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THE ALEXANDROV PROBLEM OF DISTANCE PRESERVING MAPPING IN LINEAR N-NORMED SPACES

Jinmei Gao

School of Mathematics, Qingdao University,
Qingdao 266071, China
e-mail: gaojinmei@eyou.com

Abstract. In this paper , the author has studied the Alexandrov problem of area preserving mappings in linear n-normed spaces and has provided some remarks for the generalization of earlier results of H.Y. Chu, C.G. Park and W.G. Park. In addition the author has introduced the concept of linear (n,p)-normed spaces and for such spaces she has solved the Alexandrov problem .

1. Introduction

Let (X, d_X) and (Y, d_Y) be metric spaces. A mapping $f : X \to Y$ is called an isometry if f satisfies $d_Y(f(x), f(y)) = d_X(x, y)$ for all $x, y \in X$. For some fixed number $r > 0$ suppose that f preserves distance r i.e., for all $x, y \in X$ with $d_X(x, y) = r$, we have $d_Y(f(x), f(y)) = r$. Then r is called a conservative distance for the mapping f.

In 1970, A.D. Alexandrov [1] raised the well-known problem :"Whether or not a mapping with distance one preserving property is an isometry ?" Some results about this problem can be seen in [3], [4], [5], [6] [8], [9], [10], [11] and [12]. When X and Y are normed spaces, we may assume without loss of generality that the number $r = 1$ (see [7]).

In [2] H.Y. Chu, C.G. Park and W.G. Park introduced some new concepts provided a proof of the Th.M. Rassias and P. Semrl's theorem for linear n-normed spaces.

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Definition 1.1. [2] *Let X be a real linear space with $dimX > n$ and a function $\|\cdot, ..., \cdot\| : X^n \longrightarrow \mathbb{R}$ satisfies :*

(1) $\|x_1, ..., x_n\| = 0$ *if and only if* $x_1, ..., x_n$ *are linearly dependent,*
(2) $\|x_1, ..., x_n\| = \|x_{j_1}, ..., x_{j_n}\|$ *for every permutation* $j_1, ..., j_n$ *of* $(1, ..., n)$,
(3) $\|\alpha x_1, ..., x_n\| = |\alpha|\|x_1, ..., x_n\|$,
(4) $\|x + y, x_2, ..., x_n\| \leq \|x, x_2, ..., x_n\| + \|y, x_2, ..., x_n\|$ *for* $\alpha \in \mathbb{R}$ *and* $x, y, x_2, ..., x_n \in X$.

Then the function $\|\cdot, ..., \cdot\|$ is called the n-norm on X and (X,$\|\cdot, ..., \cdot\|$) is called the linear n-normed space.

Definition 1.2. [2] *Let X and Y be real linear n-normed spaces and $f : X \to Y$ a mapping, and $x_0, x_1, ..., x_n \in X$. We call that f satisfies the n-distance one preserving property ($nDOPP$) if $\|x_1 - x_0, ..., x_n - x_0\| = 1$ implies*

$$\|f(x) - f(x_0), ..., f(x_n) - f(x_0)\| = 1.$$

f is said to be a n-isometry if

$$\|f(x) - f(x_0), ..., f(x_n) - f(x_0)\| = \|x_1 - x_0, ..., x_n - x_0\|.$$

f is said to be a n-Lipschitz mapping if there is a $k \geq 0$ such that

$$\|f(x_1) - f(x_0), ..., f(x_n) - f(x_0)\| \leq k\|x_1 - x_0, ..., x_n - x_0\|.$$

The smallest such k is called the n-Lipschitz constant.

Theorem 1.3. [2] *Let f be a n-Lipschitz mapping with the n-Lipschitz constant $k \leq 1$. Assume that if $x_0, x_1, ..., x_m$ are m-collinear, then $f(x_0)$, $f(x_1)$,..., $f(x_n)$ are m-collinear, $m = 2, ..., n$, and that f satisfies ($nDOPP$). Then f is a n-isometry .*

The aim of this paper is to provide some remarks on the Alexandrov problem in linear n-normed spaces for the generalization of earlier results in [2]. In addition the author introduces the concept of linear (n,p)-normed spaces and for such spaces solves the corresponding Alexandrov problem .

2. Notes on the Alexandrov problem in linear n-normed spaces

Definition 2.1. *We call a mapping $f : X \to Y$ is locally n-Lipschitz mapping if there is a $k \geq 0$ such that*

$$\|f(x_1) - f(x_0), ..., f(x_n) - f(x_0)\| \leq k\|x_1 - x_0, ..., x_n - x_0\|,$$

whenever $\|x_1 - x_0, ..., x_n - x_0\| \leq 1$.

We only consider in this paper the n-Lipschitz constant $k \leq 1$.

Lemma 2.2. *If a mapping $f : X \to Y$ is locally n-Lipschitz, then f is a n-Lipschitz mapping.*

Proof. We may assume that $\|x_1 - x_0, ..., x_n - x_0\| > 1$, then there is an $n_0 \in \mathbb{N}$ such that $n_0 - 1 < \|y - x, z - x\| \leq n_0$. Let $w_i = x_0 + \frac{i}{n_0}(x_1 - x_0)$ where $i = 0, 1, ..., n_0$. Then

$$\begin{aligned} \|w_i - w_{i-1}, x_2 - w_{i-1}, ..., x_n - w_{i-1}\| &= \|w_i - w_{i-1}, x_2 - x_0, ..., x_n - x_0\| \\ &= \frac{\|x_1 - x_0, ..., x_n - x_0\|}{n_0} \\ &\leq 1 \end{aligned}$$

and

$$\begin{aligned} & \|f(w_i) - f(w_{i-1}), f(x_2) - f(x_0), ..., f(x_n) - f(x_0)\| \\ = \ & \|f(w_i) - f(w_{i-1}), f(x_2) - f(w_{i-1}), ..., f(x_n) - f(w_{i-1})\| \\ \leq \ & \|w_i - w_{i-1}, x_2 - w_{i-1}, ..., x_n - w_{i-1}\| \\ = \ & \frac{\|x_1 - x_0, ..., x_n - x_0\|}{n_0}. \end{aligned}$$

Hence

$$\begin{aligned} & \|f(x_1) - f(x_0), f(x_2) - f(x_0), ..., f(x_n) - f(x_0)\| \\ = \ & \|\sum_{i=1}^{n_0}(f(w_i) - f(w_{i-1})), f(x_2) - f(x_0), ..., f(x_n) - f(x_0)\| \\ \leq \ & \sum_{i=1}^{n_0} \|f(w_i) - f(w_{i-1}), f(x_2) - f(x_0), ..., f(x_n) - f(x_0)\| \\ \leq \ & \sum_{i=1}^{n_0} \frac{\|x_1 - x_0, ..., x_n - x_0\|}{n_0} \\ = \ & \|x_1 - x_0, ..., x_n - x_0\|. \end{aligned}$$

□

Remark 1. *Assume that f is locally n-Lipschitz and $x_0, x_1, ..., x_n$ are n-collinear. Then $f(x_0), f(x_1), ..., f(x_n)$ are n-collinear.* Indeed, $x_0, x_1, ..., x_n$ are n-collinear if and only if $\|x_1 - x_0, ..., x_n - x_0\| = 0$. Since

$$\|f(x_1) - f(x_0), ..., f(x_n) - f(x_0)\| \leq \|x_1 - x_0, ..., x_n - x_0\|,$$

we have

$$\|f(x_1) - f(x_0), ...t, f(x_n) - f(x_0)\| = 0,$$

it follows that $\|f(x_1) - f(x_0), ..., f(x_n) - f(x_0)\|$ are n-collinear.

So the Theorem 1.3 in [2] can be simplified as follows:

Theorem 2.3. *Let f be a n-Lipschitz mapping which satisfies $(nDOPP)$. And assume that if x_0, x_1 and x_2 are collinear then $f(x_0), f(x_1)$ and $f(x_2)$ are collinear. Then f is a n-isometry.*

In [2], a condition (*) was defined as follows : for every $x_0, x_1, ..., x_n \in X$ with $\|x_1 - x_0, x_2 - x_0, ..., x_n - x_0\| \neq 0$ there exists a $w \in X$ such that

$$\|x_0 - w, x_1 - w, ..., x_{n-1} - w\| = 1$$

and

$$\|x_1 - w, x_2 - w, ..., x_n - w\| = 1.$$

The condition (*) is a necessity in the proof of the results in [2]. In fact, through the following lemma we can see that the condition can be led out from the proof.

Lemma 2.4. *Assume that if x, y and z are collinear then $f(x), f(y)$ and $f(z)$ are collinear and that f satisfies $(nDOPP)$. Then f preserving the area $\frac{1}{k}$ for each $k \in \mathbb{N}$.*

Proof. Let $\|x_1 - x_0, ..., x_n - x_0\| = \frac{1}{k}$ and

$$u_i = x_1 + i(x_1 - x_0), v_i = x + i(x_2 - x_0)(i = 0, 1, ...k).$$

Then, it is easy to see that x_0, u_i and u_{i-1} are collinear, so $f(x_0), f(u_i)$ and $f(u_{i-1})$ are collinear by assumption. Then for $i = 1, ..., k$, we have

$$\begin{aligned} & \|u_i - u_{i-1}, v_k - u_{i-1}, x_3 - u_{i-1}, ..., x_n - u_{i-1}\| \\ = & \|u_i - u_{i-1}, v_k - x_0, x_3 - x_0, ..., x_n - x_0\| \\ = & 1 \end{aligned}$$

and

$$\begin{aligned} & \|f(u_i) - f(u_{i-1}), f(v_k) - f(u_{i-1}), ..., f(x_n) - f(u_{i-1})\| \\ = & \|f(u_i) - f(u_{i-1}), f(v_k) - f(x_0), ..., f(x_n) - f(x_0)\| \\ = & 1. \end{aligned}$$

It follows that $f(u_i) - f(u_{i-1}) = f(u_{i+1}) - f(u_i)$. Note that

$$f(x_0) = f(x_0) + 0 \cdot (f(x_1) - f(x_0))$$

and

$$f(x_1) = f(x_0) + 1 \cdot (f(x_1) - f(x_0)).$$

Therefore

$$f(u_i) = f(x) + i \cdot (f(x_1) - f(x_0))(i = 0, 1, ..., k).$$

Similarly, we have

$$f(v_i) = f(x_0) + i \cdot (f(x_2) - f(x_0))(i = 0, 1, ..., k).$$

Since $\|u_k - x, v_k - x\| = k$, we get

$$\begin{aligned} k &= \|f(u_k) - f(x_0), f(v_k) - f(x_0), ..., f(x_n) - f(x_0)\| \\ &= \|k(f(x_1) - f(x)), k(f(x_2) - f(x_0)), ..., f(x_n) - f(x_0)\| \\ &= k^2\|f(x_1) - f(x_0), f(x_2) - f(x_0), ..., f(x_n) - f(x_0)\| \end{aligned}$$

Thus $\|f(x_1) - f(x_0), f(x_2) - f(x_0), ..., f(x_n) - f(x_0)\| = \frac{1}{k}$. □

By the same argument as in the proof of Theorem 2.12 in [2], we get the following lemma.

Lemma 2.5. *Assume that if $x_0, x_1, ..., x_n$ are m-collinear then $f(x_0)$, $f(x_1)$,..., $f(x_n)$ are m-collinear (m=2,...,n), and that if $y_1 - y_2 = \alpha(y_3 - y_2)$ for some $\alpha \in (0, 1]$ then $f(y_1) - f(y_2) = \beta(f(y_3) - f(y_2))$. If f satisfies $(nDOPP)$ then f is a n-Lipschitz mapping.*

A direct application of Theorem 2.3 and Lemma 2.5 yields the following result.

Theorem 2.6. *Let f be a mapping satisfies $(nDOPP)$. And assume that if $x_0, x_1, ..., x_n$ are m-collinear then $f(x_0)$, $f(x_1)$,...,$f(x_n)$ are m-collinear, and that if $y_1 - y_2 = \alpha(y_3 - y_2)$ for some $\alpha \in (0, 1]$ then $f(y_1) - f(y_2) = \beta(f(y_3) - f(y_2))$. Then f is a n-isometry.*

Lemma 2.7. *If there exist $\rho > 0, N > 1$ with $\rho \in \mathbb{R}, N \in \mathbb{N}$ and a mapping $f : X \to Y$ satisfies the following conditions:*

(1) *if $\|x_1 - x_0, x_2 - x_0, ..., x_n - x_0\| = \rho$, then*

$$\|f(x_1) - f(x_0), ..., f(x_n) - f(x_0)\| \le \rho,$$

(2) *if $\|x_1 - x_0, x_2 - x_0, ..., x_n - x_0\| = N\rho$, then*

$$|f(x_1) - f(x_0), ..., f(x_n) - f(x_0)\| \ge N\rho,$$

then f satisfies the n-distance ρ preserving property.

Proof. Let $\|x_1 - x_0, x_2 - x_0, ..., x_n - x_0\| = \rho$, and $w_i = x_0 + i(x_1 - x_0)$ where $i = 0, 1, ..., N$. Then we obtain

$$\|w_N - x, x_2 - x_0, ..., x_n - x_0\| = N\rho,$$

$$\begin{aligned} &\|w_i - w_{i-1}, x_2 - w_{i-1}, ..., x_n - w_{i-1}\| \\ =\ &\|w_i - w_{i-1}, x_2 - x_0, ..., x_n - x_0\| \\ =\ &\rho, \end{aligned}$$

and

$$\begin{aligned} & \|f(w_i)-f(w_{i-1}),f(x_2)-f(x_0),...,f(x_n)-f(x_0)\| \\ = & \|f(w_i)-f(w_{i-1}),f(x_2)-f(w_{i-1}),...,f(x_n)-f(w_{i-1})\| \\ \leq & \rho. \end{aligned}$$

Hence

$$\begin{aligned} N\rho & \leq \|f(w_N)-f(x_0),f(x_2)-f(x_0),...,f(x_n)-f(x_0)\| \\ & \leq \sum_{i=1}^{N}\|f(w_{n_i})-f(w_{n_{i-1}}),f(x_2)-f(x_0),...,f(x_n)-f(x_0)\| \\ & \leq N\rho. \end{aligned}$$

It follows that

$$\begin{aligned} & \|f(x_1)-f(x_0),f(x_2)-f(x_0),...,f(x_n)-f(x_0)\| \\ = & \|f(w_i)-f(w_{i-1}),f(x_2)-f(x_0),...,f(x_n)-f(x_0)\| \\ = & \rho. \end{aligned}$$

□

Theorem 2.8. *Let X,Y be linear $n-$normed spaces. Let $f:X\to Y$ be a mapping. And assume that if x,y and z are collinear, then $f(x),f(y)$ and $f(z)$ are collinear, and there exist $\rho\in\mathbb{R}$ with $\rho>0$, and $N\in\mathbb{N}$ with $N>1$ f satisfies the following conditions:*

(1) *if $\|x_1-x_0,x_2-x_0,...,x_n-x_0\|=\rho$, then*
$$\|f(x_1)-f(x_0),...,f(x_n)-f(x_0)\|\leq\rho,$$
(2) *if $\|x_1-x_0,x_2-x_0,...,x_n-x_0\|=N\rho$, then*
$$\|f(x_1)-f(x_0),...,f(x_n)-f(x_0)\|\geq N\rho.$$

Then f is a n-isometry.

Proof. It is easy to see that f is locally n-lipschitz and satisfies n-distance ρ preserving property. Therefore f is a n-isometry by Theorem 2.3 □

3. The Alexandrov problem in linear (n,p)-normed spaces

Definition 3.1. *Let X be a real linear space with $dimX\geq n$ and a function $\|\cdot,...,\cdot\|:X^n\longrightarrow\mathbb{R}$ satisfies:*

(1) $\|x_1,...,x_n\|=0$ *if and only if* $x_1,...,x_n$ *are linearly dependent ,*
(2) $\|x_1,...,x_n\|=\|x_{j_1},...,x_{j_n}\|$ *for every permutation* $j_1,...,j_n$ *of* $(1,...,n)$,
(3) $\|\alpha x_1,...,x_n\|=|\alpha|^p\|x_1,...,x_n\|(0<p\leq1)$,
(4) $\|x+y,x_2,...,x_n\|\leq\|x,x_2,...,x_n\|+\|y,x_2,...,x_n\|$,

for $\alpha \in \mathbb{R}$ and $x, y, x_2, ..., x_n \in X$. Then, the function $\|\cdot, ..., \cdot\|$ is called the (n, p)-norm on X, and $(X, \|\cdot, ..., \cdot\|)$ is called the linear (n, p)-normed space.

Definition 3.2. *$(X, \|\cdot, ..., \cdot\|)$ is said to be a p-strictly convex linear (n, p)-normed space if for any $x, y, x_2, ...x_n \in X$, $x_i \neq \alpha x + \beta y, \alpha, \beta \in \mathbb{R}, i = 2, ..., n$, $\|x + y, x_2, ..., x_n\|^{\frac{1}{p}} = \|x, x_2, ..., x_n\|^{\frac{1}{p}} + \|y, x_2, ...x_n\|^{\frac{1}{p}}$ implies $x = \lambda y$ for some $\lambda > 0$.*

Lemma 3.3. *If $(X, \|\cdot, ..., \cdot\|)$ is a p-strictly convex linear (n, p)-normed space, then $(X, \|\cdot, ..., \cdot\|^{\frac{1}{p}})$ is a linear n-normed space.*

Proof. It is easy to see that $\|\cdot, ..., \cdot\|^{\frac{1}{p}}$ satisfies the condition (1),(2) and (3) in the definition of n-norm. We only need to prove that $\|\cdot, ..., \cdot\|^{\frac{1}{p}}$ satisfies the condition (4). First, we will show that for fixed $x_2, ..., x_n \in X$ the set $O_X = \{x \in X : \|x, x_2, ..., x_n\| < 1\}$ is convex, it is enough to show that for any $x \neq y \in X, \|x, x_2, ..., x_n\| = 1, \|y, x_2, ..., x_n\| = 1$, we have $\|\lambda x + (1 - \lambda)y, x_2, ..., x_n\| < 1$, where $0 < \lambda < 1$.

(I) $\|\lambda x + (1 - \lambda)y, x_2, ..., x_n\| \neq 1$.
Otherwise, we get

$$\begin{aligned} 1 &= \|\lambda x + (1 - \lambda)y, x_2, ..., x_n\|^{\frac{1}{p}} \\ &= \|\lambda x, x_2, ..., x_n\|^{\frac{1}{p}} + \|(1 - \lambda)y, x_2, ..., x_n\|^{\frac{1}{p}}. \end{aligned}$$

It follows that $x = y$ from the definition, which is a contradiction.

(II) It is also impossible that $\|\lambda x + (1 - \lambda)y, x_2, ..., x_n\| > 1$. Otherwise, let

$$w = \frac{\lambda x + (1 - \lambda)y}{\|\lambda x + (1 - \lambda)y, x_2, ..., x_n\|^{\frac{1}{p}}},$$

$$x_1 = \frac{x}{\|\lambda x + (1 - \lambda)y, x_2, ..., x_n\|^{\frac{1}{p}}}, y_1 = \frac{y}{\|\lambda x + (1 - \lambda)y, x_2, ..., x_n\|^{\frac{1}{p}}}.$$

Then $w = \lambda x_1 + (1-\lambda)y_1, \|w, x_2, ..., x_n\| = 1$. Let $\phi(t) = \|tx_1 + (1-t)y_1\|, t \in \mathbb{R}$. Then, $\phi(t)$ is continuous on $\mathbb{R}$ and $\phi(0), \phi(1) < 1$. Letting $t \to +\infty$ or $t \to -\infty$, then from the theorem of middle value we can find x', y' and $0 < \mu < 1$ such that $\|x', x_2, ..., x_n\| = 1, \|y', x_2, ..., x_n\| = 1$ and $w = \mu x' + (1 - \mu)y'$ which contradicts with (I).

Since O_X is an open convex set, there exists a subadditive and positive homogenous function $p(x) : X \to \mathbb{R}$ such that $O_X = \{x \in X : p(x) < 1\}$ and

$$\{x \in X : \|x, x_2, ..., x_n\| \leq 1\} = \{x \in X : p(x) \leq 1\}.$$

Therefore

$$\{x \in X : \|x, x_2, ..., x_n\| = 1\} = \{x \in X : p(x) = 1\}.$$

For any $x \in X$, we have

$$\|\frac{x}{\|x, x_2, ..., x_n\|^{\frac{1}{p}}}, x_2, ..., x_n\| = 1.$$

Hence we ahve $p(x) = \|x, x_2, ..., x_n\|^{\frac{1}{p}}$. It follows that $\|\cdot, \cdot\|^{\frac{1}{p}}$ is a n-norm. □

Theorem 3.4. *(X,$\|\cdot, ..., \cdot\|$) is a p-strictly convex linear (n, p)-normed space if and only if (X,$\|\cdot, ..., \cdot\|^{\frac{1}{p}}$) is a strictly convex linear n-normed space.*

Proof. It is clear by the Lemma and Definition above. □

Lemma 3.5. *Let X, Y be linear (n, p)-normed spaces. And assume that if x, y and z are collinear, then $f(x), f(y)$ and $f(z)$ are collinear and that f satisfies $(AOPP)$. Then f preserves the area 2^{np}, for each $n \in \mathbb{N}$.*

Proof. The proof is carried out by induction on n.

For $\|x_1 - x_0, ..., x_n - x_0\| = 1$ we have $\|f(x_1) - f(x_0), ..., f(x_n) - f(x_0)\| = 1$ from the assumption.

Assume that for $\|x_1 - x_0, ..., x_n - x_0\| = 2^{(n-1)p}$, the conclusion

$$\|f(x_1) - f(x_0), ..., f(x_n) - f(x_0)\| = 2^{(n-1)p}$$

is established. We will show that

$$\|f(x_1) - f(x_0), ..., f(x_n) - f(x_0)\| = 2^{np}$$

holds for $\|x_1 - x_0, ..., x_n - x_0\| = 2^{np}$.

Let $w = \frac{x_0+x_1}{2}$. Then it is easy to see x_0, x_1 and w are collinear, which implies $f(x_0), f(x_1)$ and $f(w)$ are collinear, that is $f(w) - f(x_0) = \alpha(f(x_1) - f(w))$. We also have

$$\|w - x_0, ..., x_n - x_0\| = 2^{(n-1)p},$$

$$\|x_1 - w, x_2 - w, ..., x_n - w\| = \|x_1 - w, x_2 - x_0, ..., x_n - x_0\| = 2^{(n-1)p}$$

and

$$\|f(w) - f(x_0), f(x_2) - f(x_0), ..., f(x_n) - f(x_0)\| = 2^{(n-1)p},$$

$$\begin{aligned} & \|f(x_1) - f(w), f(x_2) - f(x_0), ..., f(x_n) - f(x_0)\| \\ = & \|f(x_1) - f(w), f(x_2) - f(w) ..., f(x_n) - f(w)\| \\ = & 2^{(n-1)p}. \end{aligned}$$

Hence $\alpha = 1, f(w) - f(x_0) = f(x_1) - f(w)$. And also, we have

$$\begin{aligned} \|f(x_1) - f(x_0), ..., f(x_n) - f(x_0)\| &= \|2(f(w) - f(x_0)), ..., f(x_n) - f(x_0)\| \\ &= 2^p \|f(w) - f(x_0), ..., f(x_n) - f(x_0)\| \\ &= 2^{np}. \end{aligned}$$

□

Theorem 3.6. *Let X be a linear (n,p)-normed space and Y be a linear p-strictly convex (n,p)-normed space and $f : X \to Y$ be locally n-Lipschitz and satisfies $(nDOPP)$. And assume that if x, y and z are collinear, then $f(x), f(y)$ and $f(z)$ are collinear. Then f is a n-isometry.*

Proof. (I) If $\|x_1 - x_0, ..., x_n - x_0\| \leq 1$, then we claim that

$$\|f(x_1) - f(x_0), ..., f(x_n) - f(x)\| = \|x_1 - x_0, ..., x_n - x_0\|.$$

Suppose that $\|f(x_1) - f(x_0), ..., f(x_n) - f(x)\| < \|x_1 - x_0, ..., x_n - x_0\|$ and let $w = x_0 + \frac{x_1 - x_0}{\|x_1 - x_0, ..., x_n - x_0\|^{\frac{1}{p}}}$. Then we have $\|w - x_0, ..., x_n - x_0\| = 1$ and

$$\begin{aligned} \|x_1 - w, x_2 - w, ..., x_n - w\|^{\frac{1}{p}} &= \|x_1 - w, x_2 - x_0, ..., x_n - x_0\|^{\frac{1}{p}} \\ &= 1 - \|x_1 - x_0, ..., x_n - x_0\|^{\frac{1}{p}} \\ &< 1. \end{aligned}$$

By the assumption, we get

$$\|f(w) - f(x_0), ..., f(x_n) - f(x_0)\| = 1$$

and

$$\begin{aligned} &\|f(x_1) - f(w), f(x_2) - f(x_0), ..., f(x_n) - f(x_0)\|^{\frac{1}{p}} \\ = \ &\|f(x_1) - f(w), f(x_2) - f(w), ..., f(x_n) - f(w)\|^{\frac{1}{p}} \\ \leq \ &1 - \|x_1 - x_0, ..., x_n - x_0\|^{\frac{1}{p}}. \end{aligned}$$

It follows that

$$\begin{aligned} 1 &= \|f(w) - f(x_0), ..., f(x_n) - f(x_0)\|^{\frac{1}{p}} \\ &= \|f(w) - f(x_0), ..., f(x_n) - f(x_0)\|^{\frac{1}{p}} \\ &\leq \|f(w) - f(x_1), ..., f(x_n) - f(x)\|^{\frac{1}{p}} \\ &\quad + \|f(x_1) - f(x_0), ..., f(x_n) - f(x - 0)\|^{\frac{1}{p}} \\ &< \|y - x, z - x\|^{\frac{1}{p}} + (1 - \|y - x, z - x\|^{\frac{1}{p}}) \\ &= 1. \end{aligned}$$

Which is a contradiction.

(II) Assume that for $\|x_1 - x_0, ..., x_n - x_0\| \le 2^{(n-1)p}$, the conclusion

$$\|f(x_1) - f(x_0), ..., f(x_n) - f(x_0)\| = \|x_1 - x_0, ..., x_n - x_0\|$$

is established.

Assume that $\|x_1 - x_0, ..., x_n - x_0\| \le 2^{np}$. Let $w = \frac{x_0+x_1}{2}$. Then x_0, x_1 and w are collinear, which implies $f(x_0), f(x_1)$ and $f(w)$ are collinear, that is

$$f(x_1) - f(w) = \alpha(f(w) - f(x_0)).$$

We also have

$$\|w - x_0, ..., x_n - x_0\| = 2^{-p}\|x_1 - x_0, ..., x_n - x_0\| \le 2^{(n-1)p}$$

and

$$\begin{aligned} \|x_1 - w, x_2 - w, ..., x_n - w\| &= \|x_1 - w, x_2 - x_0, ..., x_n - x_0\| \\ &= 2^{-p}\|x_1 - x_0, ..., x_n - x_0\| \\ &\le 2^{(n-1)p}. \end{aligned}$$

From the inductive hypothesis,

$$\|f(w) - f(x_0), ..., f(x_n) - f(x_0)\| = \|w - x_0, ..., x_n - x_0\|,$$

$$\begin{aligned} & \|f(x_1) - f(w), f(x_2) - f(x_0), ..., f(x_n) - f(x_0)\| \\ = & \|f(x_1) - f(w), f(x_2) - f(w), ..., f(x_n) - f(w)\| \\ = & \|x_1 - w, ..., x_n - w\|. \end{aligned}$$

Hence $\alpha = 1, f(x_1) - f(w) = f(w) - f(x_0)$. Therefore

$$\begin{aligned} \|f(x_1) - f(x_0), ..., f(x_n) - f(x_0)\| &= \|2(f(w) - f(x_0)), ..., f(x_n) - f(x_0)\| \\ &= 2^p\|f(w) - f(x_0), ..., f(x_n) - f(x_0)\| \\ &= 2^p\|w - x_0, ..., x_n - x_0\| \\ &= \|x_1 - x_0, ..., x_n - x_0\|. \end{aligned}$$

□

Lemma 3.7. *Let X, Y be linear (n, p)-normed spaces. And assume that if x, y and z are collinear, then $f(x), f(y)$ and $f(z)$ are collinear and that f satisfies $(nDOPP)$. Then f preserves the area k^p and $\frac{1}{k^p}$ for each $k \in \mathbb{N}$.*

Proof. If $\|x_1 - x_0, ..., x_n - x_0\| = k^p$, put

$$w_i = x_0 + \frac{i}{k}(x_1 - x_0), i = 0, 1, ..., k,$$

then we have $f(w_{i+1})-f(w_i)=f(w_i)-f(w_{i-1}), i=0,1,...,k$, from the same argument as above. Hence

$$\begin{aligned} & \|f(x_1)-f(x_0), f(x_2)-f(x_0), ..., f(x_n)-f(x_0)\| \\ = & \|\sum_{i=1}^{k}(f(w_i)-f(w_{i-1})), f(x_2)-f(x_0), ..., f(x_n)-f(x_0)\| \\ = & \|k(f(w_1)-f(w_0)), f(x_2)-f(x_0), ..., f(x_n)-f(x_0)\| \\ = & k^p. \end{aligned}$$

If $\|x_1-x_0, ..., x_n-x_0\| = \frac{1}{n^p}$, put

$$u_i = x_0 + i(x_1-x_0), v_i = x + i(x_2-x_0) \quad i=0,1,...,k,$$

then we have

$$f(u_{i+1})-f(u_i)=f(u_i)-f(u_{i-1})$$

and

$$f(v_{i+1})-f(v_i)=f(v_i)-f(v_{i-1})$$

from the same argument as above. Hence

$$\begin{aligned} k^p &= \|f(u_k)-f(x_0), f(v_k)-f(x_0), ..., f(x_n)-f(x_0)\| \\ &= \|\sum_{i=1}^{k}(f(u_i)-f(u_{i-1})), \sum_{i=1}^{k}(f(v_i)-f(v_{i-1})), ..., f(x_n)-f(x_0)\| \\ &= \|k(f(x_1)-f(x_0)), k(f(x_2)-f(x_0)), ..., f(x_n)-f(x_0)\| \\ &= k^{2p}\|f(x_1)-f(x_0), f(x_2)-f(x_0)...., f(x_n)-f(x_0)\|. \end{aligned}$$

Therefore $\|f(x_1)-f(x_0), f(x_2)-f(x_0)..., f(x_n)-f(x_0)\| = \frac{1}{k^p}$. □

Lemma 3.8. *Let X, Y be linear (n,p)-normed spaces. And assume that if $x_0, x_1, ..., x_n$ are m-collinear, then $f(x_0), f(x_1), ..., f(x_n)$ are m-collinear (m=2,...,n), and that if $y_1-y_2=\alpha(y_3-y_2)$ for some $\alpha \in (0,1]$, then $f(y_1)-f(y_2)=\beta(f(y_3)-f(y_2))$. If f satisfies $(nDOPP)$ then f is a n-Lipschitz mapping.*

Proof. If $\|x_1-x_0, x_2-x_0, ..., x_n-x_0\|=0$, then $x_0, x_1, ..., x_n$ are n-collinear, so $\|f(x_1)-f(x_0), f(x_2)-f(x_0), ..., f(x_n)-f(x_0)\|=0$ from the assumption.

Next, we will show that if

$$\|x_1-x_0, x_2-x_0, ..., x_n-x_0\| \le (\frac{s}{r})^p, \forall r, s \in \mathbb{N},$$

then

$$\|f(x_1)-f(x_0), f(x_2)-f(x_0), ..., f(x_n)-f(x_0)\| \le (\frac{s}{r})^p.$$

Let $p_i = x_0 + \frac{i}{r}\frac{1}{(\|x_1-x_0,...,x_n-x_0\|)^{\frac{1}{p}}}(x_1 - x_0)$, where $i = 1, ..., s$. Then

$$\|p_i - p_{i-1}, x_2 - p_{i-1}, ..., x_n - p_{i-1}\| = \|p_i - p_{i-1}, x_2 - x_0, ..., x_n - x_0\| = \frac{1}{r^p}.$$

Hence

$$\begin{aligned} & \|f(p_i) - f(p_{i-1}), f(x_2) - f(x_0), ..., f(x_n) - f(x_0)\| \\ = & \|f(p_i) - f(p_{i-1}), f(x_2) - f(p_{i-1}), ..., f(x_n) - f(p_{i-1})\| \\ = & \frac{1}{r^p}. \end{aligned}$$

Since $x_1 = p_{s-1} + \alpha(p_s - p_{s-1})$ for some $\alpha \in (0, 1]$, we obtain that $f(x_1) = f(p_{s-1}) + \beta(f(p_s) - f(p_{s-1}))$ for some $\beta \in (0, 1]$ from the hypothesis. Hence

$$\begin{aligned} & \|f(x_1) - f(x_0), f(x_2) - f(x_0), ..., f(x_n) - f(x_0)\| \\ & = \|(f(x_1) - f(p_{s-1})) + \sum_{i=1}^{s-1} f(p_i) - f(p_{i-1}), ..., f(x_n) - f(x_0)\| \\ & = \|(s - 1 + \beta)(f(p_1) - f(p_0)), f(x_2) - f(x_0), ..., f(x_n) - f(x_0)\| \\ & = (\frac{m-1+\beta}{r})^p \\ & \leq (\frac{m}{r})^p. \end{aligned}$$

□

A direct application of Theorem 3.8 and the above two Lemmas yields the following results.

Theorem 3.9. *Let X be a linear (n, p)-normed space and Y be a linear p-strictly convex (n, p)-normed space. And assume that if $x_0, x_1, ..., x_n$ are m-collinear, then $f(x_0), f(x_1), ..., f(x_n)$ are m-collinear (m=2,...,n), and that if $y_1-y_2 = \alpha(y_3-y_2)$ for some $\alpha \in (0, 1]$, then $f(y_1)-f(y_2) = \beta(f(y_3)-f(y_2))$. If f satisfies (nDOPP) then $f : X \to Y$ is a n-isometry.*

By the same argument as that in the above section, we may get the following result:

Theorem 3.10. *Let X, Y be linear $(n, p)-$normed spaces. Let $f : X \to Y$ be a mapping. And assume that if x, y and z are collinear, then $f(x), f(y)$ and $f(z)$ are collinear, and there exist $\rho \in \mathbb{R}$ with $\rho > 0$, and $N \in \mathbb{N}$ with $N > 1$, f satisfies the following conditions:*

(1) *if $\|x_1 - x_0, x_2 - x_0, ..., x_n - x_0\| = \rho$, then*

$$\|f(x_1) - f(x_0), ..., f(x_n) - f(x_0)\| \leq \rho,$$

(2) *if* $\|x_1 - x_0, x_2 - x_0, ..., x_n - x_0\| = \rho 2^{Np}$, *then*

$$\|f(x_1) - f(x_0), ..., f(x_n) - f(x_0)\| \geq \rho 2^{Np}.$$

Then f *is a* n*-isometry.*

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