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FIXED POINT THEOREMS FOR THE MODIFIED SIMULATION FUNCTION AND APPLICATIONS TO FRACTIONAL ECONOMICS SYSTEMS

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Abstract. In this paper, first, we prove some common fixed point theorems for the generalized contraction condition under newly defined modified simulation function which generalize and include many results in the literature. Second, we give two numerical examples with graphical representations for verifying the proposed results. Third, we discuss and study a set of common fixed point theorems for two pairs (finite families) of self-mappings. Finally, we give some applications of our results in discrete and functional fractional economic systems.

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1. Introduction

In the fixed point technique (Banach's fixed point theorem), the controllability problem is converted to a fixed point problem for an applicable nonlinear operator in a function space. An important part of this attitude is to guarantee the solvability of an invariant subset for this operator. Through the Banach fixed point result, we can get a unique solution of some nonlinear equations if we convert it into the operator form, which is a contraction operator in a complete metric space. In the past, a number of attempts have been made to generalize the contraction condition and a survey has been done by Rhoades [17]) till 1977 work.

In this direction, a new control function, named as a *simulation function* is designed by Khojasteh et al. [12], which is slightly modified and enlarged by Roldán-Lpez-de-Hierro et al [19]. Very recently, Hazarika et al. [9] has modified this notion and introduce the modified simulation function.

Definition 1.1. ([9]) The *set of modified simulation functions*, Θ is a class of functions $\theta : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ under following conditions:

(θ_1) $\theta(\xi, \zeta) < \zeta - \xi$ for all $\zeta, \xi > 0$;

(θ_2) if $\{\xi_n\}$ and $\{\zeta_n\}$ are the sequences in $(0, \infty)$ such that $\lim_{n\to\infty} \xi_n = \alpha > 0$ and $\lim_{n\to\infty} \zeta_n = \beta > 0$, then $\limsup_{n\to\infty} \theta(\xi_n, \zeta_n) < \beta - \alpha$.

We fix the notion for the set of all fixed (common and coincidence) points of a self-mapping $\mathcal{P}$ (and self-mapping $\mathcal{Q}$) on a set $\Xi \neq \emptyset$ is denoted by $Fix(\mathcal{P})$ ($CFP(\mathcal{P}, \mathcal{Q})$ and $CP(\mathcal{P}, \mathcal{Q})$).

2. Main results

2.1. **A supplementary result.** First, we prove the following result which is essential to accomplish the main results later.

Lemma 2.1. *Let (Ξ, d) be a metric space, $\mathcal{P}, \mathcal{Q}, \mathcal{S}, \mathcal{T} : \Xi \to \Xi$ be the operators satisfying the following conditions;*

(a) $\mathcal{T}(\Xi) \supseteq \mathcal{P}(\Xi)$, $\mathcal{S}(\Xi) \supseteq \mathcal{Q}(\Xi)$;

(b) *for all $p, q \in \Xi$,*

$$\theta[d(\mathcal{P}p, \mathcal{Q}q), \Lambda(p, q)] \geq 0, \tag{2.1}$$

where

$$\Lambda(p, q) = \max \left\{ \begin{array}{c} d(\mathcal{S}p, \mathcal{T}q), d(\mathcal{P}p, \mathcal{S}p), d(\mathcal{Q}q, \mathcal{T}q), \\ \frac{1}{2}[d(\mathcal{Q}q, \mathcal{S}p) + d(\mathcal{P}p, \mathcal{T}q)] \end{array} \right\} \tag{2.2}$$

and $\theta \in \Theta$. Then, for all $p_0 \in \Xi$, $\{y_n\}$ is a Cauchy sequence with

$$q_{2n} = \mathcal{P}p_{2n} = \mathcal{T}p_{2n+1}, \quad q_{2n-1} = \mathcal{Q}p_{2n-1} = \mathcal{S}p_{2n}.$$

Proof. Letting $p = p_{2n}$ and $q = p_{2n+1}$ in (2.2), we obtain

$$\begin{aligned}
&\Lambda(p_{2n}, p_{2n+1})\\
&\quad = \max\left\{ \begin{array}{c} d(\mathcal{S}p_{2n}, \mathcal{T}p_{2n+1}), d(\mathcal{P}p_{2n}, \mathcal{S}p_{2n}), d(\mathcal{Q}p_{2n+1}, \mathcal{T}p_{2n+1}), \\ \frac{1}{2}[d(\mathcal{Q}p_{2n+1}, \mathcal{S}p_{2n}) + d(\mathcal{P}p_{2n}, \mathcal{T}p_{2n+1})] \end{array} \right\}\\
&\quad = \max\left\{ \begin{array}{c} d(q_{2n-1}, q_{2n}), d(q_{2n}, q_{2n-1}), d(q_{2n+1}, q_{2n}), \\ \frac{1}{2}[d(q_{2n+1}, q_{2n-1}) + d(q_{2n}, q_{2n})] \end{array} \right\}\\
&\quad = \max\left\{ d(q_{2n-1}, q_{2n}), d(q_{2n+1}, q_{2n}) \right\}. \qquad (2.3)
\end{aligned}$$

Similarly,

$$\Lambda(p_{2n}, p_{2n-1}) = \max\{d(q_{2n-1}, q_{2n-2}),\ d(q_{2n}, q_{2n-1})\}.$$

Obviously, when $n \geq 1$, $q_{2n} = q_{2n-1}$ or $q_{2n} = q_{2n+1}$, then, from (2.1), $\{q_n\}$ is a constant sequence and so it is a Cauchy sequence.

Suppose for each $n \geq 1$, $q_n \neq q_{n-1}$. Owing (2.1) and (2.3), we have

$$\begin{aligned}
0 &\leq \theta[d(\mathcal{P}p_{2n}, \mathcal{Q}p_{2n+1}), \Lambda(p_{2n}, p_{2n+1})] \qquad (2.4)\\
&= \theta[d(q_{2n}, q_{2n+1}), \Lambda(p_{2n}, p_{2n+1})]\\
&\leq \theta[d(q_{2n}, q_{2n+1}), \max\{d(q_{2n-1}, q_{2n}), d(q_{2n+1}, q_{2n})\})].
\end{aligned}$$

If $d(q_{2n-1}, q_{2n}) < d(q_{2n+1}, q_{2n})$ for each $n \geq 1$, then, from (2.4), we have

$$\begin{aligned}
0 &\leq \theta[d(q_{2n}, q_{2n+1}), d(q_{2n+1}, q_{2n})]\\
&< d(q_{2n}, q_{2n+1}) - d(q_{2n+1}, q_{2n})\\
&= 0,
\end{aligned}$$

which is a contradiction. Thus

$$d(q_{2n}, q_{2n+1}) \leq d(q_{2n-1}, q_{2n})$$

for each $n \geq 1$. Similarly, from (2.4), we have

$$d(q_{2n+1}, q_{2n+2}) \leq d(q_{2n}, q_{2n+1}).$$

So, it follows that, for each $n \geq 1$,

$$d(q_n, q_{n+1}) \leq d(q_{n-1}, q_n).$$

Put $\sigma_n = d(q_n, q_{n+1})$. Then the sequence $\{\sigma_n\}$ is non-decreasing. Therefore, we can infer that

$$\lim_{n\to\infty} \sigma_n = \varrho.$$

Now, we claim that $\varrho = 0$. On contrary let $\varrho > 0$. Starting with some $n = n_0 \geq 1$. Therefore, using the condition (θ_2), it follows that, for each $n \geq n_0$,

$$0 \leq \limsup_{n\to\infty} \theta[\sigma_n, \sigma_{n-1}] < \varrho - \varrho = 0,$$

which is a contradiction. Then we conclude that

$$\lim_{n\to\infty} \sigma_n = 0. \tag{2.5}$$

Now, we claim that $\{q_n\}$ is a Cauchy sequence. It is enough to show that $\{q_{2n}\}$ is a Cauchy sequence. On the contrary, suppose that $\{q_{2n}\}$ is not a Cauchy sequence. Following [13, Lemma 2.1], there exits $\epsilon > 0$ and two sequences $\{j(\ell)\}$ and $\{k(\ell)\}$, with $k(\ell) > j(\ell) > \ell$ achieving $d(q_{2j(\ell)}, q_{2k(\ell)})$, $d(q_{2k(\ell)+1}, q_{2j(\ell)-1})$ and $d(q_{2k(\ell)}, q_{2j(\ell)-1}) \to \epsilon$ as $\ell \to \infty$. Further, we conclude that

$$\begin{aligned}
&d(q_{2k(\ell)}, q_{2k(\ell)+1}) + d(q_{2k(\ell)+1}, q_{2j(\ell)}) \\
&= d(q_{2k(\ell)}, q_{2k(\ell)+1}) + d(\mathcal{Q}p_{2k(\ell)}, \mathcal{P}p_{2j(\ell)-1}) \\
&\geq d(q_{2k(\ell)}, q_{2j(\ell)}).
\end{aligned}$$

Put $p = p_{2j(\ell)-1}$ and $q = p_{2k(\ell)}$ in (2.1), we obtain

$$\begin{aligned}
0 &\leq \theta[d(\mathcal{P}p_{2j(\ell)-1}, \mathcal{Q}p_{2k(\ell)}), \Lambda(p_{2j(\ell)-1}, p_{2k(\ell)})] \\
&= \theta[d(q_{2j(\ell)-1}, q_{2k(\ell)}), \Lambda(p_{2j(\ell)-1}, p_{2k(\ell)})],
\end{aligned} \tag{2.6}$$

where

$$\begin{aligned}
&\Lambda(p_{2j(\ell)-1}, p_{2k(\ell)}) \\
&= \max\left\{ \begin{array}{c} d(\mathcal{S}p_{2j(\ell)-1}, \mathcal{T}p_{2k(\ell)}), d(\mathcal{P}p_{2j(\ell)-1}, \mathcal{S}p_{2j(\ell)-1}), \\ d(\mathcal{Q}p_{2k(\ell)}, \mathcal{T}p_{2k(\ell)}), \\ \frac{1}{2}[d(\mathcal{Q}p_{2k(\ell)}, \mathcal{S}p_{2j(\ell)-1}) + d(\mathcal{P}p_{2j(\ell)-1}, \mathcal{T}p_{2k(\ell)})] \end{array} \right\} \\
&= \max\left\{ \begin{array}{c} d(q_{2j(\ell)-2}, q_{2k(\ell)-1}), d(q_{2j(\ell)-1}, q_{2j(\ell)-2}), d(q_{2k(\ell)}, q_{2k(\ell)-1}), \\ \frac{1}{2}[d(q_{2k(\ell)}, q_{2j(\ell)-2}) + d(q_{2j(\ell)-1}, q_{2k(\ell)-1})] \end{array} \right\}.
\end{aligned} \tag{2.7}$$

Passing the limit as $\ell \to \infty$ in (2.7), we have

$$\lim_{\ell\to\infty} \Lambda(p_{2j(\ell)-1}, p_{2k(\ell)}) = \max\{\epsilon, \epsilon, 0, \tfrac{1}{2}(\epsilon + \epsilon)\} = \epsilon.$$

Passing the limit as $\ell \to \infty$ in (2.6) and using the condition (θ_2), we conclude that

$$\begin{aligned}
0 &\leq \limsup_{\ell\to\infty} \theta[d(q_{2j(\ell)-1}, q_{2k(\ell)}), \Lambda(p_{2j(\ell)-1}, p_{2k(\ell)})] \\
&< \epsilon - \epsilon = 0,
\end{aligned}$$

which is a contradiction. Hence, $\{q_n\}$ is a Cauchy sequence. This completes the proof. □

2.2. **Common fixed point for two pairs of mappings.** Let (Ξ, d) be a metric space and $\mathcal{P}, \mathcal{Q} : \Xi \to \Xi$ be operators. The mapping $(\mathcal{P}, \mathcal{Q})$ is said to be

(1) *compatible* if $\lim_{n\to\infty} d(\mathcal{P}\mathcal{Q}s_n, \mathcal{Q}\mathcal{P}s_n) = 0$, when there exist a sequence $\{s_n\}$ such that
$$\lim_{n\to\infty} \mathcal{P}s_n = \lim_{n\to\infty} \mathcal{Q}s_n = s$$
for some $s \in \Xi$;

(2) *weakly compatible* if
$$\mathcal{P}\mathcal{Q}s = \mathcal{Q}\mathcal{P}s$$
whenever $\mathcal{P}s = \mathcal{Q}s$.

Theorem 2.2. *Suppose that* (Ξ, d) *is a metric space,* $\mathcal{P}, \mathcal{Q}, \mathcal{S}, \mathcal{T} : \Xi \to \Xi$ *are operators,* $\theta \in \Theta$ *satisfying* (a), (b) *of Lemma* 2.1 *and*

(c) *one of* $\mathcal{S}(\Xi)$, $\mathcal{T}(\Xi)$, $\mathcal{P}(\Xi)$ *or* $\mathcal{Q}(\Xi)$ *is a complete subspace of* Ξ.

Then $\eta \in CP(\mathcal{P}, \mathcal{S}) \cap CP(\mathcal{Q}, \mathcal{T})$*, for* $\eta \in \Xi$*. In addition, if*

(d) *the* $(\mathcal{P}, \mathcal{S})$ *and* $(\mathcal{Q}, \mathcal{T})$ *are weakly compatible,*

then $CFP(\mathcal{P}, \mathcal{Q}, \mathcal{S}, \mathcal{T})$ *is unique in* Ξ.

Proof. Start with completeness of $\mathcal{S}(\Xi)$. Then there exists $\eta \in \mathcal{S}(\Xi)$ such that
$$q_{2n-1} = \mathcal{S}p_{2n} = \mathcal{Q}p_{2n-1} \to \eta \text{ as } n \to \infty.$$
This implies that we can find $\vartheta \in \Xi$ such that
$$\mathcal{S}\vartheta = \eta. \tag{2.8}$$

Now, we propose that $\mathcal{P}\vartheta = \eta$. Suppose $d(\mathcal{P}\vartheta, \eta) > 0$. Using (2.1) and (2.8),
$$0 \le \theta[d(\mathcal{P}\vartheta, \mathcal{Q}p_{2n-1}), \Lambda(\vartheta, p_{2n-1})], \tag{2.9}$$
where
$$\begin{aligned} \Lambda(\vartheta, p_{2n-1}) &= \max\left\{ \begin{array}{c} d(\mathcal{S}\vartheta, \mathcal{T}p_{2n-1}), d(\mathcal{P}\vartheta, \mathcal{S}\vartheta), d(\mathcal{Q}p_{2n-1}, \mathcal{T}p_{2n-1}), \\ \frac{1}{2}[d(\mathcal{Q}p_{2n-1}, \mathcal{S}\vartheta) + d(\mathcal{P}\vartheta, \mathcal{T}p_{2n-1})] \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} d(\eta, q_{2n-2}), d(\mathcal{P}\vartheta, \eta), d(q_{2n-1}, q_{2n-2}), \\ \frac{1}{2}[d(q_{2n-1}, \eta) + d(A\vartheta, q_{2n-2})] \end{array} \right\}. \end{aligned} \tag{2.10}$$
Letting $n \to \infty$ in (2.10) and using (2.5),
$$\lim_{n\to\infty} \Lambda(\vartheta, p_{2n-1}) = d(\mathcal{P}\vartheta, \eta)$$
and hence, from (2.9) and the condition (θ_2),
$$\begin{aligned} 0 &\le \limsup_{n\to\infty} \theta[d(\mathcal{P}\vartheta, \mathcal{Q}p_{2n-1}), \Lambda(\vartheta, p_{2n-1})] \\ &< d(\mathcal{P}\vartheta, \eta) - d(\mathcal{P}\vartheta, \eta) = 0, \end{aligned}$$

which is a contradiction. Thus we have $\mathcal{P}\vartheta = \eta$. Since $\eta = \mathcal{P}\vartheta \in \mathcal{A}(\Xi) \subseteq \mathcal{T}(\Xi)$, there occurs $\nu \in \Xi$ satisfying $\eta = \mathcal{T}\nu$.

Now, we claim that $\mathcal{Q}\nu = \eta$. Using (2.1) and the condition (θ_1), we attain

$$\begin{aligned}
0 &\leq \theta[d(\mathcal{P}\vartheta, \mathcal{Q}\nu), \Lambda(\vartheta, \nu))] \\
&= \theta\left[d(\eta, \mathcal{Q}\nu), \max\left\{ \begin{array}{c} d(\mathcal{S}\vartheta, \mathcal{T}\nu), d(\mathcal{P}\vartheta, \mathcal{S}\vartheta), d(\mathcal{Q}\nu, \mathcal{T}\nu), \\ \frac{1}{2}[d(\mathcal{Q}\nu, \mathcal{S}\vartheta) + d(\mathcal{P}\vartheta, \mathcal{T}\nu)] \end{array} \right\}\right] \\
&= \theta[d(\eta, \mathcal{Q}\nu), d(\mathcal{Q}\nu, \eta)] \\
&< d(\eta, \mathcal{Q}\nu) - d(\eta, \mathcal{Q}\nu) \\
&= 0,
\end{aligned}$$

which is a contradiction and hence $\mathcal{Q}\nu = \eta$. Thus, on summarizing this, we arrive at

$$\mathcal{P}\vartheta = \mathcal{S}\vartheta = \eta, \ \ \mathcal{Q}\nu = \mathcal{T}\nu = \eta,$$

that is, $\eta \in CP(\mathcal{P}, \mathcal{S}) \cap CP(\mathcal{Q}, \mathcal{T})$, for $\eta \in \Xi$.

Similarly, we can conclude if one of $\mathcal{T}(\Xi)$, $\mathcal{P}(\Xi)$ or $\mathcal{Q}(\Xi)$ is a complete subspace of Ξ. Further, by the weakly compatibility of $(\mathcal{P}, \mathcal{S})$, we have

$$\mathcal{P}\eta = \mathcal{P}\mathcal{S}\vartheta = \mathcal{S}\mathcal{P}\vartheta = \mathcal{S}\eta$$

and hence $\eta \in CP(\mathcal{P}, \mathcal{S})$.

To prove $\mathcal{P}\eta = \eta$. Let $\mathcal{P}\eta \neq \eta$. It follows from (2.1) and the condition (θ_1) that

$$\begin{aligned}
0 &\leq \theta[d(\mathcal{P}\eta, \mathcal{Q}\nu), \Lambda(\eta, \nu)] \\
&= \theta[d(\mathcal{P}\eta, \eta), \Lambda(\eta, \nu)] \\
&= \theta\left[d(\mathcal{P}\eta, \mathcal{Q}\nu), \max\left\{ \begin{array}{c} d(\mathcal{S}\eta, \mathcal{T}\nu), d(\mathcal{P}\eta, \mathcal{S}\eta), d(\mathcal{Q}\nu, \mathcal{T}\nu), \\ \frac{1}{2}[d(\mathcal{Q}\nu, \mathcal{S}\eta) + d(\mathcal{P}\eta, \mathcal{T}\nu)] \end{array} \right\}\right] \\
&= \theta[d(\mathcal{P}\eta, \eta), d(\mathcal{P}\eta, \eta)] \\
&< d(\mathcal{P}\eta, \eta) - d(\mathcal{P}\eta, \eta) \\
&= 0,
\end{aligned}$$

which is a contradiction and hence $\mathcal{P}\eta = \eta$. Since $\mathcal{P}\eta = \mathcal{S}\eta = \eta$, it follows that $\eta \in CFP(\mathcal{P}, \mathcal{S})$.

Similarly, if $\mathcal{Q}$ and $\mathcal{T}$ are weakly compatible, we propose that $\eta \in CFP(\mathcal{Q}, \mathcal{T})$. Finally, let $\mu \in \Xi$ be a different common fixed point of $\mathcal{P}, \mathcal{Q}, \mathcal{S}$ and $\mathcal{T}$ with $\mu \neq \eta$. Using the condition (θ_1),

$$
\begin{aligned}
0 &\leq \theta[d(\mathcal{P}\eta, \mathcal{Q}\mu), \Lambda(\eta, \mu)] \\
&= \theta\left[d(\eta, \mu), \max\left\{ \begin{array}{c} d(\mathcal{S}\eta, \mathcal{T}\mu), d(\mathcal{P}\eta, \mathcal{S}\eta), d(\mathcal{Q}\mu, \mathcal{T}\mu), \\ \frac{1}{2}[d(\mathcal{Q}\mu, \mathcal{S}\eta) + d(\mathcal{P}\eta, \mathcal{T}\mu)] \end{array} \right\}\right] \\
&= \theta[d(\eta, \mu), d(\eta, \mu)] \\
&< d(\eta, \mu) - d(\eta, \mu) \\
&= 0
\end{aligned}
$$

and hence the result follows by the contradiction. □

3. Consequences of Theorem 2.2

Some very interesting fixed point results can be derived from the condition (2.1) of Theorems 2.2, on various form of functions $\theta \in \Theta$. We state just a few examples as corollaries (where $\Lambda(u, v)$ is given in (2.2)) out of which some of them are new and rest of them include existing results of the literature.

Corollary 3.1. (Generalization of [6]) *Let all of the conditions of Theorem* 2.2 *except the condition* (b) *is replaced by the following condition:*

$$d(\mathcal{P}p, \mathcal{Q}q) \leq \lambda\ \Lambda(p, q) \tag{3.1}$$

for all $p, q \in \Xi$ and for some $\lambda \in (0, 1)$. Then the underlying mappings have similar conclusion in Ξ.

Proof. On setting $\theta : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ by $\theta(\xi, \zeta) = \lambda\zeta - \xi$ for all $\xi, \zeta \in \mathbb{R}_+$ with $0 < \lambda < 1$ in (2.1), we have the conclusion. □

Corollary 3.2. (Generalizations of [18],[21]) *Let all of the conditions of Theorem* 2.2 *except the condition* (b) *is replaced by the following condition:*

$$d(\mathcal{P}p, \mathcal{Q}q) \leq \Lambda(p, q) - \varphi(\Lambda(p, q)) \tag{3.2}$$

for all $p, q \in \Xi$, where $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a lower semi-continuous function such that $\varphi(\xi) = 0$ if and only if $\xi = 0$. Then the underlying mappings have similar conclusion in Ξ.

Proof. On defining $\theta : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ by $\theta(\xi, s) = \zeta - \varphi(\zeta) - \xi$ for all $\xi, \zeta \in \mathbb{R}_+$, where $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a lower semi-continuous function such that $\varphi(\xi) = 0$ if and only if $\xi = 0$ in (2.1), we have the conclusion. □

Corollary 3.3. *Let all of the conditions of Theorem* 2.2 *except the condition* (b) *is replaced by the following condition:*

$$\psi(d(\mathcal{P}p, \mathcal{Q}q)) \leq \varphi(\Lambda(p, q)) \tag{3.3}$$

for all $p, q \in \Xi$, where $\psi, \varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are two continuous functions such that $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$ and $\varphi(t) < t \le \psi(t)$ for all $t > 0$. Then the underlying mappings have similar conclusion in Ξ.

Proof. If, in the equation (2.1), we define $\theta : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ by

$$\theta(\xi, \zeta) = \varphi(\zeta) - \psi(\xi)$$

for all $\xi, \zeta \in \mathbb{R}_+$, where $\psi, \varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are two continuous functions such that $\psi(\xi) = \varphi(\xi) = 0$ if and only if $\xi = 0$ and $\varphi(t) < t \le \psi(t)$ for all $\xi > 0$, we have the conclusion. □

Corollary 3.4. (Generalization of [3]) *Let all of the conditions of Theorem* 2.2 *except the condition* (b) *is replaced by the following condition:*

$$d(\mathcal{P}p, \mathcal{Q}q) \le \varphi(\Lambda(p, q)) \tag{3.4}$$

for all $p, q \in \Xi$, where $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a upper semi-continuous function with $\varphi(s) < s$ for all $t > 0$ and $\varphi(s) = 0$ if and only if $s = 0$. Then the underlying mappings have similar conclusion in Ξ.

Proof. If we define $\theta : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ by

$$\theta(\xi, \zeta) = \varphi(\zeta) - \xi$$

for all $\xi, \zeta \in \mathbb{R}_+$, where $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a upper semi-continuous function with $\varphi(\zeta) < \zeta$ for all $\zeta > 0$ and $\varphi(\zeta) = 0$ if and only if $\zeta = 0$, then (2.1) is converted to (3.4) and so the result follows from Theorem 2.2. This completes the proof. □

Corollary 3.5. (Generalization of [8]) *Suppose that (Ξ, d) is a metric space, $\mathcal{P}, \mathcal{Q}, \mathcal{S}, \mathcal{T} : \Xi \to \Xi$ are operators satisfying the condition* (a) *of Lemma* 2.1, *the conditions* (c), (d) *of Theorem* 2.2 *and the condition* (b) *of Lemma* 2.1 *is replaced by the following condition:*

$$d(\mathcal{P}p, \mathcal{Q}q) \le \Lambda(u, v)\varphi(\Lambda(p, q)) \tag{3.5}$$

for all $p, q \in \Xi$, where $\varphi : \mathbb{R}_+ \longrightarrow [0, 1)$ is a function with

$$\limsup_{t \to \tau^+} \varphi(\zeta) < 1$$

for all $\tau > 0$. Then $\mathcal{P}, \mathcal{Q}, \mathcal{S}, \mathcal{T}$ admit a unique common fixed point in Ξ.

Proof. If we define $\theta : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ by

$$\theta(\xi, \zeta) = \zeta\varphi(\zeta) - \xi$$

for all $\xi, \zeta \in \mathbb{R}_+$, where $\varphi : \mathbb{R}_+ \longrightarrow [0, 1)$ is a function with

$$\limsup_{\zeta \to \tau^+} \varphi(\zeta) < 1$$

for all $\tau > 0$, then (2.1) becomes to (3.5) and so we have the conclusion. □

4. Numerical examples

In this section, we give some numerical examples to illustrate the main results.

Example 4.1. Let $\Xi = [0, +\infty)$ be a metric space with $d(\nu, \vartheta) = |\nu - \vartheta|$. Let $\mathcal{P}, \mathcal{Q}, \mathcal{S}, \mathcal{T} : \Xi \to \Xi$ be defined by

$$\mathcal{P}u = \arctan \alpha u, \qquad \mathcal{Q}u = \arctan \frac{\beta u}{2},$$
$$\mathcal{S}u = e^{\gamma u} - 1, \qquad \mathcal{T}u = e^{\delta u} - 1,$$

where $\alpha, \beta, \gamma, \delta > 0$ and $\max\{\alpha, \beta\} < \frac{1}{4}\min\{\gamma, \delta\}$. Then the conditions (a), (c), (d) of Theorem 2.2 are obviously satisfied. Take $\theta : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ by

$$\theta(u, v) = \lambda v - u$$

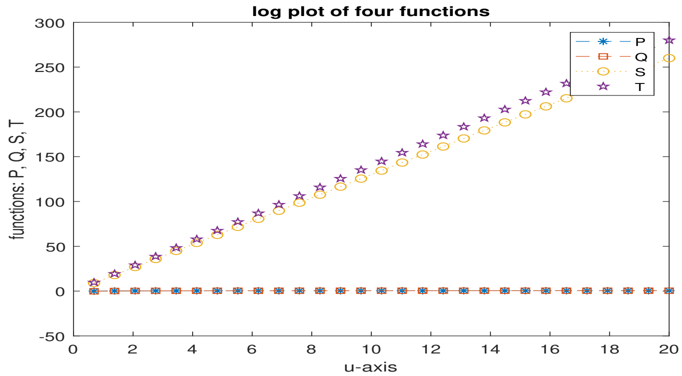
for all $u, v \in \mathbb{R}_+$ and then the condition (b) takes the form (3.1). For any $u, v \in \Xi \setminus \{0\}$, using the mean value theorem, we get

$$\begin{aligned} d(\mathcal{P}u, \mathcal{Q}v) &= \left|\arctan \alpha u - \arctan \frac{\beta v}{2}\right| \\ &\le \left|\alpha u - \frac{\beta v}{2}\right| \\ &\le \frac{1}{4}|e^{\gamma u} - e^{\delta v}| = \frac{1}{4} d(\mathcal{S}u, \mathcal{T}v) \\ &\le \frac{1}{4}\Lambda(u, v). \end{aligned} \tag{4.1}$$

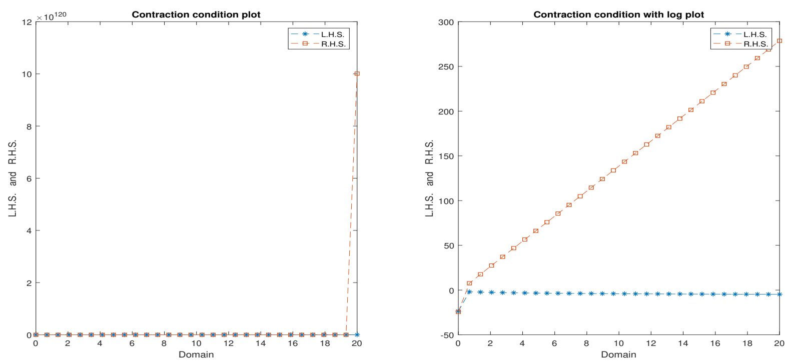
Thus all the conditions of Theorem 2.2 are satisfied and the mappings $\mathcal{P}, \mathcal{Q}, \mathcal{S}, \mathcal{T}$ have a unique common fixed point $\eta = 0$.

In the following, we have plotted (using MATLAB), the mappings $\mathcal{P}, \mathcal{Q}, \mathcal{S}, \mathcal{T}$ (log plot to visualize clearly) (Figure 1) and the left-hand (L.H.S) and the right-hand (R.H.S) calculation of the contraction condition (4.1) for particular values $\alpha = 2, \beta = 3, \gamma = 13, \delta = 14$ (Figure 2 and Figure 3).

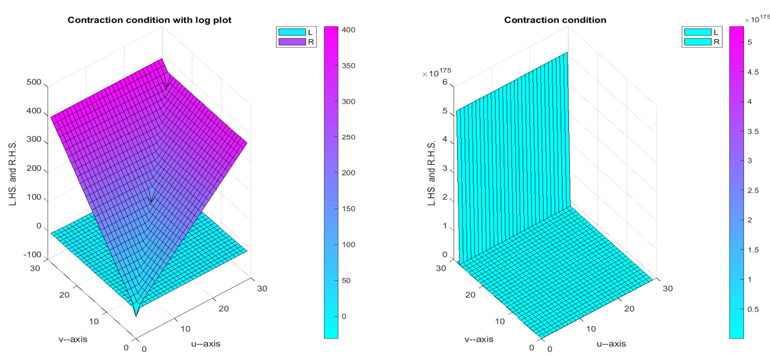
Finally, the convergence behaviours (Figure 4) of all four mappings from $q_{2n} = \mathcal{P}p_{2n} = \mathcal{T}p_{2n+1}$, $q_{2n-1} = \mathcal{Q}p_{2n-1} = \mathcal{S}p_{2n}$ has been plotted. All these graphical representation show that condition (2.1) of Theorem 2.2 is satisfied and 0 is a unique common fixed point.



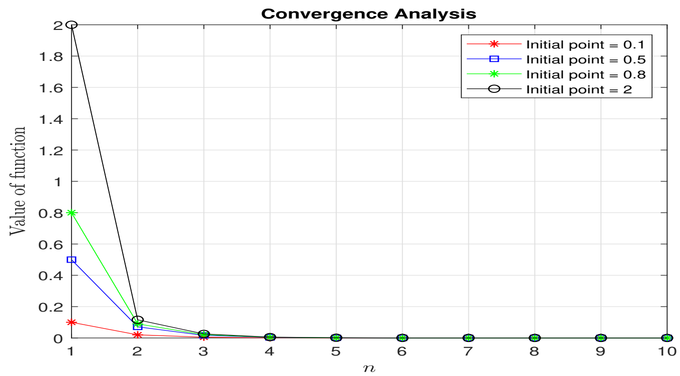
Plotting of P, Q, S, T mappings



Plotting 2D of $L.H.S.$ and $R.H.S$ of equation (4.1)



Plotting 3D of $L.H.S.$ and $R.H.S$ of equation (4.1)



Convergence analysis of mappings

Example 4.2. Let $\Xi = [0,1]$ be a metric space equipped with the metric

$$d(u,v) = \begin{cases} |u| + |v|, & \text{if } u \neq v, \\ 0, & \text{if } u = v, \end{cases}$$

for all $u, v \in \Xi$. Define the mappings $\mathcal{P}, \mathcal{Q}, \mathcal{S}, \mathcal{T} : \Xi \to \Xi$ by:

$$\mathcal{P}u = \begin{cases} 0, & \text{if } 0 \leq u \leq 1/4, \\ 1/16, & \text{if } 1/4 < u \leq 1; \end{cases} \qquad \mathcal{Q}u = 0 \text{ for } 0 \leq u \leq 1;$$

$$\mathcal{T}u = \begin{cases} u, & \text{if } 0 \leq u \leq 1/4, \\ 1, & \text{if } 1/4 < u \leq 1; \end{cases} \qquad \mathcal{S}u = \begin{cases} 0, & \text{if } u = 0, \\ 1/4, & \text{if } 0 < u \leq 1/4, \\ 1, & \text{if } 1/4 < u \leq 1. \end{cases}$$

Then, the only condition of Theorem 2.2 that has to be checked is (b)—all others are easily seen to hold true.

Take $\theta \in \Theta$ defined by

$$\theta(\xi, \zeta) = \zeta\varphi(\zeta) - \xi$$

for all $\xi, \zeta \in \mathbb{R}_+$, where $\varphi : \mathbb{R}_+ \longrightarrow [0,1)$ is a function with

$$\limsup_{\zeta \to \tau^+} \varphi(\zeta) < 1$$

for all $\tau > 0$. We will check the contractive condition (2.1), which, in this case, takes the form (3.5). Consider the following four cases:

(1) Let $0 \leq u \leq 1/4$, $0 \leq v \leq 1$. Then $d(\mathcal{P}u, \mathcal{Q}v) = 0$ and there is nothing to prove;

(2) Let $1/4 < u \leq 1$, $v = 0$. Then we have

$$d(\mathcal{P}u, \mathcal{Q}v) = 1/16, \quad \Lambda(u,v) \geq d(\mathcal{S}u, \mathcal{T}v) = 1;$$

(3) Let $1/4 < u \leq 1$, $0 < v \leq 1/4$. Then we have

$$d(\mathcal{P}u, \mathcal{Q}v) = 1/16, \ \ \Lambda(u, v) \geq d(\mathcal{S}u, \mathcal{T}v) = 1 + |v| \geq 1;$$

(4) Let $1/4 < u \leq 1$, $1/4 < v \leq 1$. Then we have

$$d(\mathcal{P}u, \mathcal{Q}v) = 1/16, \ \ \Lambda(u, v) \geq d(\mathcal{T}v, \mathcal{Q}v) = 1.$$

If we take $\varphi(t) = t/2$, then all above four cases satisfies the condition (3.5). Thus all the conditions of Theorem 2.2 are satisfied and we conclude that the mappings $\mathcal{P}, \mathcal{Q}, \mathcal{S}, \mathcal{T}$ have a unique common fixed point $\eta = 0$.

5. Results for families of self-mappings

If we select $\mathcal{P}, \mathcal{Q}, \mathcal{S}$ and $\mathcal{T}$ properly in Theorems 2.2–Corollary 3.5, we can infer some consequences for some finite family of self mappings.

Next, we employ this conception for two pairs (finite families) of self mappings in underlying space.

Theorem 5.1. *Suppose that (Ξ, d) is a metric space, $\mathcal{P}, \mathcal{Q}, \mathcal{S}, \mathcal{T}, \mathcal{M}, \mathcal{N} : \Xi \to \Xi$ are operators and $\theta \in \Theta$ satisfying the following conditions:*

(a) $\mathcal{T}\mathcal{N}(\Xi) \supseteq \mathcal{P}(\Xi)$, $\mathcal{S}\mathcal{M}(\Xi) \supseteq \mathcal{Q}(\Xi)$;

(b) *for all* $p, q \in \Xi$,

$$\theta\left[d(\mathcal{P}p, \mathcal{Q}q), \max\left\{\begin{array}{c} d(\mathcal{S}\mathcal{M}p, \mathcal{T}\mathcal{N}q), d(\mathcal{P}p, \mathcal{S}\mathcal{M}p), \\ d(\mathcal{Q}q, \mathcal{T}\mathcal{N}q), \\ \frac{1}{2}[d(\mathcal{Q}q, \mathcal{S}\mathcal{M}p) + d(\mathcal{P}p, \mathcal{T}\mathcal{N}q)] \end{array}\right\}\right] \geq 0. \qquad (5.1)$$

(c) *one of* $\mathcal{S}\mathcal{M}(\Xi), \mathcal{T}\mathcal{N}(\Xi), \mathcal{P}(\Xi)$ *or* $\mathcal{Q}(\Xi)$ *is a complete subspace of* Ξ.

Then we have the following:

(1) $CP((\mathcal{P}, \mathcal{S}\mathcal{M}) \cap CP(\mathcal{Q}, \mathcal{T}\mathcal{N}) \neq \emptyset$ *in* Ξ.

(2) *Moreover, if*

(d) *the pairs* $(\mathcal{P}, \mathcal{S}\mathcal{M})$ *and* $(\mathcal{Q}, \mathcal{T}\mathcal{N})$ *are weakly compatible.*

(3) $p^* \in CFP(\mathcal{P}, \mathcal{Q}, \mathcal{S}\mathcal{M}, \mathcal{T}\mathcal{N})$ *is unique for* $p^* \in \Xi$.

(4) *In addition,* $p^* \in \bigcap\{\mathcal{P}, \mathcal{Q}, \mathcal{S}, \mathcal{T}, \mathcal{M}, \mathcal{N}\}$ *provided*

$$\mathcal{P}\mathcal{S} = \mathcal{S}\mathcal{P}, \ \ \mathcal{P}\mathcal{M} = \mathcal{M}\mathcal{P}, \ \ \mathcal{S}\mathcal{M} = \mathcal{M}\mathcal{S},$$

$$\mathcal{Q}\mathcal{T} = \mathcal{T}\mathcal{Q}, \ \ \mathcal{Q}\mathcal{N} = \mathcal{N}\mathcal{Q}, \ \ \mathcal{T}\mathcal{N} = \mathcal{N}\mathcal{T}.$$

Proof. By Theorem 2.2, $\mathcal{P}, \mathcal{Q}, \mathcal{S}\mathcal{M}$ and $\mathcal{T}\mathcal{N}$ have a common fixed point p^* in Ξ.

Now, we demonstrate that $p^* \in CFP(\mathcal{P}, \mathcal{Q}, \mathcal{S}, \mathcal{T}, \mathcal{M}, \mathcal{N})$. Unalike, we suppose that $p^* \neq \mathcal{M}p^*$. Putting $p = \mathcal{M}p^*$ and $q = p^*$ in the (5.1), we have

$$
\begin{aligned}
0 &\le \theta\left[d(\mathcal{M}p^*, p^*), \max\left\{\begin{array}{c} d(\mathcal{SMM}p^*, \mathcal{TN}p^*), d(\mathcal{SM}p^*, \mathcal{JM}p^*), \\ d(\mathcal{TN}p^*, \mathcal{K}p^*), \\ \dfrac{d(\mathcal{SMM}p^*, \mathcal{K}p^*) + d(\mathcal{TN}p^*, \mathcal{JM}p^*)}{2s} \end{array}\right\}\right] \\
&= \theta\left[d(\mathcal{M}p^*, p^*), \max\left\{\begin{array}{c} d(\mathcal{M}p^*, p^*), d(p^*, p^*), d(p^*, p^*), \\ \dfrac{d(\mathcal{M}p^*, p^*) + d(p^*, p^*)}{2} \end{array}\right\}\right] \\
&= \theta\left[d(\mathcal{M}p^*, p^*), \max\left\{d(\mathcal{M}p^*, p^*), 0, 0, \frac{d(\mathcal{M}p^*, p^*)}{2}\right\}\right] \\
&= \theta[d(\mathcal{M}p^*, p^*), d(\mathcal{M}p^*, p^*)], \qquad (5.2)
\end{aligned}
$$

a contradiction, which implies that $p^* = \mathcal{M}p^*$. Hence $\mathcal{S}p^* = \mathcal{SM}p^* = p^*$. Therefore, we have

$$p^* = \mathcal{P}p^* = \mathcal{S}p^* = \mathcal{M}p^*.$$

Next, we affirm that $p^* \in CFP(\mathcal{Q}, \mathcal{T}, \mathcal{N})$. To get done this, we use (5.1) for $p = p^*, q = \mathcal{N}p^*$ and, with similar fashion, we can achieve $\mathcal{N}p^* = p^*$ and so $\mathcal{T}p^* = \mathcal{TN}p^* = p^*$. Thus we have

$$p^* = \mathcal{Q}p^* = \mathcal{T}p^* = \mathcal{N}p^*.$$

This completes the proof. □

Next, we use the concept of the pairwise commuting mappings to discuss the second application of Theorem 2.2.

Definition 5.2. ([15]) Let $\{\mathcal{P}_i\}_{i=1}^m$ and $\{\mathcal{Q}_k\}_{k=1}^n$ be a pair (finite families) of self mappings of Ξ and it is said to be *pairwise commuting* if the following conditions are satisfied:

(a) $\mathcal{P}_i\mathcal{P}_j = \mathcal{P}_j\mathcal{P}_i$ for each $i, j \in \mathbb{N}$ with $1 \le i, j \le m$;
(b) $\mathcal{Q}_k\mathcal{Q}_l = \mathcal{Q}_l\mathcal{Q}_k$ for each $k, l \in\in \mathbb{N}$ with $1 \le k, l \le n$;
(c) $\mathcal{P}_i\mathcal{Q}_k = \mathcal{Q}_k\mathcal{P}_i$ for each $i, k \in \mathbb{N}$ with $1 \le i \le m$, $1 \le k \le n$.

Corollary 5.3. *Let $\{\mathcal{P}_j\}_{j=1}^m$, $\{\mathcal{Q}_f\}_{f=1}^n$, $\{\mathcal{S}_\ell\}_{\ell=1}^p$ and $\{\mathcal{T}_h\}_{h=1}^q$ be two pairs (finite families) of self mappings of a metric space (Ξ, d), where*

$$\mathcal{P} = \mathcal{P}_1\mathcal{P}_2\cdots\mathcal{P}_m, \;\; \mathcal{Q} = \mathcal{Q}_1\mathcal{Q}_2\cdots\mathcal{Q}_n,$$

$$\mathcal{S} = \mathcal{S}_1\mathcal{S}_2\cdots\mathcal{S}_p, \;\; \mathcal{T} = \mathcal{T}_1\mathcal{T}_2\cdots\mathcal{T}_q$$

satisfy the inequality (2.1) *and the conditions* (a)–(d) *of Theorem* 2.2. *Then $CFP(\{\mathcal{P}_j\}_{j=1}^m, \{\mathcal{Q}_f\}_{f=1}^n, \{\mathcal{S}_\ell\}_{\ell=1}^p, \{\mathcal{T}_h\}_{h=1}^q)$ is unique if the pairs of families $(\{\mathcal{P}_j\}, \{\mathcal{S}_\ell\})$ and $(\{\mathcal{Q}_f\}, \{\mathcal{T}_h\})$ commute pairwise, where $j, \ell, f, h \in \mathbb{N}$ with $1 \le j \le m$, $1 \le \ell \le p$ $f \le j \le n$ and $1 \le h \le q$.*

Proof. The result follows from the line of arguments given in the work of Imdad et al. [15]. □

If we consider $\mathcal{P}_1 = \mathcal{P}_2 = \cdots = \mathcal{P}_m = \mathcal{P}$, $\mathcal{Q}_1 = \mathcal{Q}_2 = \cdots = \mathcal{Q}_n = \mathcal{Q}$, $\mathcal{S}_1 = \mathcal{S}_2 = \cdots = \mathcal{S}_p = \mathcal{S}$ and $\mathcal{T}_1 = \mathcal{T}_2 = \cdots = \mathcal{T}_q = \mathcal{T}$ in Corollary 5.3, we figure out below theorem that involves iterates of mappings:

Corollary 5.4. *Suppose that (Ξ, d) is a metric space, $\mathcal{P}, \mathcal{Q}, \mathcal{S}, \mathcal{T} : \Xi \to \Xi$ are mappings and $\theta \in \Theta$. For any fixed positive integers m, n, p, q, suppose that*

(a) *$\mathcal{P}^m(\Xi) \subseteq \mathcal{T}^q(\Xi)$ and $\mathcal{Q}^n(\Xi) \subseteq \mathcal{S}^p(\Xi)$;*

(b) *for all $u, v \in \Xi$,*

$$0 \le \theta\left[d(\mathcal{P}^m u, \mathcal{Q}^n v), \max\left\{ \begin{array}{c} d(\mathcal{S}^p u, \mathcal{T}^q v), d(\mathcal{P}^m u, \mathcal{S}^p u), d(\mathcal{Q}^n v, \mathcal{T}^q v), \\ \frac{1}{2}[d(\mathcal{Q}^n v, \mathcal{S}^p u) + d(\mathcal{P}^m u, \mathcal{T}^q v)] \end{array} \right\}\right]. \tag{5.3}$$

(c) *one of $\mathcal{S}^p(\Xi)$, $\mathcal{T}^q(\Xi)$, $\mathcal{P}^m(\Xi)$ or $\mathcal{Q}^n(\Xi)$ is a complete subspace of Ξ.*

Then we have the following:

(1) *$CP(\{\mathcal{P}^m, \mathcal{S}^p\} \cap \{\mathcal{Q}^m, \mathcal{T}^q\}) \neq \emptyset$ in Ξ.*

(2) *Moreover, if*

(d) *the pairs $(\mathcal{P}^m, \mathcal{S}^p)$ and $(\mathcal{Q}^n, \mathcal{T}^q)$ are weakly compatible,*

then $CFP(\mathcal{P}, \mathcal{Q}, \mathcal{S}, \mathcal{T})$ is unique if the pairs $(\mathcal{A}, \mathcal{S})$ and $(\mathcal{B}, \mathcal{T})$ are commutative.

6. Applications

Mainly, stability theory in finances may present by fixed point theorems. It has been developed to found the occurrence of the established costs, which sequentially associate with demand in all markets of an economy (the occurrence of such costs had been an open enquiry in economics). If this can be occurred, then the suggested functional has a fixed point agreeing to the theorem. Furthermore, the fixed-point theorem employed to show the slightest collection of the fixed points.

The economic system certified to the mappings utilized in the existence proofs has changed in time. It was established, sometimes indirectly, that the mappings described a dynamic price adjustment procedure leading to general equilibrium. Nevertheless, as the first outcomes regarding stability of the system, the economic explanation of the mappings was adapted and limited to the law of supply and demand as an instruction of price changes without reference to the effects of these price differences on additional demands in the following period of time.

In other words, the stability of economic system can be recognized by the fixed point. In the following examples, we deliver different economic systems.

6.1. **Functional systems.** Here we establish the uniqueness solve-ability of a common outcome of the proposed functional fractional economic equations occurring in dynamic programming. Dynamic programming plays a major role in economics because of it is speedy to reach the equilibrium point. We suppose that $\bar{B}$ and $\hat{B}$ are two different Banach spaces, $\bar{S} \subset \bar{B}$ is the space of the economic situation and $\hat{S} \subset \hat{B}$ is the space of decision.

6.1.1. *Local fractional calculus.* The concept of local fractional calculus (fractal) is developed without singular kernel in [4, 20]. It is defined to deal with non-differentiable studies in science and engineering. Recall the definition as follows: the fractal of a function $\tau(\xi)$ of order $0 < \wp \leq 1$ is formulated by

$$D^{\wp}\,\tau(\xi) \;=\; \frac{d^{\wp}\,\tau(\xi)}{\xi^{\alpha}}\Big|_{\xi=\xi_0} \;=\; \lim_{\xi\to\xi_0} \frac{d^{\wp}[\tau(\xi)-\tau(\xi_0)]}{[d(\xi-\xi_0)]^{\wp}},$$

where the formal

$$\frac{d^{\wp}[\tau(\xi)-\tau(\xi_0)]}{[d(\xi-\xi_0)]^{\wp}},$$

indicates the classical Riemann-Liouville fractional operator. A function τ is known as *local fractional continuous function* (LFCF) at ξ_0 if, for all $\vartheta > 0$, there is υ satisfies

$$|\tau(\xi)-\tau(\xi_0)| < \vartheta^{\wp}$$

whenever $|\xi-\xi_0| < \upsilon$. We refer to the space of all LFCFs by $C_{\wp}$. For any $\tau \in C_{\wp}$, the *local fractional integral* (LFI) is formulated as follows (see [5]):

$$\Upsilon_J^{\wp}\,\tau(\varsigma) = \frac{1}{\Gamma(1+\wp)} \int_a^b \tau(\varsigma)\,(d\varsigma)^{\wp}, \quad (d\varsigma)^{\wp} = \frac{\varsigma^{1-\wp}}{\Gamma(2-\wp)}\,d\varsigma^{\wp}, J=[a,b].$$

In addition, the integral (LFI) achieves the inclusion

$$\Upsilon_J^{\wp}\tau(\varsigma) \in \Big[\underline{\tau}\frac{(b-a)^{\wp}}{\Gamma(\wp+1)}, \bar{\tau}\frac{(b-a)^{\wp}}{\Gamma(\wp+1)}\Big], \tag{6.1}$$

where $\underline{\tau}$ and $\bar{\tau}$ are the upper and lower terms of τ respectively. An essential and satisfactory state for the occurrence of the LFI can be recognized by the fractal collection with a generalized Lebesgue measure zero. Lastly, suppose that $\tau(\varsigma) \in C_{\wp}(J)$. Then there is ι in J achieving

$$\Upsilon_J^{\wp}\,\tau(\varsigma) = \tau(\iota)\frac{(b-a)^{\wp}}{\Gamma(\wp+1)} \Longrightarrow \Upsilon_J^{\wp}\,1 = \frac{(b-a)^{\wp}}{\Gamma(\wp+1)}.$$

In our discussion, we suppose that the integral (LFI) ($\Upsilon_J^{\wp}$) denotes the transformation of the process of the optimal function φ with initial state x for the fractal dynamic programming (FDP):

$$\begin{aligned} \varphi_i(x) &= \sup_{y\in\hat{S}} \Phi_i\big(x, y, \varphi_i(\Upsilon_J^{\wp} t(x), y)\big), \\ \psi_i(x) &= \sup_{y\in\hat{S}} \Psi_i\big(x, y, \psi_i(\Upsilon_J^{\wp} t(x), y)\big), \end{aligned} \tag{6.2}$$

$$(x,\ \ \Upsilon_J^{\wp} \in \bar{S},\ \ y \in \hat{S},\ \ \Phi_i,\ \ \Psi_i \in \mathbb{R},\ \ i = 1, 2),$$

where x and y indicate the state and decision variables respectively.

Define the following operators:

$$\begin{aligned} \Theta_i(x) &= \sup_{y\in\hat{S}} \Phi_i\big(x, y, \vartheta(\Upsilon_J^{\wp} t(x), y)\big), \\ \Upsilon_i(x) &= \sup_{y\in\hat{S}} \Psi_i\big(x, y, \eta(\Upsilon_J^{\wp} t(x), y)\big). \end{aligned} \tag{6.3}$$

$$(i = 1, 2,\ \ x \in \bar{S},\ \ t : \bar{S} \to \bar{S}).$$

Next, we introduce occurrence result:

Theorem 6.1. *Consider the operator system* (6.3) *satisfying the following conditions:*

(a) Φ_i *and* Ψ_i *are bounded;*

(b) $|\Phi_1(x, y, \vartheta(\cdot)) - \Phi_2(x, y, \eta(\cdot))| < \Delta_{\wp}^{-1}\theta$, *where* $(x, y) \in (\bar{S}, \hat{S})$, $\theta \in \Theta$ *and* $\Delta_{\wp} = [1 + \sup_x |\Upsilon_1(\eta)(x) - \Upsilon_2(\vartheta)(x)|]/\Gamma(\wp + 1)$, *where*

$$\theta[|\Phi_1(\cdot) - \Phi_2(\cdot)|, \Lambda(\cdot, \cdot)] \ge 0,$$

and

$$\Lambda(\cdot,\cdot) = \max\left\{ \begin{array}{c} |\Upsilon_1\vartheta - \Theta_1\vartheta| \cdot |\Upsilon_2\eta - \Theta_2\eta|, \\ |\Upsilon_1\vartheta - \Theta_2\eta| \cdot |\Upsilon_2\eta - \Theta_1\vartheta| + |\Upsilon_1\vartheta - \Upsilon_2\eta|, \\ |\Upsilon_1\vartheta - \Theta_1\vartheta|, \\ |\Upsilon_2\eta - \Theta_2\eta| + \dfrac{1}{2}[|\Upsilon_1\vartheta - \Theta_2\eta| + |\Upsilon_2\eta - \Theta_1\vartheta|] \end{array} \right\};$$

(c) *For any sequences* $\{\vartheta_n\}$, $\{\eta_n\} \subset \bar{S}$, *we assume that*

$$\lim_{n\to\infty} \sup_x |\vartheta_n - \vartheta| = 0,\ \ \lim_{n\to\infty} \sup_x |\eta_n - \eta| = 0$$

such that $\eta = \Upsilon_2\vartheta_i$ *and* $\vartheta = \Upsilon_1\eta_i$, $i = 1, 2$*;*

(d) *For any* $\vartheta \in \bar{S}$, *there exist* η_1 *and* $\eta_2 \in \bar{S}$ *such that* $\Theta_1\vartheta(x) = \Upsilon_2\eta_2(x)$ *and* $\Theta_2\vartheta(x) = \Upsilon_1\eta_1(x)$ *for any* $x \in \bar{S}$*;*

(e) *For* $\Theta_1\vartheta = \Upsilon_1\vartheta$, *we have* $\Upsilon_1\Theta_1\vartheta = \Theta_1\Upsilon_1\vartheta$ *and, for* $\Theta_2\vartheta = \Upsilon_2\vartheta$, *we have* $\Upsilon_2\Theta_2\eta = \Theta_2\Upsilon_2\eta$.

Then the system of functional equations (6.2) *admits a singular common solution in* $\bar{S}$.

Proof. Obviously, our metric $d(\vartheta, \eta) = \sup_x |\vartheta(x) - \eta(x)|$ indicates a complete metric space. Moreover, in view of the conditions (a), (d), (e), Θ_i and Υ_i are weakly compatible and self mappings such that $\Theta_1(\bar{S}) \subset \Upsilon_2(\bar{S})$ and $\Theta_2(\bar{S}) \subset \Upsilon_1(\bar{S})$. The condition (a) implies that, for each $\varepsilon > 0$, there occurs $y_i \in \hat{S}$ satisfying

$$\Theta_i \vartheta_i(x) < \Phi_i(x_i, y_i, \vartheta_i) + \varepsilon, \tag{6.4}$$

where $x_i = \Upsilon_i(x, y_i)$ for each $i = 1, 2$. In addition, we have

$$\Theta_1 \vartheta_1(x) \geq \Phi_1(x, y_2, \vartheta_1(x_2)) \tag{6.5}$$

and

$$\Theta_2 \vartheta_2(x) \geq \Phi_2(x, y_1, \vartheta_2(x_1)). \tag{6.6}$$

Thus, by (6.4)-(6.6) and the condition (b), we have the following inequality:

$$\begin{aligned} \Theta_1 \vartheta_1(x) - \Theta_2 \vartheta_2(x) &\leq |\Phi_1(x, y_1, \vartheta_2(x_2)) - \Phi_2(x, y_2, \vartheta_1(x_1))| + \varepsilon \\ &\leq \Delta_\wp^{-1} \theta + \varepsilon. \end{aligned} \tag{6.7}$$

Moreover, in view of (6.4) and (6.5) together with the condition (b), we have

$$\Theta_1 \vartheta_1(x) - \Theta_2 \vartheta_2(x) \geq -\Delta_\wp^{-1} \theta - \varepsilon. \tag{6.8}$$

Combining (6.7) and (6.8), we obtain

$$|\Theta_1 \vartheta_1(x) - \Theta_2 \vartheta_2(x)| \leq \Delta_\wp^{-1} \theta + \varepsilon \tag{6.9}$$

for all $x \in \bar{S}$ and $\varepsilon > 0$ ($\varepsilon \to 0$) and so the condition (c) and Theorem 2.2 impose a single common fixed point correlate with the equilibrium point of the system (6.3). □

Example 6.2. Let $\mathbb{R} = X = Y$ be Banach spaces under the normal norm $d(x, y) = |x - y|$ and $\bar{S} = [0, 1]$ and $\hat{S} = [1, \infty)$. Define the following functions

$$\begin{cases} \varphi_i = \psi_i = \dfrac{1}{4} \dfrac{\iota x}{y}, \;\; \iota x = I^{0.5} t(x), \; t(x) \in \bar{S}, \\ \Phi_i = \Psi_i = \frac{1}{8} \frac{x}{y}, \;\; x \in \bar{S}, \; y \in \hat{S}, \\ \Theta_1 \vartheta(x) = \sup \Phi_1(x), \;\; \vartheta \in \bar{S}, \\ \Theta_2 \eta(x) = \sup \Phi_2(x), \;\; \eta \in \bar{S}, \\ \Upsilon_1 \vartheta(x) = \sup \Psi_1(x), \\ \Upsilon_2 \eta(x) = \sup \Psi_2(x), \end{cases} \tag{6.10}$$

where $\Upsilon : \bar{S} \times \hat{S} \to \bar{S}$ is defined by $\Upsilon(x,y) = \dfrac{x}{4y}$. By using (6.1), we have $\varphi_i = \psi_i \leq \dfrac{1}{4.5}$, where $\Gamma(1.5) = 0.88862$. Consequently, we obtain

$$\Theta_1\vartheta(x) = \sup \Phi_1(x) \leq \varphi_1,$$
$$\Theta_2\eta(x) = \sup \Phi_2(x) \leq \varphi_2,$$
$$\Upsilon_1\vartheta(x) = \sup \Psi_1(x) \leq \psi_1,$$
$$\Upsilon_2\eta(x) = \sup \Psi_2(x) \leq \psi_2.$$

A computation implies

$$\Delta = [1 + \sup |\Upsilon_1 - \Upsilon_2|] = 1.$$

Further, for each ϑ and η, we define two sequences $\{\vartheta_n\}$ and $\{\eta_n\}$ by

$$\vartheta_n = \Big(1 - \frac{1}{n}\Big)\vartheta, \;\; \eta_n = \Big(1 - \frac{1}{n}\eta\Big)$$

achieving the following limits:

$$\lim_{n\to\infty} \sup_{x\in\bar{S}} \vartheta_n(x) = \lim_{n\to\infty} \sup_{x\in\bar{S}} \eta_n(x) = 0.$$

In virtue of the definition of the functional, we conclude the following facts

$$\Theta_1\vartheta(x) = \Upsilon_2\eta_2(x),$$
$$\Theta_2\vartheta(x) = \Upsilon_1\eta_1(x).$$

Finally, it is clear that Φ_i and Ψ_i, $i = 1,2$ are bounded and satisfy

$$|\Phi_1\vartheta(x) - \Phi_2\eta(x)| \leq \frac{1}{8}|\vartheta - \eta|.$$

Hence, in view of Theorem 6.1, the system (6.10) has a single common fixed point agree with the stability point.

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