|t|)^m \leq \begin{cases} 2^{m/2}(1+|t|^2)^{m/2} & \text{if } m \geq 0, \\ (1+|t|^2)^{m/2} & \text{if } m < 0. \end{cases}$$

Therefore,

$$(1+|t|)^m \leq \max(1, 2^{m/2})(1+|t|^2)^{m/2} = C_m(1+|t|^2)^{m/2},$$

where  $C_m = \max(1, 2^{m/2})$ . Thus

$$\begin{aligned} & \| (B_{\tau, \alpha, \beta} A_{\sigma, \alpha, \beta} \phi)(x, y) \|_{H^s(\mathbb{R} \times \mathbb{R})}^2 \\ & \leq C_{m_1} C_{m_2} \int_{\mathbb{R}} (1 + |y|^2)^{s-2l} (1 + y^2 \csc^2(\alpha - \beta))^{-l} dy \\ & \quad \times \int_{\mathbb{R}} |C_m^2 (1 + |t|^2)^{\frac{s+m_1+m_2}{2}} (\mathcal{F}_{\alpha, \beta} \phi)(t)|^2 dt. \end{aligned}$$

Now we have to show that the first integral is convergent. Consider

$$\begin{aligned} & \int_{\mathbb{R}} (1 + |y|^2)^{s-2l} (1 + y^2 \csc^2(\alpha - \beta))^{-l} dy \\ & = \int_{|y| \leq 1} (1 + |y|^2)^{s-2l} (1 + y^2 \csc^2(\alpha - \beta))^{-l} dy \\ & \quad + \int_{|y| > 1} (1 + |y|^2)^{s-2l} (1 + y^2 \csc^2(\alpha - \beta))^{-l} dy \\ & = I_1 + I_2, \end{aligned}$$

where  $I_1$  and  $I_2$  denotes first and second integrals respectively. Since  $I_1$  is bounded, we have to show the convergent for  $I_2$ .

$$I_2 = \int_{|y| > 1} (1 + |y|^2)^{s-2l} (1 + y^2 \csc^2(\alpha - \beta))^{-l} dy.$$

If  $s - 2l > 0$ , then

$$\begin{aligned} I_2 & \leq \int_{|y| > 1} 2^{s-2l} |y|^{2s-4l} (\sin^2(\alpha - \beta) + y^2)^{-l} (\csc(\alpha - \beta))^{-2l} dy \\ & \leq \int_{|y| > 1} 2^{s-2l} |y|^{2s-4l} |y|^{-2l} (\csc(\alpha - \beta))^{-2l} dy \\ & = 2^{s-2l} (\csc(\alpha - \beta))^{-2l} \int_{|y| > 1} |y|^{2s-6l} dy \\ & < \infty. \end{aligned}$$

The  $y$ -integral is convergent by choosing  $l > \frac{1+2s}{6}$ .

Now, for  $s - 2l < 0$ , let  $s - 2l = -p$ ,

$$\begin{aligned} I_2 & \leq \int_{|y| > 1} \frac{\sin^{2l}(\alpha - \beta)}{(1 + |y|^2)^p (\sin^2(\alpha - \beta) + y^2)^l} dy \\ & \leq \int_{|y| > 1} \frac{1}{|y|^{2p} y^{2l}} dy \\ & \leq \int_{|y| > 1} y^{-(2p+2l)} dy < \infty. \end{aligned}$$

The above integral is convergent by choosing  $2p + 2l > 1$ . Therefore,

$$\begin{aligned} & \| (B_{\tau,\alpha,\beta} A_{\sigma,\alpha,\beta} \phi)(x, y) \|_{H^s(\mathbb{R} \times \mathbb{R})}^2 \\ & \leq C_1 \int_{\mathbb{R}} |(1 + |t|^2)^{\frac{s+m_1+m_2}{2}} (\mathcal{F}_{\alpha,\beta} \phi)(t)|^2 dt \\ & \leq C_1 \| \phi \|_{H^{s+m_1+m_2}(\mathbb{R})}^2, \end{aligned}$$

where  $C_1$  is a positive constant. This completes the proof.  $\square$

#### REFERENCES

- [1] L. Almeida, *The fractional Fourier transform and time frequency representations*, IEEE Trans. Signal Process, **42**(11) (1994), 3084–3091.
- [2] R.S. Pathak and A. Prasad, *A generalized pseudo-differential operator on Gelfand-Shilov space and Sobolev space*, Indian J. Pure Appl. Math., **37**(4) (2006), 223–235.
- [3] M.W. Wong, *An Introduction to pseudo-differential operators*, 2nd edition, World Scientific, Sinfapore, (1999).
- [4] S. Zaidman, *Distributions and pseudo-differential operators*, Longman, Essex, England (1991).
- [5] A.I. Zayed, *A convolution and product theorem for the fractional Fourier transform*, IEEE Signal Process. Lett., **5**(4) (1998), 101–103.
- [6] A.I. Zayed, *Fractional Fourier transform of generalized functions*, Integral transforms Spec. Funct., **7**(3-4) (1998), 299–312.