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ON THE STABILITY OF FUNCTIONAL EQUATIONS IN 2-NORMED SPACES

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Abstract. In this paper, we discuss and achieve the Hyers-Ulam stability of the quadratic functional equation in the framework of linear 2-normed spaces, therefore we generalize the corresponding theorems.

1. Introduction

The first stability problem was raised by S.M. Ulam [11] during his talk at the university if Wisconsin in 1940. Given a group G_1, a metric group (G_2, d) and a positive number ε, does there exist a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies

$$d(f(xy), f(x)f(y)) < \delta$$

for all $x, y \in G_1$, then a homomorphism $h : G_1 \to G_2$ exists with

$$d(f(x), h(x)) < \varepsilon$$

for all $x, \in G_1$? For general functional equations, the concept of stability for functional equations arises when the equation is replaced by an inequality which act as a perturbation of the equation. If the answer is affirmative, the functional equation for homomorphism will be called stable.

The first affirmative result concerning the stability of functional equations was presented by D.H. Hyers [4]. He dealt with $\varepsilon-$additive mappings $f : E_1 \to$

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E_2 between Banach spaces, i.e. f satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| < \varepsilon, \forall x, y \in E_1.$$

Th.M. Rassias [6] considered a generalized version of the previous result which permitted the Cauchy difference to become unbounded. That is, he assumed that

$$\|f(x+y) - f(x) - f(y)\| < \varepsilon(\|x\|^p + \|y\|^p), \forall x, y \in E_1, 0 \leq p < 1$$

This result was later extended to all $p \neq 1$ and generalized by Z. Gajda [3] and Th.M. Rassias [7].

For a real constant c, the quadratic function $f(x) = cx^2$ satisfies the equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.1}$$

Hence, equation (1.1) is called *the quadratic functional equation*, and the solution of the quadratic functional equation (1.1) is called a *quadratic function.*

F. Skof [10] proved the Hyers-Ulam stability of the quadratic functional equation for functions $f : E_1 \to E_2$, where E_1 is a normed space and E_2 a Banach space. P.W.Cholewa [1] demonstrated that Skof's theorem is also valid if E_1 is replaces by an Abelian group G.

Theorem 1.1 ([1]). *Let G be a Abelian group, E be a Banach space and a function $f : G \to E$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2[f(x) + f(y)]\| \leq \delta,$$

for some $\delta \geq 0$ and for all $x, y \in G$. Then there exists a unique quadratic function $Q : G \to E$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{2}\delta,$$

for all $x \in X$.

S. Czerwik [2] proved the modified Hyers-Ulam stability of the quadratic functional equation for the case $p > 2$ and $p < 2$ separately, that is, assume that f satisfies

$$\|f(x+y) + f(x-y) - 2[f(x) + f(y)]\| \leq \delta + \theta(\|x\|^p + \|y\|^p), p < 2$$

and

$$\|f(x+y) + f(x-y) - 2[f(x) + f(y)]\| \leq \theta(\|x\|^p + \|y\|^p), p > 2.$$

If $p = 2$, the quadratic functional equation 1.1 is not stable. (see [9]).

A. White [12] introduced the concepts of Cauchy sequence and convergent sequence in linear 2-normed spaces and the notion of 2-Banach spaces.

In this paper, we investigate and give a generalization of Hyers-Ulam stability for quadratic functional equation in the framework of linear 2-normed spaces.

2. Main results

Theorem 2.1. *Let E be a linear 2-normed space and F a 2-Banach space. The dimension of E and F is greater than one, and let $f : E \to F$ be a surjective mapping. Assume that there exists $\delta \geq 0$ and $\alpha, \beta \in \mathbb{R}, \alpha + \beta \neq 2$, such that*

$$\|f(x+y) + f(x-y) - 2[f(x) + f(y)], f(z)\| \leq \delta\|x, z\|^{\alpha}\|y, z\|^{\beta} \tag{$*$}$$

for all $x, y, z \in E$. Then there exists a unique function $g : E \to F$ such that

$$g(x+y) + g(x-y) = 2g(x) + 2g(y)$$

and

$$\|f(x) - g(x), f(z)\| \leq \varepsilon\|x, z\|^{\alpha+\beta},$$

where

$$\varepsilon = \begin{cases} 4^{-1}\frac{\delta}{1-2^{\alpha+\beta-2}} & \alpha + \beta < 2, \\ 2^{-(\alpha+\beta)}\frac{\delta}{1-2^{2-\alpha-\beta}} & \alpha + \beta > 2. \end{cases}$$

Proof. (I) For the case $\alpha + \beta < 2$. Apply the induction assumption

$$\|f(x) - 4^{-n}f(2^n x), f(z)\| \leq 4^{-1}\delta\|x, z\|^{\alpha+\beta}\sum_{i=0}^{n-1} 2^{i(\alpha+\beta-2)}. \tag{2.1}$$

Put $x = y$ in $(*)$ and dividing by 4, we can get that the induction assumption is true for $n = 1$. Assume now that the induction assumption is true for the case n, we want to show that the assumption is true for the case $(n+1)$.

$$\begin{aligned}
&\|f(x) - 4^{-n-1}f(2^{n+1}x), f(z)\| \\
&\quad \leq \|f(x) - 4^{-n}f(2^n x), f(z)\| + \|4^{-n}f(2^n x) - 4^{-n-1}f(2^{n+1}x), f(z)\| \\
&\quad \leq 4^{-1}\delta\|x, z\|^{\alpha+\beta}\sum_{m=0}^{n-1} 2^{m(\alpha+\beta-1)} + 4^{-n}4^{-1}\delta\|2^n x, z\|^{p} \\
&\quad = 4^{-1}\delta\|x, z\|^{\alpha+\beta}\sum_{i=0}^{n} 2^{i(\alpha+\beta-2)}.
\end{aligned}$$

Thus the assumption is true for any positive integer n. It follows that

$$\|f(x) - 4^{-n}f(2^n x), f(z)\| < 4^{-1}\delta\|x, z\|^{\alpha+\beta}\frac{1}{1 - 2^{\alpha+\beta-2}}.$$

Put $g_n(x)=\frac{f(2^nx)}{4^n}$, for $n>m>0$, we have

$$\begin{aligned} g_m(x)-g_n(x) &= \frac{f(2^mx)}{4^m}-\frac{f(2^nx)}{4^n} \\ &= \frac{f(2^mx)-4^{-(n-m)}f(2^{n-m}\cdot 2^mx)}{4^m}. \end{aligned}$$

we can apply (2.1) to get

$$\begin{aligned} \|g_m(x)-g_n(x),f(z)\| &= \|\frac{f(2^mx)-4^{-(n-m)}f(2^{n-m}\cdot 2^mx)}{4^m},f(z)\| \\ &\leq 4^{-1}\delta\|x,z\|^{\alpha+\beta}\sum_{i=m}^{n-m}2^{i(\alpha+\beta-2)}. \end{aligned}$$

It follows that for each $x\in E,\{g_n(x)\}_n$ is a Cauchy sequence in F. Since F is a 2-Banach space, there is a limit

$$g(x)=\lim_{n\to\infty}g_n(x),$$

which satisfies

$$\|f(x)-g(x),f(z)\|\leq\varepsilon\|x,z\|^{\alpha+\beta},$$

for all $x,z\in E$, where $\varepsilon=4^{-1}\delta\frac{1}{1-2^{\alpha+\beta-2}}$.
From $(*)$ we get

$$\|f(2^nx+2^ny)+f(2^nx-2^ny)-2[f(2^nx)+f(2^ny)],f(z)\|\leq\delta(\|2^nx,z\|^{\alpha}\|2^ny,z\|^{\beta})$$

and

$$\begin{aligned} & \|g_n(x+y)+g_n(x-y)-2[g_n(x)+g_n(y)],f(z)\| \\ = & \|4^{-n}f(2^nx+2^ny)+4^{-n}f(2^nx-2^ny)-2[4^{-n}f(2^nx)+4^{-n}f(2^ny)],f(z)\| \\ \leq & \delta 2^{n(\alpha+\beta-2)}\|x,z\|^{\alpha}\|y,z\|^{\beta}. \end{aligned}$$

Letting $n\to\infty$, we obtain $g(x+y)+g(x-y)=2g(x)+2g(y)$.

Next we show the uniqueness of g. Assume that there exists another mapping $h:E\to F$, a constant $\varepsilon_1\geq 0$ such that

$$\|f(x)-h(x),f(z)\|\leq\varepsilon_1\|x,z\|^{\alpha+\beta}.$$

Since $g(x),h(x)$ are quadratic functions, we have

$$g(nx)=n^2g(x),\quad h(nx)=n^2h(x).$$

Hence

$$\begin{aligned} \|g(nx)-h(nx),f(z)\| &\leq \|f(nx)-g(nx),f(z)\|+\|f(nx)-h(nx),f(z)\| \\ &\leq \varepsilon\|nx,z\|^{\alpha+\beta}+\varepsilon_1\|nx,z\|^{\alpha+\beta} \\ &= (\varepsilon+\varepsilon_1)n^{\alpha+\beta}\|x,z\|^{\alpha+\beta}. \end{aligned}$$

Therefore

$$\begin{aligned} \|g(x) - h(x), f(z)\| &= \frac{1}{n^2}\|g(nx) - h(nx), f(z)\| \\ &\leq (\varepsilon + \varepsilon_1)n^{\alpha+\beta-2}\|x, z\|^{\alpha+\beta}. \end{aligned}$$

Thus

$$\lim_{n\to\infty} \|g(x) - h(x), f(z)\| = 0,$$

for all $x, z \in E$. Since f is surjective, $g(x) \equiv h(x)$ for all $x \in E$.

(II) For the case $\alpha + \beta > 2$. We can prove the induction assumption

$$\|f(x) - 4^n f(2^{-n}x), f(z)\| \leq \delta 2^{-(\alpha+\beta)} \sum_{i=0}^{n-1} 2^{i(2-\alpha-\beta)}\|x, z\|^{\alpha+\beta},$$

for all $x, z \in E$ and any positive integer n. Put $g_n(x) = 4^n f(2^{-n}x)$, by the same argument as above we can show that sequence $\{g_n(x)\}_n$ is a Cauchy sequence in 2-Banach space F, and the limit function

$$g(x) = \lim_{n\to\infty} g_n(x)$$

is the unique linear mapping satisfying

$$\|f(x) - g(x), f(z)\| \leq \varepsilon\|x, z\|^{\alpha+\beta},$$

$$g(x + y) + g(x - y) = 2g(x) + 2g(y),$$

for all $x, y, z \in E$, where $\varepsilon = 2^{-(\alpha+\beta)}\delta\frac{1}{1-2^{2-\alpha-\beta}}$. □

From the above Theorem, we get several direct corollaries in the following:

Corollary 2.2. *Let E be a linear 2-normed space and F a 2-Banach space. The dimension of E and F is greater than one, and let $f : E \to F$ be a mapping. If there exists $\delta \geq 0$ and $\alpha \neq 1$ such that*

$$\|f(x + y) + f(x - y) - 2[f(x) + f(y)], f(z)\| \leq \delta\|x, z\|^\alpha\|y, z\|^\alpha,$$

for all $x, y, z \in E$, then there is a unique nonlinear mapping $g : E \to F$ such that

$$g(x + y) + g(x - y) = 2[g(x) + g(y)]$$

and

$$\|f(x) - g(x), f(z))\| \leq \varepsilon\|x, z\|^{2\alpha},$$

for all $x, y, z \in E$, where

$$\varepsilon = \begin{cases} \frac{\delta}{4-4^\alpha} & \alpha < 1, \\ 4^\alpha\frac{\delta}{4^\alpha-4} & \alpha > 1. \end{cases}$$

Corollary 2.3. *Let E be a linear 2-normed space and F a 2-Banach space. The dimension of E and F is greater than one, and let $f : E \to F$ be a mapping. If there exists $\delta \geq 0$ such that*

$$\|f(x+y) + f(x-y) - 2[f(x) + f(y)], f(z)\| \leq \delta$$

for all $x, y, z \in E$, then there is a unique nonlinear mapping $g : E \to F$ such that

$$g(x+y) + g(x-y) = 2[g(x) + g(y)]$$

and

$$\|f(x) - g(x), f(z))\| \leq \varepsilon,$$

for all $x, y, z \in E$ where $\varepsilon = \frac{\delta}{3}$

Following Th.M. Rassias and P. Semrl [8], a function $H : [0, \infty)^2 \to [0, \infty)$ is called *homogeneous of degree p* if it satisfies $H(tu, tv) = t^p H(u, v)$ for all $t, u, v \in [0, \infty)$.

Theorem 2.4. *Let E be a linear 2-normed space and F a 2-Banach space. The dimension of E and F is greater than one, and H is a monotonically increasing symmetric homogeneous function of degree $p \geq 0, p \neq 2$. Let a function $f : E \to F$ satisfy the inequality*

$$\|f(x+y) + f(x-y) - 2[f(x) + f(y)], f(z)\| \leq H(\|x, z\|, \|y, z\|), \qquad (**)$$

for some $\delta \geq 0$ and for all $x, y, z \in E$. Then there is a unique quadratic function $g : E \to F$ such that

$$\|f(x) - g(x), f(z))\| \leq \frac{1}{4 - 2^p} H(1, 1)\|x, z\|^p,$$

for all $x, y, z \in E$.

Proof. Case I. $p < 2$.
We first claim that for any $n \in \mathbb{N}$ and all $x, y, z \in E$

$$\|f(x) - 4^{-n} f(2^n x), f(z)\| \leq 4^{-1} H(1, 1)\|x, z\|^p \sum_{i=0}^{n-1} 2^{i(p-2)}. \qquad (2.2)$$

It is clear that the assumption is true for $n = 1$. Assume now that the assumption is true for the case n, we want to show that the assumption is true for the case $(n+1)$.

$$\begin{aligned}
&\|f(x)-4^{-n-1}f(2^{n+1}x),f(z)\|\\
&\quad\le \|f(x)-4^{-n}f(2^nx),f(z)\|+\|4^{-n}f(2^nx)-4^{-n-1}f(2^{n+1}x),f(z)\|\\
&\quad\le 4^{-1}H(1,1)\|x,z\|^p\sum_{i=0}^{n-1}2^{i(p-2)}+4^{-1}H(1,1)\|x,z\|^p2^{n(p-2)}\\
&\quad= 4^{-1}H(1,1)\|x,z\|^p\sum_{i=0}^{n}2^{i(p-2)}.
\end{aligned}$$

Thus the assumption is true for any positive integer n. It follows that

$$\|f(x)-4^{-n}f(2^nx),f(z)\|<\frac{1}{4-2^p}H(1,1)\|x,z\|^p.$$

Put $g_n(x)=\frac{f(2^nx)}{4^n}$, for $n>m>0$ we have

$$\begin{aligned}
g_m(x)-g_n(x) &= \frac{f(2^mx)}{4^m}-\frac{f(2^nx)}{4^n}\\
&= \frac{f(2^mx)-4^{m-n}f(2^{n-m}\cdot 2^mx)}{4^m}.
\end{aligned}$$

we can apply (2.2) to get

$$\begin{aligned}
\|g_m(x)-g_n(x),f(z)\| &= \|\frac{f(2^mx)-4^{m-n}f(2^{n-m}\cdot 2^mx)}{4^m},f(z)\|\\
&\le 4^{-1}H(1,1)\|x,z\|^p\sum_{i=m}^{n-m-1}2^{i(p-2)}.
\end{aligned}$$

It follows that for each $x\in E,\{g_n(x)\}_n$ is a Cauchy sequence in F. Since F is a 2-Banach space, there is a limit

$$g(x)=\lim_{n\to\infty}g_n(x)$$

which satisfies

$$\|f(x)-g(x),f(z)\|\le\frac{1}{4-2^p}H(1,1)\|x,z\|^{p-2}$$

for all $x,z\in E$. From $(**)$, we get

$$\|f(2^nx+2^ny)+f(2^nx-2^ny)-2[f(2^nx)+f(2^ny)],f(z)\|\le H(\|2^nx,z\|,\|2^ny,z\|)$$

and

$$\begin{aligned}
&\|g_n(x+y)+g_n(x-y)-2[g_n(x)+g_n(y)],f(z)\|\\
&= \frac{1}{4^n}\|\|\|f(2^nx+2^ny)+f(2^nx-2^ny)-2[f(2^nx)+f(2^ny)],f(z)\\
&\le 2^{n(\alpha-2)}H(\|x,z\|,\|y,z\|).
\end{aligned}$$

Letting $n \to \infty$, we obtain $g(x+y)+g(x-y)=2g(x)+2g(y)$.

Next we show the uniqueness of g. Assume that there exists another quadratic function $h : E \to F$ satisfy the assumption, then there exists a monotonically increasing symmetric homogeneous function H' of degree $p \geq 0, p \neq 2$ such that

$$\|f(x)-h(x), f(z))\| \leq \frac{1}{4-2^p} H'(1,1)\|x,z\|^p,$$

for all $x, y, z \in E$. Since $g(x), h(x)$ are quadratic function, we have

$$g(nx)=n^2 g(x), \quad h(nx)=n^2 h(x).$$

Hence

$$\begin{aligned}\|g(x)-h(x), f(z)\| &= \frac{1}{n^2}\|g(nx)-h(nx), f(z)\| \\ &\leq \frac{1}{n^2}(\|f(nx)-g(nx), f(z)\|+\|f(nx)-h(nx), f(z)\|) \\ &\leq n^{p-2}\frac{1}{4-2^p}\|x,z\|^p[H(1,1)+H'(1,1)].\end{aligned}$$

Thus

$$\lim_{n\to\infty}\|g(x)-h(x), f(z)\| = 0, \forall x, z \in E.$$

Since f is surjective, $g(x) \equiv h(x)$ for all $x \in E$.

Case II. $p > 2$.

Similarly we can prove the induction assumption

$$\|f(x)-4^n f(2^{-n}x), f(z)\| \leq H(1,1)\|x,z\|^p(2^{-p}+\sum_{i=1}^{n-1} 2^{i(2-p)}),$$

for all $x, z \in E$ and any positive integer n. Put $g_n(x)=4^n f(2^{-n}x)$ then we claim that

$$\|f(x)-g_n(x), f(z)\| \leq H(1,1)\|x,z\|^p \sum_{i=0}^{n-1} 2^{i(2-p)-p}.$$

From the same argument as above we can show that sequence $\{g_n(x)\}_n$ is a Cauchy sequence in 2-Banach space F, and the limit function

$$g(x)=\lim_{n\to\infty} g_n(x)$$

is the unique quadratic function g satisfying

$$\|f(x)-g(x), f(z)\| \leq H(1,1)\|x,z\|^p \frac{1}{2^p-4},$$

for all $x, y, z \in E$. □

Inspired by the idea of G. Isac and Th.M. Rassias [5], we give another generalization of Theorem 2.1.

Theorem 2.5. *Let E be a linear 2-normed space and F a 2-Banach space. The dimension of E and F is greater than one, and function $\psi : [0,\infty) \to [0,\infty)$ satisfy the following conditions:*

(1) $\lim_{t\to\infty} \frac{\psi(t)}{t^2} = 0$;
(2) $\psi(ts) \leq \psi(t)\psi(s), \forall s,t \in [0,\infty)$;
(3) $\psi(t) < 2t, \forall t \in [0,\infty)$.

If a surjective mapping $f : E \to F$ satisfies the inequality

$$\|f(x+y) + f(x-y) - 2[f(x) + f(y)], f(z)\| \leq \theta(\psi(\|x,z\|) + \psi(\|y,z\|)),$$

for some $\theta \geq 0$ and for all $x, y, z \in E$, then there exists a unique quadratic function $g : E \to F$ such that

$$\|f(x) - g(x), f(z)\| \leq \theta\psi(\|x\|)\frac{2}{4-\psi(2)}, \ \forall x \in E. \tag{2.3}$$

Proof. We will prove first that

$$\|f(x) - \frac{1}{4^n}f(2^n x), f(z)\| \leq \frac{\theta}{2}\psi(\|x\|)\sum_{i=0}^{n-1}(\frac{\psi(2)}{4})^i, \tag{2.4}$$

for any $n \in \mathbb{N}$ and all $x, z \in E$. The proof of (2.4) follows by induction on n. Put $x = y$ in (2.3) and dividing by 4 yielding the validity of (2.4) for $n = 1$. Assume now that (2.4) holds for n, we want to prove it for the case $n+1$. Using the triangle inequality, we get

$$\begin{aligned}
&\|f(x) - \frac{1}{4^{n+1}}f(2^{n+1}x), f(z)\| \\
&= \|f(x) - \frac{1}{4^n}f(2^n x), f(z)\| + \frac{1}{4^n}\|f(2^n x) - \frac{1}{4}f(2^{n+1}x), f(z)\| \\
&\leq \frac{\theta}{2}\psi(\|x\|)\sum_{i=0}^{n-1}(\frac{\psi(2)}{4})^i + \frac{\theta}{2}\psi(\|x\|)(\frac{\psi(2)}{4})^n \\
&= \frac{\theta}{2}\psi(\|x\|)\sum_{i=0}^{n}(\frac{\psi(2)}{4})^i.
\end{aligned}$$

Which ends the proof of (2.4). It follows that

$$\|f(x) - \frac{1}{4^n}f(2^n x), f(z)\| \leq \theta\psi(\|x\|)\frac{1}{2-\psi(2)}, \ \forall n \in \mathbb{N}.$$

For $m > n > 0$ we obtain

$$\begin{aligned}
&\|\frac{1}{4^m}f(2^m x) - \frac{1}{4^n}f(2^n x), f(z)\| \\
&= \frac{1}{4^n}\|\frac{1}{4^{m-n}}f(2^{m-n}2^n x) - f(2^n x), f(z)\| \\
&\leq \theta\frac{1}{4^n}\psi(\|2^n x\|)\sum_{i=0}^{m-n-1}(\frac{\psi(2)}{4})^i \\
&\leq \theta(\frac{\psi(2)}{4})^n\psi(\|x\|)\sum_{i=0}^{m-n-1}(\frac{\psi(2)}{4})^i \\
&= \theta\psi(\|x\|)\sum_{i=n}^{m-1}(\frac{\psi(2)}{4})^i \\
&< \theta\psi(\|x\|)\frac{4}{4-\psi(2)}(\frac{\psi(2)}{4})^n.
\end{aligned}$$

By (3), we get that the sequence $\{4^{-n}f(2^n x)\}$ is a Cauchy sequence and convergent, since F is complete. Set $g(x) = \lim_{n\to\infty} 4^{-n}f(2^n x)$, $\forall x \in E$. We will prove that $G(x)$ is a quadratic function. It then follows from the assumption that

$$\begin{aligned}
\|f(2^n x + 2^n y) &+ f(2^n x - 2^n y) + 2(f(2^n x) + f(2^n y)), f(z)\| \\
&\leq \theta(\psi(\|2^n x, z\|) + \psi(\|2^n y, z\|)) \\
&\leq \theta(\psi(2))^n(\psi(\|x, z\|) + \psi(\|y, z\|)).
\end{aligned}$$

Which implies that

$$\begin{aligned}
\frac{1}{4^n}\|f(2^n x + 2^n y) &+ f(2^n x - 2^n y) + 2(f(2^n x) + f(2^n y)), f(z)\| \\
&\leq \theta(\frac{\psi(2)}{4})^n(\psi(\|x, z\|) + \psi(\|y, z\|)).
\end{aligned}$$

Letting $n \to \infty$, since f is a surjective mapping, we conclude that $g(x)$ is a quadratic function. We claim that $g(x)$ is the unique such quadratic function. Suppose there exists another $h : E \to F$ satisfying

$$\|f(x) - g(x), f(z)\| \leq \theta'\psi'(\|x\|)\frac{2}{4-\psi'(2)},\ \forall x \in E,$$

where $\theta' \geq 0$ is a constant and $\psi' : [0, \infty) \to [0, \infty)$ satisfies the conditions in the assumption. Then

$$\begin{aligned} \|g(x) - h(x), f(z)\| &= \frac{1}{n^2}(\|g(nx) - h(nx), f(z)\|) \\ &\leq \frac{1}{n^2}(\|g(nx) - f(nx), f(z)\| + \|h(nx) - f(nx), f(z)\|) \\ &\leq \frac{\psi(n)}{n^2}\theta\psi(\|x\|)\frac{2}{4-\psi(2)} + \frac{\psi'(n)}{n^2}\theta'\psi'(\|x\|)\frac{2}{4-\psi'(2)}. \end{aligned}$$

In view of (1) and the last inequality, we conclude that $g(x) = h(x)$. □

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References

[1] P. W. Cholewa, *Remarks on the stability of functional equations*, Aequationes math., **27** (1984), 76-86.

[2] S. Czerwik, *On the stability of the quadratic mapping in normed spaces*, Abh. Math. Sem. Univ. Hamburg., **62** (1992), 59-64.

[3] Z. Gajda, *On stability of additive mappings*, Internat. J. Math. Math. Sci., **14** (1991), 431-434.

[4] D. H. Hyers,*On the stability of the linear functional equation*, Proc.Nat. Acad. Sci., U.S.A., **27** (1941), 222-224.

[5] G. Isac and Th.M. Rassias, *On the Hyers-Ulam stability of $\psi-$ additive mappings*, J. Approx. Theory, **72** (1993), 131-137.

[6] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., **72** (1978), 297-300.

[7] Th. M. Rassias, *Communication, 27th International Symposium on Functional Equations*, Bielsko-Biala, Katowice, Krokow, Poland, 1989.

[8] Th. M. Rassias and P. Semrl, *On the behavior of mappings which do not satisfy Hyers-Ulam stability*, Proc. Amer. Math. Soc., **114** (1992), 989-993.

[9] Th. M. Rassias, *On the stability of functional equations and a problem of Ulam*, Acta Appl. Math., **62** (2000), 23-130.

[10] F. Skof, *Proprieta locali e approssimazione di operatori*, Rend.Sem.Mat.Fis. Milano., **53** (1983), 113-129.

[11] S. M. Ulam, *Problems in modern mathematics*, Chapter VI, Science Editions, Wiley, New York, 1960.

[12] A. White, *2-Banach space*, Math. Nachr., **42** (1969), 44-60.