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QUALITATIVE ANALYSIS OF A PROPORTIONAL CAPUTO FRACTIONAL PANTOGRAPH DIFFERENTIAL EQUATION WITH MIXED NONLOCAL CONDITIONS

Bounmy Khaminsou[1], **Chatthai Thaiprayoon**[2], **Weerawat Sudsutad**[3] **and Sayooj Aby Jose**[4]

[1]Department of Mathematics, Faculty of Science, Burapha University
Chonburi, 22000, Thailand
e-mail: kbbounmy@yahoo.com

[2]Department of Mathematics, Faculty of Science, Burapha University
Chonburi, 22000, Thailand
e-mail: chatthai@buu.ac.th

[3]Department of General Education, Faculty of Science and Health Technology
Navamindradhiraj University, Bangkok, 10300, Thailand;
Department of Applied Statistics, Faculty of Applied Science
King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand
e-mail: weerawat@nmu.ac.th;wrw.sst@gmail.com

[4]Ramanujan Centre for Higher Mathematics, Alagappa University
Karaikudi-630 004, India;
Department of Mathematics, Alagappa University, Karaikudi-630 004, India
e-mail: sayooaby999@gmail.com

Abstract. In this paper, we investigate existence, uniqueness and four different types of Ulam's stability, that is, Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability of the solution for a class of nonlinear fractional Pantograph differential equation in term of a proportional Caputo fractional derivative with mixed nonlocal conditions. We construct sufficient conditions for the existence and uniqueness of solutions by utilizing well-known classical fixed point theorems such as Banach contraction principle, Leray-Schauder nonlinear alternative and Krasnosel'skiĭ's fixed point theorem. Finally, two examples are also given to point out the applicability of our main results.

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[0]Corresponding author: W. Sudsutad(weerawat@nmu.ac.th).

1. Introduction

The fractional calculus has origin during the same time as that of the ordinary calculus. It has been concerned with derivatives and integrals of arbitrary order that can be non integer of functions. In recent years, several researchers have exposed attention in the field of qualitative theory of fractional differential equations, which will be used to describe phenomena of real world problems. For more details, see the monographs [4, 18, 19, 27, 29, 31, 34, 39, 47] and the references therein.

Several interesting and important area concerning of research for fractional differential equations are devoted to the existence theory and stability analysis of the solutions. In recent years, there are many researchers have discussed the existence, uniqueness and different types of Ulam–Hyers (UH) stability of solutions of initial and boundary value problems for fractional differential equations. The UH stability is the essential and special type of stability analysis that researchers studied in the field of mathematical analysis. The concept of Ulam stability of functional equations was firstly initiated by Ulam [42, 43] and Hyers [23] presented the partial answer to the question of Ulam in the case of Banach space. Thereafter, this type of stability is called the UH stability. In 1950, the Hyers stability was generalized by Aoki [10]. Rassias [36, 37] provided an interesting generalization of the UH stability of linear and nonlinear mappings. The UH stability was initially applied to linear differential equation by Obloza [33]. We refer the reader to see monographs [1, 3, 5, 11, 12, 14, 17, 26, 30, 32, 45, 46]. It is to be noted that, the above said areas of interest (existence and stability) have been fabulously deliberated within Riemann–Liouville, Caputo, Hilfer or Hadamard derivatives.

Recently, Jarad *et al.* [24] introduced a new type of fractional derivative operator so called generalized proportional fractional (GPF) derivatives extended by local derivatives [9]. The characteristic of the new derivative is that it involves two fractional order, preserves the semigroup property, possesses nonlocal character and upon limiting cases it converges to the original function and its derivative. The GPF derivative is well behaved and has a various helpful over the classical derivatives in the sense that it generalizes previously defined derivatives in the literature. We list some recent papers which have been refined in frame of GPF derivative and other related works [2, 7, 8, 25, 35, 40, 41].

Nonlocal boundary value problems have become a rapidly growing area of research. The study of this type of problems is driven not only by a theoretical interest, but also by the fact that several phenomena in engineering, physics and life sciences can be modelled in this way. The idea of nonlocal conditions dates back to the work of Hilb [22]. However, the systematic investigation of a certain class of spatial nonlocal problems was carried out by Bitsadze and Samarskii [15]. We refer the reader to [13, 16] and the references cited therein for a motivation regarding nonlocal conditions.

In [12], the authors considered a fractional differential equations with mixed nonlocal fractional derivatives, integrals and multi-point conditions of the form

$$\begin{cases} {}^{c}\mathfrak{D}^{\alpha}x(t) = f(t, x(t)), \qquad t \in (0, T], \\ \displaystyle\sum_{i=1}^{m} \gamma_i x(\eta_i) + \sum_{j=1}^{n} \lambda_j {}^{c}\mathfrak{D}^{\beta_j} x(\xi_i) + \sum_{r=1}^{k} \sigma_r \mathcal{I}^{\delta_r} x(\phi_r) = A, \end{cases} \tag{1.1}$$

where $x \in C^1([0,T], \mathbb{R})$ is a continuous function, ${}^{c}\mathfrak{D}^{\alpha}$, ${}^{c}\mathfrak{D}^{\beta_j}$ denote the Caputo fractional derivatives of order α and β_j, respectively, $0 < \beta_j \leq \alpha \leq 1$ for $j = 1, 2, \ldots, n$, $\mathcal{I}^{\delta_r}$ is the Riemann–Liouville fractional integral operator of order $\delta_r > 0$ for $r = 1, 2, \ldots, k$, γ_i, λ_j, σ_r, $A \in \mathbb{R}$, η_i, ξ_j, $\phi_r \in [0, T]$, $i = 1, 2, \ldots, m$ and $f \in C([0,T] \times \mathbb{R}, \mathbb{R})$.

The existence and uniqueness results were obtained by applying Schaefer's fixed point theorem and Banach's contraction mapping principle. In addition, the authors established different kinds of Ulam stability for the purposed problem.

In [44], the authors studied the existence, uniqueness and Ulam–Hyers–Rassias stability for a class of ψ-Hilfer fractional differential equations described by

$$\begin{cases} {}^{H}\mathfrak{D}_{a^+}^{\alpha,\rho;\psi} x(t) = f(t, x(t), {}^{H}\mathfrak{D}_{a^+}^{\alpha,\rho;\psi} x(t)), \qquad t \in J = (a, T], \\ \mathcal{I}_{a^+}^{1-\gamma;\psi} x(a) = x_a, \qquad \alpha \leq \gamma = \alpha + \rho - \alpha\rho, \qquad T > a, \end{cases} \tag{1.2}$$

where ${}^{H}\mathfrak{D}_{a^+}^{\alpha,\rho;\psi}$ is the ψ-Hilfer fractional derivative of order $\alpha \in (0,1]$ and type $\rho \in [0,1]$, $\mathcal{I}_{a^+}^{1-\gamma;\psi}$ is the Riemann–Liouville fractional integral of order $1-\gamma$ with respect to the function ψ, $f \in C(J \times \mathbb{R}^2, \mathbb{R})$ and $x_a \in \mathbb{R}$.

Harikrishman *et al.* [21] discussed existence, uniqueness of nonlocal initial value problems for Pantograph equations with ψ-Hilfer fractional derivative of the form

$$\begin{cases} {}^{H}\mathfrak{D}_{a^+}^{\alpha,\rho;\psi} x(t) = f(t, x(t), x(\lambda t)), \quad t \in J = (a, b], \quad 0 < \lambda < 1, \\ \mathcal{I}_{a^+}^{1-\gamma;\psi} x(a) = \displaystyle\sum_{i=1}^{k} c_i x(\tau_i), \qquad \tau_i \in (a, b], \qquad \alpha \leq \gamma = \alpha + \rho - \alpha\rho, \end{cases} \tag{1.3}$$

where ${}^{H}\mathfrak{D}_{a^+}^{\alpha,\rho;\psi}$ is the ψ-Hilfer fractional derivative of order $\alpha \in (0,1)$ and type $\rho \in [0,1]$, $\mathcal{I}_{a^+}^{1-\gamma;\psi}$ is the Riemann–Liouville fractional integral of order $1-\gamma$ with respect to the continuous function ψ such that $\psi' > 0$ and $f \in C(J \times \mathbb{R}^2, \mathbb{R})$.

In [6], the authors established existence, uniqueness and Ulam–Hyers stability of implicit Pantograph fractional differential equations involving ψ-Hilfer fractional derivatives of the form

$$\begin{cases} {}^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi}x(t) = f(t, x(t), x(\lambda t), {}^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi}x(\lambda t)), \quad t \in J,\ 0 < \lambda < 1, \\ \mathcal{I}_{0^+}^{1-\gamma;\psi}x(0^+) = \sum_{i=1}^{m} b_i\, \mathcal{I}_{0^+}^{\beta;\psi}x(\xi_i), \quad \xi_i \in J,\ \alpha \le \gamma = \alpha + \rho - \alpha\rho, \end{cases} \tag{1.4}$$

where ${}^{H}\mathfrak{D}_{0^+}^{\alpha,\rho;\psi}$ is the ψ-Hilfer fractional derivative of order $\alpha \in (0,1)$ and type $\rho \in [0,1]$, $\mathcal{I}_{0^+}^{1-\gamma;\psi}$ and $\mathcal{I}_{0^+}^{\beta;\psi}$ are the ψ-Riemann–Liouville fractional integral of order $1-\gamma$ and $\beta > 0$, respectively, with respect to the continuous function ψ such that $\psi' \neq 0$ and $f \in C(J \times \mathbb{R}^3, \mathbb{R})$, $J = [0, T]$, $b_i \in \mathbb{R}$ and $0 < \xi_1 \le \xi_2 \le \cdots \le \xi_m < T$, $T > 0$.

Motivated by the papers [6, 12, 21, 44] and some familiar results on fractional Pantograph differential equations, we discuss the existence results and different types of Ulam stability such as Ulam–Hyers, generalized Ulam–Hyers, Ulam–Hyers–Rassias and generalized Ulam–Hyers–Rassias stability for the generalized proportional fractional Pantograph differential equations with mixed nonlocal conditions of the form:

$$\begin{cases} {}_{a}^{C}D^{\alpha,\rho}x(t) = f(t, x(t), x(\lambda t), {}_{a}^{C}D^{\alpha,\rho}x(\lambda t)), \quad t \in [a, T],\ 0 < \lambda < 1, \\ \sum_{i=1}^{m} \gamma_i x(\eta_i) + \sum_{j=1}^{n} \kappa_j {}_{a}^{C}D^{\beta_j,\rho}x(\xi_j) + \sum_{r=1}^{k} \sigma_r {}_{a}I^{\delta_r,\rho}x(\theta_r) = A, \end{cases} \tag{1.5}$$

where ${}_{a}^{C}D^{q,\rho}$ is the Caputo GPF derivative of order $q = \{\alpha, \beta_j\}$ with $0 < \beta_j < \alpha \le 1$, for $j = 1, 2, \ldots, n$, $0 < \rho \le 1$, the notation ${}_{a}I^{\delta_r,\rho}$ is the Riemann-Liouville GPF integral of order $\delta_r > 0$ for $r = 1, 2, \ldots, k$, $\rho > 0$, the given constants $\gamma_i, \kappa_j, \sigma_r \in \mathbb{R}$, the points $\eta_i, \xi_j, \theta_r \in [a, T]$, $i = 1, 2, \ldots, m$, and $f : [a, T] \times \mathbb{R}^3 \to \mathbb{R}$ is a given continuous function, $T > a \ge 0$.

The paper is organized as follows: In Section 2, we recall some basic and essential definitions and lemmas. In Section 3 the existence and uniqueness results for the problem (1.5) are obtained, via Banach contraction principle, Leray-Schauder nonlinear alternative and Krasnosel'skiĭ's fixed point theorems. In Section 4, we discuss the Ulam-Hyers, generalized Ulam-Hyers, Ulam-Hyers-Rassias and generalized Ulam-Hyers-Rassias stability results. Finally, some examples are given in Section 5 to illustrate the benefit of our main results.

2. Preliminaries

In this section, we recall some definitions and properties of generalized proportional fractional derivatives and fractional integrals that will be used throughout the remaining part of this paper. For more details, see; [20, 24, 38].

Definition 2.1. ([24]) The generalized proportional fractional (GPF) integral of a function f of order $\alpha > 0$ with $\rho \in (0,1]$ is defined as

$$({}_aI^{\alpha,\rho} f)(t) = \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1} f(s)ds, \tag{2.1}$$

where $\Gamma(\cdot)$ is represent the Gamma function [29].

Definition 2.2. ([24]) The Caputo type generalized proportional fractional derivative of a function f of order α with $\rho \in (0,1]$ is defined as

$$({}_a^C D^{\alpha,\rho} f)(t) = \frac{1}{\rho^{n-\alpha}\Gamma(n-\alpha)} \int_a^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{n-\alpha-1} D^{n,\rho} f(s)ds, \tag{2.2}$$

where $n = [\alpha]+1$, $[\alpha]$ represents the integer part of the real number α and $D^{n,\rho} f(t) = (D^\rho f(t))^n$ with $D^\rho f(t) = (1-\rho)f(t) - \rho f'(t)$.

Lemma 2.3. ([24]) *For $\rho \in (0,1]$ and $n = [\alpha]+1$, we have*

$$({}_a{}^C D^{\alpha,\rho}\ {}_aI^{\alpha,\rho} f(s))(t) = f(t)$$

and

$$({}_aI^{\alpha,\rho}\ {}_a^C D^{\alpha,\rho} f)(t) = f(t) - e^{\frac{\rho-1}{\rho}(t-a)} \sum_{k=0}^{n-1} \frac{D^{k,\rho} f(a)}{\rho^k k!}(t-a)^k. \tag{2.3}$$

Proposition 2.4. ([24]) *Let $\alpha \geq 0$, $\beta > 0$. Then, for any $\rho \in (0,1]$ and $n = [\alpha]+1$, we have*

(i) $\left({}_aI^{\alpha,\rho} e^{\frac{\rho-1}{\rho}t}(t-a)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)\rho^\alpha} e^{\frac{\rho-1}{\rho}x}(x-a)^{\beta+\alpha-1}$, $\alpha > 0$.

(ii) $\left({}_a^C D^{\alpha,\rho} e^{\frac{\rho-1}{\rho}t}(t-a)^{\beta-1}\right)(x) = \frac{\rho^\alpha\Gamma(\beta)}{\Gamma(\beta-\alpha)} e^{\frac{\rho-1}{\rho}x}(x-a)^{\beta-\alpha-1}$, $\beta > n$.

(iii) $\left({}_a^C D^{\alpha,\rho} e^{\frac{\rho-1}{\rho}t}(t-a)^k\right)(x) = 0$, $\quad k = 0,1,\ldots,n-1$.

Let $\mathbb{E} = C([a,T],\mathbb{R})$ be the Banach space of all continuous functions from $[a,T]$ into $\mathbb{R}$ equipped with the norm $\|x\|_{\mathbb{E}} = \sup_{t\in[a,T]}\{|x(t)|\}$.

In order to transform the purpose problem into a fixed point problem, (1.5) must be converted to an equivalent Volterra integral equation. We provide the following lemma which is an important in our main results.

Lemma 2.5. *Let $h : [a,T] \to \mathbb{R}$ be a continuous function, $0 < \beta_j < \alpha \le 1$, $j = 1,\dots,n$ and ρ, δ_r, > 0, $r = 1,2,\dots,k$. Then, the function $x \in \mathbb{E}$ is a solution to the following linear generalized proportional fractional equation equipped with mixed nonlocal conditions of the form:*

$$\begin{cases} {}_a^C D^{\alpha,\rho} x(t) = h(t), \quad t \in [a,T], \\ \displaystyle\sum_{i=1}^{m} \gamma_i x(\eta_i) + \sum_{j=1}^{n} \kappa_j {}_a^C D^{\beta_j,\rho} x(\xi_j) + \sum_{r=1}^{k} \sigma_r I^{\delta_r,\rho} x(\theta_r) = A, \end{cases} \tag{2.4}$$

if and only if x satisfies the following integral equation

$$\begin{aligned} x(t) &= {}_aI^{\alpha,\rho}h(t) + \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg(A - \sum_{i=1}^{m}\gamma_i {}_aI^{\alpha,\rho}h(\eta_i) - \sum_{j=1}^{n}\kappa_j {}_aI^{\alpha-\beta_j,\rho}h(\xi_j) \\ &\quad - \sum_{r=1}^{k}\sigma_r {}_aI^{\alpha+\delta_r,\rho}h(\theta_r)\Bigg), \end{aligned} \tag{2.5}$$

where

$$\Omega := \sum_{i=1}^{m}\gamma_i e^{\frac{\rho-1}{\rho}(\eta_i-a)} + \sum_{r=1}^{k}\frac{\sigma_r(\theta_r-a)^{\delta_r}e^{\frac{\rho-1}{\rho}(\theta_r-a)}}{\rho^{\delta_r}\Gamma(1+\delta_r)} \neq 0. \tag{2.6}$$

Proof. Let x be a solution of the problem (2.4). By using Lemma 2.3, the integral equation can be written as

$$x(t) = {}_aI^{\alpha,\rho}h(t) + c_1 e^{\frac{\rho-1}{\rho}(t-a)}, \tag{2.7}$$

where arbitrary constants $c_1 \in \mathbb{R}$.

Taking the operators ${}_a^C D^{\beta_j,\rho}$ and ${}_aI^{\delta_r,\rho}$ into (2.7) with Proposition 2.4 (i), we obtain

$$\begin{aligned} {}_a^C D^{\beta_j,\rho}x(t) &= {}_aI^{\alpha-\beta_j,\rho}h(t), \\ {}_aI^{\delta_r,\rho}x(t) &= {}_aI^{\delta_r,\rho}h(t) + c_1\frac{(t-a)^{\delta_r}e^{\frac{\rho-1}{\rho}(t-a)}}{\rho^{\delta_r}\Gamma(1+\delta_r)}. \end{aligned}$$

Applying the given boundary condition in (2.4), we have

$$\begin{aligned} A &= \sum_{i=1}^{m}\gamma_i {}_aI^{\alpha,\rho}h(\eta_i) + \sum_{j=1}^{n}\kappa_j {}_aI^{\alpha-\beta_j,\rho}h(\xi_j) + \sum_{r=1}^{k}\sigma_r {}_aI^{\alpha+\delta_r,\rho}h(\theta_r) \\ &\quad + c_1\left(\sum_{i=1}^{m}\gamma_i e^{\frac{\rho-1}{\rho}(\eta_i-a)} + \sum_{r=1}^{k}\frac{\sigma_r(\theta_r-a)^{\delta_r}e^{\frac{\rho-1}{\rho}(\theta_r-a)}}{\rho^{\delta_r}\Gamma(1+\delta_r)}\right). \end{aligned}$$

Solving the above equation, it follows that

$$c_1 = \frac{1}{\Omega}\left(A - \sum_{i=1}^{m}\gamma_{ia}I^{\alpha,\rho}h(\eta_i) - \sum_{j=1}^{n}\kappa_{ja}I^{\alpha-\beta_j,\rho}h(\xi_j) - \sum_{r=1}^{k}\sigma_{ra}I^{\alpha+\delta_r,\rho}h(\theta_r)\right),$$

where Ω is defined by (2.6). Inserting these value of c_1 in (2.7), we get (2.5).

Conversely, it is easily to shown by direct computation that the solution $x(t)$ is given by (2.5) satisfies the problem (2.4) under the given conditions. This completes the proof. □

Fixed point theorems play a major role in establishing the existence theory for the problem (1.5). We collect here some well-known fixed point theorems used in this paper.

Lemma 2.6. **([20], Banach contraction principle)** *Let D be a non-empty closed subset of a Banach space E. Then any contraction mapping T from D into itself has a unique fixed point.*

Lemma 2.7. **([20], Nonlinear alternative for single-valued maps)** *Let E be a Banach space, C be a closed, convex subset of M, X be an open subset of C, and $0 \in X$. Suppose that $F : \overline{X} \to C$ is a continuous, compact (that is, $F(\overline{X})$ is a relatively compact subset of C) map. Then either*

(i) *F has a fixed point in $\overline{X}$, or*
(ii) *there is $x \in \partial X$ (the boundary of X in C) and $\varrho \in (0,1)$ with $x = \varrho F(x)$.*

Lemma 2.8. **([28], Krasnoselskii's fixed point theorem)** *Let M be a closed, bounded, convex, and nonempty subset of a Banach space.Let A, B be the operators such that*

(i) *$Ax + By \in \mathcal{M}$ whenever $x, y \in \mathcal{M}$;*
(ii) *A is compact and continuous;*
(iii) *B is contraction mapping.*

Then there exists $z \in \mathcal{M}$ such that $z = Az + bz$.

3. Existence results

For simplicity, we set

$$F_x(t) = f(t, x(t), x(\lambda t), F_x(\lambda t)).$$

Throughout this paper, the expression ${}_aI^{\alpha,\rho}F_x(s)(c)$ means that

$${}_aI^{q,\rho}F_x(s)(c) := \frac{1}{\rho^q\Gamma(q)}\int_a^c e^{\frac{\rho-1}{\rho}(c-s)}(c-s)^{q-1}F_x(s)ds,$$

where $q = \{\alpha, \alpha - \beta_j, \alpha + \delta_r\}$ and $c = \{t, \eta_i, \xi_j, \theta_r\}$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$, $r = 1, 2, \dots, k$.

In view of Lemma 2.5, an operator $\mathcal{Q} : \mathbb{E} \to \mathbb{E}$ is defined by

$$\begin{aligned}(\mathcal{Q}x)(t) &= {}_aI^{\alpha,\rho}F_x(s)(t) + \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg(A - \sum_{i=1}^{m}\gamma_{i\,a}I^{\alpha,\rho}F_x(s)(\eta_i) \\ &\quad - \sum_{j=1}^{n}\kappa_{j\,a}I^{\alpha-\beta_j,\rho}F_x(s)(\xi_j) - \sum_{r=1}^{k}\sigma_{r\,a}I^{\alpha+\delta_r,\rho}F_x(s)(\theta_r)\Bigg), \quad (3.1)\end{aligned}$$

where the operators $\mathcal{Q}_1$, $\mathcal{Q}_2 : \mathbb{E} \to \mathbb{E}$ are defined by

$$(\mathcal{Q}_1x)(t) = {}_aI^{\alpha,\rho}F_x(s)(t), \quad (3.2)$$

$$\begin{aligned}(\mathcal{Q}_2x)(t) &= \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg(A - \sum_{i=1}^{m}\gamma_{i\,a}I^{\alpha,\rho}F_x(s)(\eta_i) - \sum_{j=1}^{n}\kappa_{j\,a}I^{\alpha-\beta_j,\rho}F_x(s)(\xi_j) \\ &\quad - \sum_{r=1}^{k}\sigma_{r\,a}I^{\alpha+\delta_r,\rho}F_x(s)(\theta_r)\Bigg), \quad (3.3)\end{aligned}$$

which implies $\mathcal{Q}x = \mathcal{Q}_1x + \mathcal{Q}_2x$. It should be noticed that the problem (1.5) has solutions if and only if the operator $\mathcal{Q}$ has fixed points. In the following subsection, we establish the existence results of solutions for the problem (1.5), which is studied by applying Banach contraction principle, Leray-Schauder nonlinear alternative and Krasnosel'skiǐ's fixed point theorem.

The first existence and uniqueness result of a solution for the problem (1.5) will be proved by using Banach contraction principle (Banach's fixed point theorem).

Theorem 3.1. *Assume that $f : [a, T] \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function such that*

(H_1) *there exist constants $L_1 > 0$ and $0 < 2L_1\Lambda_1 + L_2 < 1$ such that*

$$|f(t, u_1, v_1, w_1) - f(t, u_2, v_2, w_2)| \le L_1\left(|u_1 - u_2| + |v_1 - v_2|\right) + L_2|w_1 - w_2|$$

for any u_i, v_i, $w_i \in \mathbb{R}$, $i = 1, 2$ and $t \in [a, T]$.

If

$$\frac{2L_1\Lambda_1}{1 - L_2} < 1, \quad (3.4)$$

then the problem (1.5) *has a unique solution* ($x \in \mathbb{E}$) *on* $[a, T]$, *where*

$$\Lambda_1 = \frac{(T-a)^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \frac{1}{|\Omega|}\Bigg(\sum_{i=1}^{m}\frac{|\gamma_i|(\eta_i-a)^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \sum_{j=1}^{n}\frac{|\kappa_j|(\xi_j-a)^{\alpha-\beta_j}}{\rho^{\alpha-\beta_j}\Gamma(\alpha-\beta_j+1)} + \sum_{r=1}^{k}\frac{|\sigma_r|(\theta_r-a)^{\alpha+\delta_r}}{\rho^{\alpha+\delta_r}\Gamma(\alpha+\delta_r+1)}\Bigg). \tag{3.5}$$

Proof. Firstly, we transform the problem (1.5) into a fixed point problem, $x = \mathcal{Q}x$, where the operator $\mathcal{Q}$ is defined as in (3.1). It is clear that the fixed points of the operator $\mathcal{Q}$ are solutions of the problem (1.5). Applying the Banach contraction principle, we shall show that the operator $\mathcal{Q}$ has a fixed point which is the unique solution of the problem (1.5).

Let $\sup_{t\in[a,T]} |f(t, 0, 0, 0)| := M_1 < \infty$. Next, we set

$$B_{R_1} := \{x \in \mathbb{E} : \|x\|_{\mathbb{E}} \le R_1\}$$

with

$$R_1 \ge \frac{M_1\Lambda_1|\Omega| + |A|(1-L_2)}{|\Omega|[1-(2L_1\Lambda_1+L_2)]}, \qquad 2L_1\Lambda_1 + L_2 < 1, \tag{3.6}$$

where Ω and Λ_1 are given by (2.6) and (3.5), respectively. Observe that B_{R_1} is bounded, closed, and convex subset of $\mathbb{E}$. The proof is divided into two steps:

Step I. To show that $\mathcal{Q}B_{R_1} \subset B_{R_1}$.

For any $x \in B_{R_1}$, we have

$$|(\mathcal{Q}x)(t)| \le {}_aI^{\alpha,\rho}|F_x(s)|(t) + \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{|\Omega|}\Bigg(|A| + \sum_{i=1}^{m}|\gamma_i|{}_aI^{\alpha,\rho}|F_x(s)|(\eta_i) + \sum_{j=1}^{n}|\kappa_j|{}_aI^{\alpha-\beta_j,\rho}|F_x(s)|(\xi_j) + \sum_{r=1}^{k}|\sigma_r|{}_aI^{\alpha+\delta_r,\rho}|F_x(s)|(\theta_r)\Bigg).$$

It follows from condition (H_1) that

$$\begin{aligned} |F_x(t)| &\le |f(t, x(t), x(\lambda t), F_x(\lambda t)) - f(t, 0, 0, 0)| + |f(t, 0, 0, 0)| \\ &\le 2L_1|x(t)| + L_2|F_x(t)| + M_1 \\ &\le \frac{2L_1|x(t)| + M_1}{1-L_2}. \end{aligned}$$

This implies that

$$\begin{aligned} |(\mathcal{Q}x)(t)| &\leq {}_aI^{\alpha,\rho}\left(\frac{2L_1|x(s)|+M_1}{1-L_2}\right)(t) \\ &\quad +\frac{e^{\frac{\rho-1}{\rho}(t-a)}}{|\Omega|}\Bigg[|A|+\sum_{i=1}^{m}|\gamma_i|_aI^{\alpha,\rho}\left(\frac{2L_1|x(s)|+M_1}{1-L_2}\right)(\eta_i) \\ &\quad +\sum_{j=1}^{n}|\kappa_j|_aI^{\alpha-\beta_j,\rho}\left(\frac{2L_1|x(s)|+M_1}{1-L_2}\right)(\xi_j) \\ &\quad +\sum_{r=1}^{k}|\sigma_r|_aI^{\alpha+\delta_r,\rho}\left(\frac{2L_1|x(s)|+M_1}{1-L_2}\right)(\theta_r)\Bigg]. \end{aligned}$$

By using $0 < e^{\frac{\rho-1}{\rho}(u-s)} \leq 1$ for any $a \leq s < u \leq T$, we obtain

$$\begin{aligned} |(\mathcal{Q}x)(t)| &\leq \left(\frac{2L_1R_1+M_1}{1-L_2}\right)\Bigg(\frac{(T-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)} \\ &\quad +\frac{1}{|\Omega|}\Bigg(\sum_{i=1}^{m}\frac{|\gamma_i|(\eta_i-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}+\sum_{j=1}^{n}\frac{|\kappa_j|(\xi_j-a)^{\alpha-\beta_j}}{\rho^{\alpha-\beta_j}\Gamma(\alpha-\beta_j+1)} \\ &\quad +\sum_{r=1}^{k}\frac{|\sigma_r|(\theta_r-a)^{\alpha+\delta_r}}{\rho^{\alpha+\delta_r}\Gamma(\alpha+\delta_r+1)}\Bigg)\Bigg)+\frac{|A|}{|\Omega|} \\ &= \left(\frac{2L_1R_1+M_1}{1-L_2}\right)\Lambda_1+\frac{|A|}{|\Omega|} \leq R_1, \end{aligned}$$

which implies that $\|\mathcal{Q}x\|_{\mathbb{E}} \leq R_1$. Therefore, $\mathcal{Q}B_{R_1} \subset B_{R_1}$.

Step II. To show that the operator $\mathcal{Q} : \mathbb{E} \to \mathbb{E}$ is contraction.
For any x, $y \in \mathbb{E}$ and for each $t \in [a, T]$, we have

$$\begin{aligned} |(\mathcal{Q}x)(t)-(\mathcal{Q}y)(t)| &\leq {}_aI^{\alpha,\rho}|F_x(s)-F_y(s)|(T) \\ &\quad +\frac{e^{\frac{\rho-1}{\rho}(T-a)}}{|\Omega|}\Bigg(\sum_{i=1}^{m}|\gamma_i|_aI^{\alpha,\rho}|F_x(s)-F_y(s)|(\eta_i) \\ &\quad +\sum_{j=1}^{n}|\kappa_j|_aI^{\alpha-\beta_j,\rho}|F_x(s)-F_y(s)|(\xi_j) \\ &\quad +\sum_{r=1}^{k}|\sigma_r|_aI^{\alpha+\delta_r,\rho}|F_x(s)-F_y(s)|(\theta_r)\Bigg) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
|F_x(t) - F_y(t)| &\leq |f(t, x(t), x(\lambda t), F_x(\lambda t)) - f(t, x(t), x(\lambda t), F_y(\lambda t))| \\
&\leq L_1(|x(t) - y(t)| + |x(\lambda t) - y(\lambda t)|) + L_2|F_x(\lambda t) - F_y(\lambda t)| \\
&\leq 2L_1|x(t) - y(t)| + L_2|F_x(t) - F_y(t)| \\
&\leq \frac{2L_1}{1 - L_2}|x(t) - y(t)|. \qquad (3.8)
\end{aligned}$$

Then, by substituting (3.8) in (3.7), we get

$$\begin{aligned}
&|(\mathcal{Q}x)(t) - (\mathcal{Q}y)(t)| \\
&\leq {}_aI^{\alpha,\rho}\left(\frac{2L_1}{1 - L_2}|x(s) - y(s)|\right)(T) \\
&\quad + \frac{e^{\frac{\rho-1}{\rho}(T-a)}}{|\Omega|}\left[\sum_{i=1}^{m}|\gamma_i|_aI^{\alpha,\rho}\left(\frac{2L_1}{1 - L_2}|x(s) - y(s)|\right)(\eta_i)\right. \\
&\quad + \sum_{j=1}^{n}|\kappa_j|_aI^{\alpha-\beta_j,\rho}\left(\frac{2L_1}{1 - L_2}|x(s) - y(s)|\right)(\xi_j) \\
&\quad \left. + \sum_{r=1}^{k}|\sigma_r|_aI^{\alpha+\delta_r,\rho}\left(\frac{2L_1}{1 - L_2}|x(s) - y(s)|\right)(\theta_r)\right] \\
&\leq \frac{2L_1}{1 - L_2}\left[\frac{(T - a)^\alpha}{\rho^\alpha\Gamma(\alpha + 1)} + \frac{1}{|\Omega|}\left(\sum_{i=1}^{m}\frac{|\gamma_i|(\eta_i - a)^\alpha}{\rho^\alpha\Gamma(\alpha + 1)}\right.\right. \\
&\quad \left.\left. + \sum_{j=1}^{n}\frac{|\kappa_j|(\xi_j - a)^{\alpha-\beta_j}}{\rho^{\alpha-\beta_j}\Gamma(\alpha - \beta_j + 1)} + \sum_{r=1}^{k}\frac{|\sigma_r|(\theta_r - a)^{\alpha+\delta_r}}{\rho^{\alpha+\delta_r}\Gamma(\alpha + \delta_r + 1)}\right)\right]\|x - y\|_{\mathbb{E}} \\
&= \frac{2L_1\Lambda_1}{1 - L_2}\|x - y\|_{\mathbb{E}},
\end{aligned}$$

which implies that $\|\mathcal{Q}x - \mathcal{Q}y\|_{\mathbb{E}} \leq (2L_1\Lambda_1)/(1 - L_2)\|x - y\|_{\mathbb{E}}$. As $(2L_1\Lambda_1)/(1 - L_2) < 1$, hence, the operator $\mathcal{Q}$ is a contraction map. Therefore, by the Banach contraction principle (Lemma 2.6), the problem (1.5) has a unique solution in $\mathbb{E}$. The proof is completed. □

The second existence result is based on the Leray-Schauder nonlinear alternative.

Theorem 3.2. *Assume that*

(H_2) *there exists a continuous nondecreasing function* $\psi : [0, \infty) \to [0, \infty)$, $p \in C([a, T], \mathbb{R}^+)$ *and* $q \in C([a, T], \mathbb{R}^+ \cup \{0\})$ *such that*

$$|f(t, x(t), x(\lambda t), {}_a^CD^{\alpha,\rho}x(\lambda t))| \leq p(t)\psi(|x(t)|) + q(t)\left|{}_a^CD^{\alpha,\rho}x(\lambda t)\right|,$$

for all $(t,x) \in [a,T] \times \mathbb{R}$*, where* $p_0 = \sup_{t\in[a,T]}\{p(t)\}$ *and* $q_0 = \sup_{t\in[a,T]}\{q(t)\}$ *with* $q_0 < 1$*.*

(H_3) *there exists a positive constant* N_1 *such that*

$$\frac{N_2}{\frac{|A|}{|\Omega|} + p_0\left(\frac{2-q_0}{1-q_0}\right)\psi(N_2)\Lambda_1} > 1,$$

where Λ_1 *is defined by* (3.5)*.*

Then the problem (1.5) *has at least one solution on* $[a,T]$*.*

Proof. Let the operator Q defined by (3.1). Firstly, we shall show that Q maps bounded sets (balls) into bounded sets in $\mathbb{E}$. For a constant $R_2 > 0$, let $B_{R_2} := \{x \in \mathbb{E} : \|x\|_{\mathbb{E}} \le R_2\}$ be a bounded ball in $\mathbb{E}$, we have, for $t \in [a,T]$,

$$\begin{aligned} |(\mathcal{Q}x)(t)| \le\ & {}_aI^{\alpha,\rho}|F_x(s)|(T) + \frac{e^{\frac{\rho-1}{\rho}(T-a)}}{|\Omega|}\Bigg(|A| + \sum_{i=1}^{m}|\gamma_i|_aI^{\alpha,\rho}|F_x(s)|(\eta_i) \\ & + \sum_{j=1}^{n}|\kappa_j|_aI^{\alpha-\beta_j,\rho}|F_x(s)|(\xi_j) + \sum_{r=1}^{k}|\sigma_r|_aI^{\alpha+\delta_r,\rho}|F_x(s)|(\theta_r)\Bigg). \end{aligned}$$

It follows from (H_2) that

$$\begin{aligned} \left|{}_a^CD^{\alpha,\rho}x(t)\right| &\le p(t)\psi(|x(t)|) + q(t)\left|{}_a^CD^{\alpha,\rho}x(\lambda t)\right| \\ &\le p(t)\psi(|x(t)|) + q(t)\left|{}_a^CD^{\alpha,\rho}x(t)\right|. \end{aligned}$$

This implies that

$$\left|{}_a^CD^{\alpha,\rho}x(t)\right| \le \frac{p(t)\psi(|x(t)|)}{1-q(t)}.$$

By using $0 < e^{\frac{\rho-1}{\rho}(u-s)} \le 1$ for any $a \le s < u \le T$, we get

$$\begin{aligned} |(\mathcal{Q}x)(t)| \le\ & \frac{|A|}{|\Omega|} + p_0\left(\frac{2-q_0}{1-q_0}\right)\psi(\|x\|_{\mathbb{E}})\Bigg[\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_a^T (T-s)^{\alpha-1}ds \\ & + \frac{1}{|\Omega|}\Bigg(\sum_{i=1}^{m}\frac{|\gamma_i|}{\rho^\alpha\Gamma(\alpha)}\int_a^{\eta_i}(\eta_i-s)^{\alpha-1}ds \\ & + \sum_{j=1}^{n}\frac{|\kappa_j|}{\rho^{\alpha-\beta_j}\Gamma(\alpha-\beta_j)}\int_a^{\xi_j}(\xi_j-s)^{\alpha-\beta_j-1}ds \\ & + \sum_{r=1}^{k}\frac{|\sigma_r|}{\rho^{\alpha+\delta_r}\Gamma(\alpha+\delta_r)}\int_a^{\theta_r}(\theta_r-s)^{\alpha+\delta_r-1}ds\Bigg)\Bigg] \\ =\ & \frac{|A|}{|\Omega|} + p_0\left(\frac{2-q_0}{1-q_0}\right)\psi(\|x\|_{\mathbb{E}})\Lambda_1, \end{aligned}$$

which leads to

$$\|\mathcal{Q}x\|_{\mathbb{E}} \leq \frac{|A|}{|\Omega|} + +p_0 \left(\frac{2-q_0}{1-q_0}\right) \psi(R_2)\Lambda_1 := K_1.$$

Next, we shall show that the operator Q maps bounded sets into equicontinuous sets of $\mathbb{E}$. Let points $t_1, t_2 \in [a, T]$ with $t_1 < t_2$ and $x \in B_{R_2}$. Then we have

$$\begin{aligned}
&|(\mathcal{Q}x)(t_2) - (\mathcal{Q}x)(t_1)| \\
&\leq |{}_aI^{\alpha,\rho}F_x(s)(t_2) - {}_aI^{\alpha,\rho}F_x(s)(t_1)| \\
&\quad + \frac{\left|e^{\frac{\rho-1}{\rho}(t_2-a)} - e^{\frac{\rho-1}{\rho}(t_1-a)}\right|}{|\Omega|} \Bigg(|A| + \sum_{i=1}^{m} |\gamma_i| {}_aI^{\alpha,\rho}|F_x(s)|(\eta_i) \\
&\quad + \sum_{j=1}^{n} |\kappa_j| {}_aI^{\alpha-\beta_j,\rho}|F_x(s)|(\xi_j) + \sum_{r=1}^{k} |\sigma_r| {}_aI^{\alpha+\delta_r,\rho}|F_x(s)|(\theta_r) \Bigg) \\
&\leq p_0 \left(\frac{2-q_0}{1-q_0}\right) \psi(\|x\|_{\mathbb{E}}) \Bigg(\frac{1}{\rho^\alpha \Gamma(\alpha)} \int_a^{t_1} \Big| e^{\frac{\rho-1}{\rho}(t_2-s)}(t_2-s)^{\alpha-1} \\
&\quad - e^{\frac{\rho-1}{\rho}(t_1-s)}(t_1-s)^{\alpha-1} \Big| ds + \frac{(t_2-t_1)^\alpha}{\rho^\alpha \Gamma(\alpha+1)} \Bigg) \\
&\quad + \frac{p_0 \left(\frac{2-q_0}{1-q_0}\right) \psi(\|x\|_{\mathbb{E}})}{|\Omega|} \left| e^{\frac{\rho-1}{\rho}(t_2-a)} - e^{\frac{\rho-1}{\rho}(t_1-a)} \right| \Bigg[|A| + \sum_{i=1}^{m} \frac{|\gamma_i|(\eta_i-a)^\alpha}{\rho^\alpha \Gamma(\alpha+1)} \\
&\quad + \sum_{j=1}^{n} \frac{|\kappa_j|(\xi_j-a)^{\alpha-\beta_j}}{\rho^{\alpha-\beta_j}\Gamma(\alpha-\beta_j+1)} + \sum_{r=1}^{k} \frac{|\sigma_r|(\theta_r-a)^{\alpha+\delta_r}}{\rho^{\alpha-\delta_r}\Gamma(\alpha+\delta_r+1)} \Bigg].
\end{aligned}$$

Clearly, which independent of $x \in B_{R_2}$ the inequality,

$$|(\mathcal{Q}x)(t_2) - (\mathcal{Q}x)(t_1)| \to 0$$

as $t_2 \to t_1$. Therefore it follows from the Arzelá-Ascoli theorem the operator $\mathcal{Q} : \mathbb{E} \to \mathbb{E}$ is completely continuous.

Finally, we shall show that there exists an open set $\mathcal{X} \subseteq \mathbb{E}$ with $x \neq \varrho\mathcal{Q}(x)$ for $\varrho \in (0,1)$ and $x \in \partial\mathcal{X}$.

Let $x \in \mathbb{E}$ be a solution of $x = \varrho\mathcal{Q}x$ for $\varrho \in [0,1]$. Then, for $t \in [a, T]$, we obtain

$$\begin{aligned} |x(t)| &= |\varrho(\mathcal{Q}x)(t)| \\ &\leq \frac{|A|}{|\Omega|} + p_0\left(\frac{2-q_0}{1-q_0}\right)\psi(\|x\|_{\mathbb{E}})\Bigg[\frac{(T-s)^\alpha}{\rho^\alpha\Gamma(\alpha+1)} + \frac{1}{|\Omega|}\Bigg(\sum_{i=1}^{m}\frac{|\gamma_i|(\eta_i-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)} \\ &\quad + \sum_{j=1}^{n}\frac{|\kappa_j|(\xi_j-a)^{\alpha-\beta_j}}{\rho^{\alpha-\beta_j}\Gamma(\alpha-\beta_j+1)} + \sum_{r=1}^{k}\frac{|\sigma_r|(\theta_r-a)^{\alpha+\delta_r}}{\rho^{\alpha+\delta_r}\Gamma(\alpha+\delta_r+1)}\Bigg)\Bigg] \\ &= \frac{|A|}{|\Omega|} + p_0\left(\frac{2-q_0}{1-q_0}\right)\psi(\|x\|_{\mathbb{E}})\Lambda_1, \end{aligned}$$

which on taking the norm for $t \in [a,T]$, implies that

$$\|x\|_{\mathbb{E}} \leq \frac{|A|}{|\Omega|} + p_0\left(\frac{2-q_0}{1-q_0}\right)\psi(\|x\|_{\mathbb{E}})\Lambda_1.$$

Consequently, we get

$$\frac{\|x\|_{\mathbb{E}}}{\frac{|A|}{|\Omega|} + p_0\left(\frac{2-q_0}{1-q_0}\right)\psi(\|x\|_{\mathbb{E}})\Lambda_1} \leq 1.$$

In view of (H_3), there exists N_2 such that $\|x\|_{\mathbb{E}} \neq N_2$. Let us set

$$\mathcal{X} = \{x \in \mathbb{E} : \|x\|_{\mathbb{E}} < N_2\} \quad \text{and} \quad \mathcal{Y} = \mathcal{X} \cap B_{R_2}.$$

Note that the operator $\mathcal{Q} : \overline{\mathcal{Y}} \to \mathbb{E}$ is continuous and completely continuous. From the choice of $\mathcal{Y}$, there is no $x \in \partial\mathcal{Y}$ such that $x = \varrho\mathcal{Q}x$ for some $\varrho \in (0,1)$. Hence, by the nonlinear alternative of Leray-Schauder type (Lemma 2.7), we deduce that $\mathcal{Q}$ has fixed point $x \in \overline{\mathcal{Y}}$ which implies that the problem (1.5) has at least one solution on $[a,T]$. This completes the proof. □

By using Krasnoselskii's fixed point theorem, the final existence theorem will be obtained.

Theorem 3.3. *Let $f : [a,T] \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function satisfying (H_1). In addition we assume that*

(H_4) $|f(t,u,v,w)| \leq g(t)$, $\forall (t,u,v,w) \in [a,T] \times \mathbb{R}^3$ *and* $g \in C([a,T],\mathbb{R}^+)$.

Then the problem (1.5) *has at least one solution on* $[a,T]$ *provided*

$$\frac{1}{|\Omega|}\Bigg(\sum_{i=1}^{m}\frac{|\gamma_i|(\eta_i-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)} + \sum_{j=1}^{n}\frac{|\kappa_j|(\xi_j-a)^{\alpha-\beta_j}}{\rho^{\alpha-\beta_j}\Gamma(\alpha-\beta_j+1)} + \sum_{r=1}^{k}\frac{|\sigma_r|(\theta_r-a)^{\alpha+\delta_r}}{\rho^{\alpha+\delta_r}\Gamma(\alpha+\delta_r+1)}\Bigg) < 1. \tag{3.9}$$

Proof. Setting $\sup_{t\in[a,T]}|g(t)| = \|g\|_{\mathbb{E}}$ and cloosing

$$R_3 \geq \frac{|A|}{|\Omega|} + \|g\|_{\mathbb{E}}\Lambda_1. \tag{3.10}$$

We consider $B_{R_3} := \{x \in \mathbb{E} : \|x\|_{\mathbb{E}} \leq R_3\}$ and the operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ on B_{R_3} are defined by (3.2) and (3.3), respectively, for $t \in [a,T]$. Note that $\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2$.

For any $x, y \in B_{R_3}$, we obtain

$$\begin{aligned}
&\|\mathcal{Q}_1 x + \mathcal{Q}_2 y\|_{\mathbb{E}} \\
&\leq \sup_{t\in[a,t]}\Bigg\{ {}_aI^{\alpha,\rho}|F_x(s)|(t) + \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{|\Omega|}\Bigg(|A| + \sum_{i=1}^{m}|\gamma_i|{}_aI^{\alpha,\rho}|F_x(s)|(\eta_i) \\
&\quad + \sum_{j=1}^{n}|\kappa_j|{}_aI^{\alpha-\beta_j,\rho}|F_x(s)|(\xi_j) + \sum_{r=1}^{k}|\sigma_r|{}_aI^{\alpha+\delta_r,\rho}|F_x(s)|(\theta_r)\Bigg)\Bigg\} \\
&\leq \frac{|A|}{|\Omega|} + \|g\|_{\mathbb{E}}\Bigg\{\frac{(T-a)^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \frac{1}{|\Omega|}\Bigg(\sum_{i=1}^{m}\frac{|\gamma_i|(\eta_i-a)^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} \\
&\quad + \sum_{j=1}^{n}\frac{|\kappa_j|(\xi_j-a)^{\alpha-\beta_j}}{\rho^{\alpha-\beta_j}\Gamma(\alpha-\beta_j+1)} + \sum_{r=1}^{k}\frac{|\sigma_r|(\theta_r-a)^{\alpha+\delta_r}}{\rho^{\alpha+\delta_r}\Gamma(\alpha+\delta_r+1)}\Bigg)\Bigg\} \\
&= \frac{|A|}{|\Omega|} + \|g\|_{\mathbb{E}}\Lambda_1 \leq R_3.
\end{aligned}$$

This implies that $\mathcal{Q}_1 x + \mathcal{Q}_2 x \in B_{R_3}$ which satisfies assumption (i) of Lemma 2.8. It is easy to see, using (3.9), that the operator $\mathcal{Q}_2$ is a contraction mapping and also assumption (iii) of Lemma 2.8 holds.

To show that assumption (ii) of Lemma 2.8 is satisfied. Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $\mathbb{E}$ where $n \to \infty$. Then for each $t \in [a,T]$, we get

$$\begin{aligned}
|(\mathcal{Q}_1 x_n)(t) - (\mathcal{Q}_1 x)(t)| &\leq {}_aI^{\alpha,\rho}|F_{x_n}(s) - F_x(s)|(t) \\
&\leq \frac{(T-a)^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\|F_{x_n} - F_x\|_{\mathbb{E}}.
\end{aligned}$$

Since f is continuous, it implies that F_x is also continuous. Therefore, we obtain $\|\mathcal{Q}_1 x_n - \mathcal{Q}_1 x\|_{\mathbb{E}} \to 0$, as $n \to \infty$. Thus, this shows that the operator $\mathcal{Q}_1 x$ is continuous. Also, the set $\mathcal{Q}_1 B_{R_3}$ is uniformly bounded as

$$\|\mathcal{Q}_1 x\|_{\mathbb{E}} \leq \frac{(T-a)^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\|g\|_{\mathbb{E}}.$$

Next, we prove the compactness of the operator $\mathcal{Q}_1$.

Setting $\sup_{(t,u,v,w)\in[a,t]\times\mathbb{R}^3}|f(t,u,v,w)| = f^* < \infty$, then for each $t_1, t_2 \in [a,T]$ with $t_1 \le t_2$, we have

$$\begin{aligned}
&|(\mathcal{Q}_1 x)(t_2) - (\mathcal{Q}_1 x)(t_1)| \\
&= |{}_aI^{\alpha,\rho}F_x(s)(t_2) - {}_aI^{\alpha,\rho}F_x(s)(t_1)| \\
&\le \frac{1}{\rho^\alpha\Gamma(\alpha)}\int_a^{t_1}\left|e^{\frac{\rho-1}{\rho}(t_2-s)}(t_2-s)^{\alpha-1} - e^{\frac{\rho-1}{\rho}(t_1-s)}(t_1-s)^{\alpha-1}\right||F_x(s)|ds \\
&\quad + \frac{1}{\rho^\alpha\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}|F_x(s)|ds \\
&\le \frac{f^*}{\rho^\alpha\Gamma(\alpha+1)}\Bigg(\int_a^{t_1}\Bigg|e^{\frac{\rho-1}{\rho}(t_2-s)}(t_2-s)^{\alpha-1} \\
&\quad - e^{\frac{\rho-1}{\rho}(t_1-s)}(t_1-s)^{\alpha-1}\Bigg||F_x(s)|ds + |t_2-t_1|^\alpha\Bigg),
\end{aligned}$$

which is independent of x and $|(\mathcal{Q}_1 x)(t_2) - (\mathcal{Q}_1 x)(t_1)| \to 0$ as $t_2 \to t_1$. Therefore, the set $\mathcal{Q}_1 B_{\overline{R}_3}$ is equicontinuous, the operator $\mathcal{Q}_1$ maps bounded subsets into reltively compact subsets, it follows that the set $\mathcal{Q}_1 B_{\overline{R}_3}$ is relatively compact. Then, by the Arzelá-Ascoli theorem, the operator $\mathcal{Q}_1$ is compact on $B_{\overline{R}_3}$. Thus all the assumptions of Lemma 2.8 are satisfied. So, the conclusion of Lemma 2.8 implies that the problem (1.5) has at least one solution on $[a,T]$. The proof is completed. □

4. Ulam-Hyers stability results

In this section, we are analyzing the different kind of Ulam stability such as Ulam-Hyers stable, generalized Ulam-Hyers stable, Ulam-Hyers-Rassias stable and generalized Ulam-Hyers-Rassias stable of the boundary value problem (1.5).

Here we mention that in this paper the definitions of stability have been adopted from [38].

Definition 4.1. ([38]) The problem (1.5) is said to be Ulam-Hyers stable if there exists a constant $\Phi \in \mathbb{R}^+ \setminus \{0\}$ such that for each $\varrho > 0$ and solution $z \in \mathbb{E}^1 = C^1([a,T],\mathbb{R})$ of the inequality

$$\left|{}_a^CD^{\alpha,\rho}z(t) - f(t, z(t), z(\lambda t), ({}_a^CD^{\alpha,\rho}z)(\lambda t))\right| \le \varrho, \quad t \in [a,T], \tag{4.1}$$

there exists a solution $x \in \mathbb{E}^1$ of the problem (1.5) such that

$$|z(t) - x(t)| \le \Phi\varrho, \quad t \in [a,T]. \tag{4.2}$$

Definition 4.2. ([38]) The problem (1.5) is said to be generalized Ulam-Hyers stable if there exists a function $\Phi_f \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\Phi_f(0) = 0$ such that for each solution $z \in \mathbb{E}^1$ of inequality (4.1) there exists a solution $x \in \mathbb{E}^1$ of the problem (1.5) such that

$$|z(t) - x(t)| \leq \Phi_f(\varrho), \quad t \in [a, T]. \tag{4.3}$$

Definition 4.3. ([38]) The problem (1.5) is said to be Ulam-Hyers-Rassias stable with respect to $\Phi_f \in C([a, T], \mathbb{R}^+)$ if there exists a real number $C_{f,\Phi} > 0$ such that for each solution $z \in \mathbb{E}^1$ of the inequality

$$\left|{}_a^C D^{\alpha,\rho} z(t) - f(t, z(t), z(\lambda t), ({}_a^C D^{\alpha,\rho} z)(\lambda t))\right| \leq \varrho \Phi_f(t), \quad t \in [a, T], \tag{4.4}$$

there exists a solution $x \in \mathbb{E}^1$ of the problem (1.5) such that

$$|z(t) - x(t)| \leq C_{f,\Phi} \varrho \Phi_f(t), \quad t \in [a, T]. \tag{4.5}$$

Definition 4.4. ([38]) The problem (1.5) is said to be generalized Ulam-Hyers-Rassias stable with respect to $\Phi_f \in C([a, T], \mathbb{R}^+)$ if there exists a real number $C_{f,\Phi} > 0$ such that for each solution $z \in \mathbb{E}^1$ of the inequality

$$\left|{}_a^C D^{\alpha,\rho} z(t) - f(t, z(t), z(\lambda t), ({}_a^C D^{\alpha,\rho} z)(\lambda t))\right| \leq \Phi_f(t), \quad t \in [a, T], \tag{4.6}$$

there exists a solution $x \in \mathbb{E}^1$ of the problem (1.5) such that

$$|z(t) - x(t)| \leq C_{f,\Phi} \Phi_f(t), \quad t \in [a, T]. \tag{4.7}$$

Remark 4.5. It is clear that

(i) Definition 4.1 $\Rightarrow$ Definition 4.2;
(ii) Definition 4.3 $\Rightarrow$ Definition 4.4;
(iii) Definition 4.3 for $\Phi_f(\cdot) = 1 \Rightarrow$ Definition 4.1.

Remark 4.6. A function $z \in \mathbb{E}^1$ is a solution of the inequality (4.1) if and only if there exists a function $\Psi \in C([a, T], \mathbb{R})$ (dependent on z) such that

(i) $|\Psi(t)| \leq \varrho$, $\forall t \in [a, T]$.
(ii) ${}_a^C D^{\alpha,\rho} z(t) = f(t, z(t), z(\lambda t), ({}_a^C D^{\alpha,\rho} z)(\lambda t)) + \Psi(t), \quad t \in [a, T]$.

By Remark 4.6, the solution of the problem

$${}_a^C D^{\alpha,\rho} z(t) = f(t, z(t), z(\lambda t), ({}_a^C D^{\alpha,\rho} z)(\lambda t) + \Psi(t), \quad t \in [a, T],$$

can be written by

$$\begin{aligned} z(t) &= {}_aI^{\alpha,\rho}F_z(s)(t) + \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg(A - \sum_{i=1}^{m}\gamma_{ia}I^{\alpha,\rho}F_z(s)(\eta_i) \\ &\quad - \sum_{j=1}^{n}\kappa_{ja}I^{\alpha-\beta_j,\rho}F_z(s)(\xi_j) - \sum_{r=1}^{k}\sigma_{ra}I^{\alpha+\delta_r,\rho}F_z(s)(\theta_r)\Bigg) \\ &\quad + {}_aI^{\alpha,\rho}\Psi(s)(t) - \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg(\sum_{i=1}^{m}\gamma_{ia}I^{\alpha,\rho}\Psi(s)(\eta_i) \\ &\quad + \sum_{j=1}^{n}\kappa_{ja}I^{\alpha-\beta_j,\rho}\Psi(s)(\xi_j) + \sum_{r=1}^{k}\sigma_{ra}I^{\alpha+\delta_r,\rho}\Psi(s)(\theta_r)\Bigg). \end{aligned} \tag{4.8}$$

Firstly, we present an important lemma that will be used in the proofs of Ulam–Hyers stability and generalized Ulam–Hyers stability.

Lemma 4.7. *If $z \in \mathbb{E}^1$ satisfies the inequality (4.1), then the function z is a solution of the following inequality*

$$|z(t) - (\mathcal{Q}z)(t)| \le \Lambda_1\varrho, \quad 0 < \varrho \le 1, \tag{4.9}$$

where Λ_1 is given by (3.5).

Proof. From Remark 4.6 with (4.8), we obtain

$$\begin{aligned} &|z(t) - (\mathcal{Q}z)(t)| \\ &= \Bigg|{}_aI^{\alpha,\rho}\Psi(s)(t) - \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg(\sum_{i=1}^{m}\gamma_{ia}I^{\alpha,\rho}\Psi(s)(\eta_i) + \sum_{j=1}^{n}\kappa_{ja}I^{\alpha-\beta_j,\rho}\Psi(s)(\xi_j) \\ &\quad + \sum_{r=1}^{k}\sigma_{ra}I^{\alpha+\delta_r,\rho}\Psi(s)(\theta_r)\Bigg)\Bigg| \\ &\le {}_aI^{\alpha,\rho}|\Psi(s)|(T) + \frac{1}{|\Omega|}\Bigg(\sum_{i=1}^{m}|\gamma_i|_aI^{\alpha,\rho}|\Psi(s)|(\eta_i) + \sum_{j=1}^{n}|\kappa_j|_aI^{\alpha-\beta_j,\rho}|\Psi(s)|(\xi_j) \\ &\quad + \sum_{r=1}^{k}|\sigma_r|_aI^{\alpha+\delta_r,\rho}|\Psi(s)|(\theta_r)\Bigg) \end{aligned}$$

$$\le \varrho\Bigg[\frac{(T-a)^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}+\frac{1}{|\Omega|}\Bigg(\sum_{i=1}^{m}\frac{|\gamma_i|(\eta_i-a)^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}+\sum_{j=1}^{n}\frac{|\kappa_j|(\xi_j-a)^{\alpha-\beta_j}}{\rho^{\alpha-\beta_j}\Gamma(\alpha-\beta_j+1)}$$
$$+\sum_{r=1}^{k}\frac{|\sigma_r|(\theta_r-a)^{\alpha+\delta_r}}{\rho^{\alpha+\delta_r}\Gamma(\alpha+\delta_r+1)}\Bigg)\Bigg]$$
$$=\Lambda_1\varrho,$$

where Λ_1 is given by (3.5), from which inequality (4.9) is obtained. □

Now, we present the Ulam-Hyers stability and generalized Ulam-Hyers stability results.

Theorem 4.8. *Assume that $f:[a,T]\times\mathbb{R}^3\to\mathbb{R}$ is a continuous function. If (H_1) is satisfied with*

$$\frac{2L_1\Lambda_1}{1-L_2}<1.$$

Then the problem (1.5) is Ulam-Hyers stable as well as generalized Ulam-Hyers stable on $[a,T]$.

Proof. Let $z\in\mathbb{E}^1$ be a solution of the inequality (4.1) and let x be the unique solution of the problem (1.5),

$$\begin{cases} {}_a^CD^{\alpha,\rho}x(t)=f(t,x(t),x(\lambda t),{}_a^CD^{\alpha,\rho}x(\lambda t)), \quad t\in(a,T], \quad 0<\lambda<1,\\[2ex] \displaystyle\sum_{i=1}^{m}\gamma_i x(\eta_i)+\sum_{j=1}^{n}\kappa_j{}_a^CD^{\beta_j,\rho}x(\xi_j)+\sum_{r=1}^{k}\sigma_r{}_aI^{\delta_r,\rho}x(\theta_r)=A.\end{cases}$$

By applying the triangle inequality, $|u-v|\le|u|+|v|$, and Lemma 4.7, we have

$$|z(t)-x(t)|=\Bigg|z(t)-{}_aI^{\alpha,\rho}F_x(s)(t)-\frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg(A-\sum_{i=1}^{m}\gamma_i{}_aI^{\alpha,\rho}F_x(s)(\eta_i)$$
$$-\sum_{j=1}^{n}\kappa_j{}_aI^{\alpha-\beta_j,\rho}F_x(s)(\xi_j)-\sum_{r=1}^{k}\sigma_r{}_aI^{\alpha+\delta_r,\rho}F_x(s)(\theta_r)\Bigg)\Bigg|$$
$$=|z(t)-(\mathcal{Q}z)(t)+(\mathcal{Q}z)(t)-(\mathcal{Q}x)(t)|$$
$$\le|z(t)-(\mathcal{Q}z)(t)|+|(\mathcal{Q}z)(t)-(\mathcal{Q}x)(t)|$$
$$\le\Lambda_1\varrho+\frac{2L_1\Lambda_1}{1-L_2}|z(t)-x(t)|,$$

where Λ_1 is defined by (3.5). This yields that

$$|z(t)-x(t)|\le\frac{\Lambda_1\varrho}{1-\frac{2L_1\Lambda_1}{1-L_2}}.$$

By setting

$$\Phi = \frac{\Lambda_1}{1 - \frac{2L_1\Lambda_1}{1-L_2}}, \tag{4.10}$$

we end up with

$$|z(t) - x(t)| \leq \Phi\varrho.$$

Hence, the problem (1.5) is Ulam-Hyers stable. Moreover, if we set $\Phi_f(\varrho) = \Phi\varrho$ such that $\Phi_f(0) = 0$, then the problem (1.5) is generalized Ulam-Hyers stable. This completes the proof. □

Remark 4.9. A function $z \in \mathbb{E}^1$ is a solution of the inequality (4.4) if and only if there exists a function $\Theta \in C([a,T],\mathbb{R})$ (dependent on z) such that

(i) $|\Theta(t)| \leq \varrho\Psi_\Theta(t)$, $\forall t \in [a,T]$.
(ii) ${}_a^C D^{\alpha,\rho} z(t) = f(t, z(t), z(\lambda t), {}_a^C D^{\alpha,\rho} z(\lambda t)) + \Theta(t), \quad t \in [a,T]$.

By Remark 4.9, the solution of the problem

$${}_a^C D^{\alpha,\rho} z(t) = f(t, z(t), z(\lambda t), {}_a^C D^{\alpha,\rho} z(\lambda t)) + \Theta(t), \quad t \in [a,T],$$

can be written by

$$\begin{aligned} z(t) \;=\; & {}_aI^{\alpha,\rho}F_z(s)(t) + \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\left(A - \sum_{i=1}^{m}\gamma_{i\,a}I^{\alpha,\rho}F_z(s)(\eta_i)\right. \\ & \left. - \sum_{j=1}^{n}\kappa_{j\,a}I^{\alpha-\beta_j,\rho}F_z(s)(\xi_j) - \sum_{r=1}^{k}\sigma_{r\,a}I^{\alpha+\delta_r,\rho}F_z(s)(\theta_r)\right) \\ & + {}_aI^{\alpha,\rho}\Theta(s)(t) - \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\left(\sum_{i=1}^{m}\gamma_{i\,a}I^{\alpha,\rho}\Theta(s)(\eta_i)\right. \\ & \left. + \sum_{j=1}^{n}\kappa_{j\,a}I^{\alpha-\beta_j,\rho}\Psi(s)(\xi_j) + \sum_{r=1}^{k}\sigma_{r\,a}I^{\alpha+\delta_r,\rho}\Theta(s)(\theta_r)\right). \end{aligned} \tag{4.11}$$

Lemma 4.10. *Let $z \in \mathbb{E}^1$ be a solution of inequality* (4.4)*. Then the function z satisfies the inequality*

$$|z(t) - (\mathcal{Q}z)(t)| \leq \Lambda_1\Psi_\Theta(t)\varrho, \quad 0 < \varrho \leq 1, \tag{4.12}$$

where Λ_1 is given by (3.5)*.*

Proof. From Remark 4.9, we obtain the inequality

$$
\begin{aligned}
o\,|z(t)-(\mathcal{Q}z)(t)| &= \Bigg| {}_aI^{\alpha,\rho}\Theta(s)(t) - \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg(\sum_{i=1}^{m}\gamma_i {}_aI^{\alpha,\rho}\Theta(s)(\eta_i) \\
&\quad + \sum_{j=1}^{n}\kappa_j {}_aI^{\alpha-\beta_j,\rho}\Theta(s)(\xi_j) \\
&\quad + \sum_{r=1}^{k}\sigma_r {}_aI^{\alpha+\delta_r,\rho}\Theta(s)(\theta_r)\Bigg)\Bigg| \\
&\le {}_aI^{\alpha,\rho}|\Theta(s)|(T) + \frac{1}{|\Omega|}\Bigg(\sum_{i=1}^{m}|\gamma_i| {}_aI^{\alpha,\rho}|\Theta(s)|(\eta_i) \\
&\quad + \sum_{j=1}^{n}|\kappa_j| {}_aI^{\alpha-\beta_j,\rho}|\Theta(s)|(\xi_j) \\
&\quad + \sum_{r=1}^{k}|\sigma_r| {}_aI^{\alpha+\delta_r,\rho}|\Theta(s)|(\theta_r)\Bigg) \\
&\le \Bigg[\frac{(T-a)^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \frac{1}{|\Omega|}\Bigg(\sum_{i=1}^{m}\frac{|\gamma_i|(\eta_i-a)^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} \\
&\quad + \sum_{j=1}^{n}\frac{|\kappa_j|(\xi_j-a)^{\alpha-\beta_j}}{\rho^{\alpha-\beta_j}\Gamma(\alpha-\beta_j+1)} \\
&\quad + \sum_{r=1}^{k}\frac{|\sigma_r|(\theta_r-a)^{\alpha+\delta_r}}{\rho^{\alpha+\delta_r}\Gamma(\alpha+\delta_r+1)}\Bigg)\Bigg]\Psi_{\Theta}(t)\varrho \\
&= \Lambda_1\Psi_{\Theta}(t)\varrho,
\end{aligned}
$$

where Λ_1 is given by (3.5), which leads to inequality in (4.9). □

Next, we are ready to prove Ulam-Hyers-Rassias and generalized Ulam-Hyers Rassias stability results.

Theorem 4.11. *Assume that* $f : [a,T]\times\mathbb{R}^3 \to \mathbb{R}$ *is a continuous function. If* (H_1) *is satisfied with with*

$$\frac{2L_1\Lambda_1}{1-L_2} < 1.$$

Then the problem (1.5) *is Ulam-Hyers-Rassias stable as well as generalized Ulam-Hyers Rassias stable on* $[a,T]$.

Proof. Let $z \in \mathbb{E}^1$ be a solution of the inequality (4.4) and let x be the unique solution of the problem (1.5). By applying the triangle inequality and Lemma 4.7 with (4.11), we get

$$\begin{aligned}|z(t)-x(t)| &= \Bigg| z(t) - {}_aI^{\alpha,\rho}F_x(s)(t) \\ &\quad - \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg(A - \sum_{i=1}^{m}\gamma_{i\,a}I^{\alpha,\rho}F_x(s)(\eta_i) \\ &\quad - \sum_{j=1}^{n}\kappa_{j\,a}I^{\alpha-\beta_j,\rho}F_x(s)(\xi_j) - \sum_{r=1}^{k}\sigma_{r\,a}I^{\alpha+\delta_r,\rho}F_x(s)(\theta_r)\Bigg)\Bigg| \\ &= |z(t) - (\mathcal{Q}z)(t) + (\mathcal{Q}z)(t) - (\mathcal{Q}x)(t)| \\ &\le |z(t) - (\mathcal{Q}z)(t)| + |(\mathcal{Q}z)(t) - (\mathcal{Q}x)(t)| \\ &\le \Lambda_1\Psi_\Theta(t)\varrho + \frac{2L_1\Lambda_1}{1-L_2}|z(t)-x(t)|,\end{aligned}$$

where Λ_1 is defined by (3.5), which implies that

$$|z(t)-x(t)| \le \frac{\Lambda_1\Psi_\Theta(t)\varrho}{1-\frac{2L_1\Lambda_1}{1-L_2}}.$$

By setting

$$C_{f,\Phi} := \frac{\Lambda_1}{1-\frac{2L_1\Lambda_1}{1-L_2}},$$

we get the following inequality

$$|z(t)-x(t)| \le C_{f,\Phi}\varrho\Psi_\Theta(t).$$

Hence, the problem (1.5) is Ulam-Hyers Rassias stable. Moreover, if we set

$$\Phi_f(t) = \varrho\Psi_\Theta(t),$$

with $\Phi_f(0) = 0$, then the problem (1.5) is generalized Ulam-Hyers Rassias stable. The proof is completed. □

5. Examples

In this section, we present two examples which illustrate the validity and applicability of main results.

Example 5.1. Consider the following nonlinear GPF Pantograph differential equation via mixed nonlocal conditions of the form:

$$\begin{cases} {}^{C}D^{\frac{2}{3},\frac{1}{2}}x(t) = \dfrac{2+|x(t)|+|x(\frac{3}{2}t)|+|{}^{C}D^{\frac{2}{3},\frac{1}{2}}x(\frac{3}{2}t)|}{95e^{2t}\cos 2t\left(1+|x(t)|+|x(\frac{3}{2}t)|+|{}^{C}D^{\frac{2}{3},\frac{1}{2}}x(\frac{3}{2}t)|\right)}, \\ \displaystyle\sum_{i=1}^{2}\left(\frac{i+1}{2}\right)x\left(\frac{2i+1}{3}\right)+\sum_{j=1}^{3}\left(\frac{2j-1}{5}\right){}^{C}D^{\frac{2j+1}{10},\frac{1}{2}}x\left(\frac{j}{2}\right) \\ \qquad\qquad\displaystyle +\sum_{r=1}^{2}\left(\frac{r}{3}\right)I^{\frac{r}{r+1},\frac{1}{2}}x\left(\frac{r+1}{2r}\right)=1. \end{cases} \tag{5.1}$$

Here $\alpha = 2/3$, $\rho = 1/2$, $\lambda = 3/2$, $a = 0$, $T = 2$, $m = 2$, $n = 3$, $k = 2$, $\gamma_i = (i+1)/2$, $\eta_i = (2i+1)/3$, $i = 1,2$, $\kappa_j = (2j-1)/5$, $\beta_j = (2j+1)/10$, $\xi_j = j/2$, $j = 1,2$, $\sigma_r = r/3$, $\delta_r = r/(r+1)$, $\theta_r = (r+1)/2r$, $r = 1,2$,

From the given all datas, we obtain that $\Omega \approx 1.3039822 \neq 0$, $\Lambda_1 \approx 9.7044$ and

$$f(t,u,v,w) = \frac{2+|u|+|v|+|w|}{95e^{2t}\cos 2t\,(1+|u|+|v|+|w|)}.$$

For x_1, x_2, y_1, y_2, z_1, $z_2 \in \mathbb{R}$ and $t \in [0,2]$, we have

$$|f(t,x_1,y_1,z_1)-f(t,x_2,y_2,z_2)| \leq \frac{1}{95e^{2t}\cos 2t}\left(|x_1-x_2|+|y_1-y_2|+|z_1-z_2|\right).$$

The assumptions (H_1) is satisfied with $L_1 = L_2 = \frac{1}{95}$. Hence

$$\frac{2L_1\Lambda_1}{1-L_2} \approx 0.206476 < 1.$$

Since, all the assumptions of Theorem 3.1 are satisfied, the problem (5.1) has a unique solution on $[0,2]$. Furthermore, we can also compute that

$$\Phi := \frac{\Lambda_1}{1-\frac{2L_1\Lambda_1}{1-L_2}} \approx 12.22949 > 0.$$

Hence, by Theorem 4.8, the problem (5.1) is both Ulam-Hyers and also generalized Ulam-Hyers stable.

Example 5.2. Consider the following nonlinear GPF Pantograph differential equation via mixed nonlocal conditions of the form:

$$\begin{cases} {}^{C}D^{\frac{1}{2},\frac{1}{3}}x(t) = \dfrac{1}{4^{t+3}\left(1+|x(t)|+|x(\frac{1}{6}t)|+|{}^{C}D^{\frac{1}{2},\frac{1}{3}}x(\frac{1}{6}t)|\right)}, \\ \displaystyle\sum_{i=1}^{3}\left(\frac{i+1}{2i}\right)x\left(\frac{i+1}{4}\right)+\sum_{j=1}^{2}\left(\frac{2j-1}{3}\right){}^{C}D^{\frac{j}{j+1},\frac{1}{2}}x\left(\frac{j}{2}\right) \\ \displaystyle\qquad\qquad +\sum_{r=1}^{2}\left(\frac{r-1}{3}\right)I^{\frac{2r}{r+1},\frac{1}{2}}x\left(\frac{r+1}{2r}\right)=5. \end{cases} \tag{5.2}$$

Here $\alpha = 1/2$, $\rho = 1/3$, $\lambda = 1/6$, $a = 0$, $T = 2$, $m = 3$, $n = 2$, $k = 2$, $\gamma_i = (i+1)/2i$, $\eta_i = (i+1)/4$, $i = 1,2,3$, $\kappa_j = (2j-1)/3$, $\beta_j = j/(j+1)$, $\xi_j = j/2$, $j = 1,2$, $\sigma_r = (r-1)/3$, $\delta_r = 2r/(r+1)$, $\theta_r = (r+1)/2r$, $r = 1,2$,

From the given all datas, we obtain that $\Omega \approx 0.809627 \neq 0$, $\Lambda_1 \approx 10.02678$ and

$$f(t,u,v,w) = \frac{1}{4^{t+3}\left(1+|u|+|v|+|w|\right)}.$$

For x_1, x_2, y_1, y_2, z_1, $z_2 \in \mathbb{R}$ and $t \in [0,2]$, we have

$$|f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2)| \leq \frac{1}{4^{t+3}}\left(|x_1-x_2|+|y_1-y_2|+|z_1-z_2|\right).$$

The assumptions (H_1) is satisfied with $L_1 = L_2 = \frac{1}{64}$. Hence

$$\frac{2L_1\Lambda_1}{1-L_2} \approx 0.31831 < 1.$$

we can also compute that

$$\Phi := \frac{\Lambda_1}{1-\frac{2L_1\Lambda_1}{1-L_2}} \approx 14.708709 > 0.$$

Hence, by Theorem 4.11, the problem (5.2) is both Ulam-Hyers-Rassias and also generalized Ulam-Hyers-Rassias stable.

6. Conclusion

In this paper, we constructed the equivalent between the purpose problem (1.5) and the Volterra fractional integral equation. Afterwards, we investigated sufficient conditions for the existence and uniqueness of solutions of the purpose problem (1.5) by using Banach contraction mapping principle, Leray-Schauder nonlinear alternative and Krasnoselskii's fixed point theorem. Moreover, we proved four different types of Ulam stability results including Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability for the problem (1.5). For the justification, two numerical examples were given to illustrate our main theoretical results.

We believe that the all results of this paper will provide considerable potential to interested researchers to develop relevant results concerning qualitative properties of nonlinear GPF differential equations. In a forthcoming work, we shall focus on studying the different types of existence results and stability analysis to an impulsive GPF differential equation with nonlocal fractional integral multi-point conditions.

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References

[1] S. Abbsa, M. Benchohra, J. Lagreg, A. Alsaedi and Y. Zhou, *Existence and Ulam stability results for fractional differential equations of Hilfer-Hadamard type,* Adv. Differ. Equ., **2017**(180) (2017).

[2] T. Abdeljawad, F. Jarad, S. Mallak and J. Alzabut, *Lyapunov type inequalities via fractional proportional derivatives and application on the free zero disc of Kilbas-Saigo generalized Mittag- Leffler functions,* European Physical J. Plus, **(2019)**, 134:247.

[3] M. Adam,, *On the Hyers–Ulam–Rassias stability of a generalized quadratic and additive functional equation* Nonlinear Funct. Anal. Appl., **14**(5) (2009), 699-705.

[4] B. Ahmad, A. Alsaedi, S.K. Ntouyas and J. Tariboon, *Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities*, Springer International Publishing AG, 2017.

[5] B. Ahmad, M. M. Matar and O. El-Salmy, *Existence of solutions and Ulam stability for Caputo type sequential fractional differential equations of order* $\alpha \in (2,3)$*,* Inter. J. Anal. Appl., **15**(1) (2017), 86–101.

[6] I. Ahmed, P. Kuman, K. Shah, P. Borisut, K. Sitthithakerngkiet and M.A. Demba, *Stability results for implicit fractional Pantograph differential equations via* ϕ*-Hilfer fractional derivative with a nonlocal Riemann–Liouville fractional integral condition.* Mathematics, **2020**, 8, 94; doi:10.3390/math8010094.

[7] J. Alzabut, T. Abdeljawad, F. Jarad and W. Sudsutad, *A Gronwall inequality via the generalized proportional fractional derivative with applications,* J. Inequ. Appl., **(2019)** 2019:101.

[8] J. Alzabut, W. Sudsutad, Z. Kayar and H. Baghani, *A new Gronwall–Bellman inequality in a frame of generalized proportional fractional proportional derivative,* Mathematics, **2019**, 7, 747, 1-15,; doi:10.3390/math7080747.
[9] D.R. Anderson, *Second-order self-adjoint differential equations using a proportional-derivative controller,* Commun. Appl. Nonlinear Anal., **24** (2017), 17-48.
[10] T. Aoki, *On the stability of the linear transformation in Banach spaces,* J. Math. Soc. Japan, **2** (1950), 64-66.
[11] A. Aphithana, S.K. Ntouyas and J. Tariboon, *Existence and Ulam–Hyers stability for Caputo conformable differential equations with four–point integral conditions,* Adv. Differ. Equ., **2019**, 2019, 139.
[12] S. Asawasamrit, W. Nithiarayaphaks, S.K. Ntouyas and J. Tariboon, *Existence and stability analysis for fractional differential equations with mixed nonlocal conditions,* Mathematics, **2019**, 7, 117.
[13] K. Balachandran and K. Uchiyama, *Existence of solutions of nonlinear integrodifferential equations of Sobolev type with nonlocal conditions in Banach spaces,* Proc. Indian Acad. Sci. (Math. Sci.) **110** (2000), 225-232.
[14] M. Benchohra and S. Bouriah, *Existence and stability results for nonlinear boundary value problem for implicit differential equations of fractional order,* Moroccan J. Pure Appl. Anal., **1**(1) (2015), 22-36.
[15] A.V. Bitsadze and A.A. Samarskii, *Some elementary generalizations of linear elliptic boundary value problems,* Soviet Math. Dokl. **10** (1969), 398-400.
[16] L. Byszewski, *Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem,* J. Math. Anal. Appl., **162** (1991), 494-505.
[17] E. Capelas de Oliveira and J. Vanterler da C. Sousa, *Ulam–Hyers-Rassias stability for a class of fractional integro-differential equations,* Results Math., **2018**, 73, 111.
[18] K. Diethelm, *The Analysis of Fractional Differential Equations*, Lecture Notes in Mathematics, Springer, New York, 2010.
[19] M. Eshaghi Gordji and F. Habibian, *Hyers–Ulam–Rassias stability of quadratic derivations on Banach algebras,* Nonlinear Funct. Anal. Appl., **14**(5) (2009), 759-766.
[20] A. Granas and J. Dugundji, *Fixed Point Theory,* Springer, New York, USA 2003.
[21] S. Harikrishman, E. Elsayed and K. Kanagarajan, *Existence and uniqueness results for fractional Pantograph equations involving ψ-Hilfer fractional derivative.* Dyn. Contin. Discret. Impuls. Syst., **2018**(25) (2018), 319-328.
[22] E. Hilb, *Zur Theorie der Entwicklungen willkurlicher Funktionen nach Eigenfunktionen,* Math. Z., **58** (1918), 1-9.
[23] D.H. Hyers, *On the stability of the linear functional equation,* Proc. Natl. Acad. Sci. USA,, **27**(4) (1941), 222-224.
[24] J. Jarad, T. Abdeljawad and J. Alzabut, *Generalized fractional derivatives generated by a class of local proportional derivatives,* Eur. Phys. J. Spec. Top., **226** (2017), 3457-3471.
[25] B. Khaminsou, C. Thaiprayoon, J. Alzabut and W. Sudsutad, *Nonlocal boundary value problems for integro-differential Langevin equation via the generalized Caputo proportional fractional derivative,* Bound. Value Prob., **(2020)**, 2020:176.
[26] A. Khan, M.I. Syam, A. Zada and H. Khan, *Stability analysis of nonlinear fractional differential equations with Caputo and Riemann-Liouville derivatives,* Eur. Phys. J. Plus, **2018** (2018) 133: 264.
[27] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of the Fractional Differential Equations*, North-Holland Mathematics Studies, 204, (2006).

[28] M.A. Krasnoselskii, *Two remarks on the method of successive approximations.* Usp. Mat. Nauk **10** (1955), 123-127.

[29] V. Lakshmikantham, S. Leela and J.V. Devi, *Theory of Fractional Dynamic Systems*, Cambridge Scientific Publishers, 2009.

[30] K. Liu, M. Fečkan, D. O'Regan and J. Wang, *Hyers–Ulam stability and existence of solutions for differential equations with Caputo–Fabrizio fractional derivative,* Mathematics, **2019**, 7, 333.

[31] K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, NewYork, 1993.

[32] T. Miura, Go Hirasawa and Sin-El Takahasi, *Note on the Hyers–Ulam–Rassias stability of the first order linear differential equation,* Nonlinear Funct. Anal. Appl., **11**(5) (2006), 851-858.

[33] M. Obloza, *Hyers stability of the linear differential equation,* Rocznik Nauk-Dydakt. Prace Mat., **13** (1993), 259-270.

[34] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1999.

[35] G. Rahman, A. Khan, T. Abdeljawad and K.S. Nisar, *The Minkowski inequalities via generalized proportional fractional integral operators,* Adv. Differ. Equ. **2019**(287) (2019).

[36] T.M. Rassias, *On the stability of linear mappings in Banach spaces,* Proc. Amer. Math. Soc., **72** (1978), 297-300.

[37] T.M. Rassias, *On a modified Hyers–Ulam sequence,* J. Math. Anal. Appl., **2003**(158) (2003), 106-113.

[38] I.A. Rus, *Ulam stabilities of ordinary differential equations in a Banach space,* Carpathian J. Math., **26** (2010), 103–107.

[39] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives*, Gordon and Breach Science, Yverdon, 1993.

[40] W. Shammakh and H.Z. Alzumi, *Existence results for nonlinear boundary value problem involving generalized proportional derivative,* Adv. Differ. Equ., **(2019)**, 2019:94.

[41] W. Sudsutad, J. Alzabut, C. Tearnbucha and C. Thaiprayoon, *On the oscillation of differential equations in frame of generalized proportional fractional derivatives,* AIMS Mathematics, **5**(2) (2020), 856-871, doi:10.3934/math.2020058.

[42] S.M. Ulam, *Problem in Modern Mathematics,* John Wiley and Sons, New York, NY, USA, 1940.

[43] S.M. Ulam, *A collection of mathematical Problems,* Interscience, New York, NY, USA, 1968.

[44] J. Vanterler da C. Sousa, and E. Capelas de Oliveira, *On the Ulam–Hyers–Rassias stability for nonlinear fractional differential equations using the ψ-Hilfer operator.* J. Fixed Point Theory Appl., **2018**, 20, 96.

[45] J. Wang, L. Lv and Y. Zhou, *Ulam stability and data dependence for fractional differential equations with Caputo derivative,* Electron. J. Qual. Theory Differ. Equ., **2011**(63) (2011), 1-10.

[46] J. Wang, Y. Zhou and M. Medved, *Existence and stability of fractional differential equations with Hadamard derivative,* Topol. Methods Nonlinear Anal., **41** (2013), 113-133.

[47] Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014.