Nonlinear Functional Analysis and Applications
Vol. 16, No. 1 (2011), pp. 1-8

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

NONLINEAR INCLUSION PROBLEMS FOR ORDERED RME SET-VALUED MAPPINGS IN ORDERED HILBERT SPACES

Hong-gang Li

Institute of Applied Mathematics Research
Chongqing University of Posts and TeleCommunications
Chongqing 400065, China
e-mail: lihg12@126.com

Abstract. A class of nonlinear inclusion problems for ordered RME set-valued mappings are introduced and studied in ordered Hilbert spaces. By using the resolvent operator associated with RME set-valued mappings, an existence theorem of solutions for this kinds of nonlinear inclusion problems for ordered extended set-valued mappings is established, an approximation algorithm is suggested, and the relation between the first valued x_0 and a solution in the nonlinear inclusion problems is discussed. In this field, the results in the instrument are obtained.

1. Introduction

Let X be a real ordered Hilbert space with a norm $\|\cdot\|$, an inner product $\langle\cdot,\cdot\rangle$, zero θ, and a partial ordered relation $\leq$ defined by the normal cone $\mathbf{P}$ of X [5]. Let $M : X \to 2^X$ be an ordered RME set-valued mapping. We consider the following problem:

Find $x \in X$ such that

$$0 \in M(x). \tag{1.1}$$

The problem (1.1) is called a class of nonlinear inclusion problems for ordered RME set-valued mappings in ordered Hilbert spaces.

Remark 1.1. For a suitable choice of M and the space X, some known classes of variational inclusions and variational inequalities in ([5], [6]) can be obtained

[0]Received July 13, 2009. Revised August 25, 2010.
[0]2000 Mathematics Subject Classification: 49J40, 47H06.
[0]Keywords: Nonlinear inclusion problems, order RME mapping, set-valued mapping, ordered Hilbert space.

as special cases of the nonlinear inclusion problems (1.1):

(i) If M is a single-valued mapping and $M(x) = A(g(x))$, then the problem (2.1) in [5] can be obtained as special case of the problem (1.1).
(ii) If M is a single-valued mapping and $M(x) = A(x) \oplus F(x, g(x))$, then the problem (1.1) in [6] is as same as the problem (1.1).

In recent years, the fixed point theory and application for the nonlinear operators have been intensively studied in ordered Banach spaces ([2], [3], [4]). And very recently, the author have introduced and studied the approximation algorithm and the approximation solution for a class of generalized nonlinear ordered variational inequalities and ordered equations(or, a new class of generalized nonlinear ordered variational inequalities and ordered equations) in ordered Banach spaces ([5], [6]).

Inspired and motivated by recent research works in this field, a class of nonlinear inclusion problems for ordered (λ, β)-monotone extended set-valued mappings are introduced and studied in order Hilbert spaces. By using the resolvent operator [1] associated with ordered RME set-valued mapping, an existence theorem of solutions for this kind nonlinear inclusion problems for ordered extended set-valued mappings is established, a approximation algorithm is suggested, and the relation of between the first valued x_0 and a solution in the nonlinear inclusion problems is discussed. The results in the instrument are obtained in the field. For details, we refer the reader to [1-10] and the references therein.

1.1. **Preliminaries.**

Let X be a real ordered Hilbert space with a norm $\|\cdot\|$, an inner product $\langle\cdot,\cdot\rangle$, zero θ, and a partial ordered relation $\leq$ defined by the normal cone $\mathbf{P}$ [5], the N be the normal constant of $\mathbf{P}$, for arbitrary $x, y \in X$, $lub\{x, y\}$ and $glb\{x, y\}$ express the least upper bound of the set $\{x, y\}$ and the greatest lower bound of the set $\{x, y\}$ on the partial ordered relation $\leq$, respectively. Here, suppose $lub\{x, y\}$ and $glb\{x, y\}$ always exist. Let us recall some concepts and results.

Definition 1.2. [9] *Let X be an ordered Hilbert space, and $\boldsymbol{P}$ be a cone of X. $\leq$ is a partial ordered relation defined by the cone $\boldsymbol{P}$, for $x, y \in X$, if holds $x \leq y$(or $y \leq x$), then x and y is said to be comparison between each other(denoted by $x \propto y$ for $x \leq y$ and $y \leq x$).*

Proposition 1.3. [2] *If $x \propto y$, then $lub\{x, y\}$ and $glb\{x, y\}$ exist, $x-y \propto y-x$, and $\theta \leq (x-y) \vee (y-x)$.*

Proposition 1.4. [2] *If for any natural number n, $x \propto y_n$, and $y_n \to y^*(n \to \infty)$, then $x \propto y^*$.*

Proposition 1.5. *Let X be an ordered Hilbert space, $\oplus$ be a XOR operator* [5], *and both α and β be tow reals. If for $x, y, u, v \in X$, defined by $x \odot y = (x - y) \wedge (y - x)$, then we have the following relations:*

(1) $x \odot y = y \odot x$;
(2) $x \odot x = \theta$;
(3) $x \odot \theta \leq \theta, \quad if \quad x \propto \theta$;
(4) $x \oplus y = -((-x) \odot (-y))$;
(5) $x \odot y = -((-x) \oplus (-y))$;
(6) $x \oplus y = (-x) \oplus (-y)$;
(7) $x \odot y = (-x) \odot (-y)$;
(8) $(x + y) \odot (u + v) \geq (x \odot u) + (y \odot v)$;
(9) $(x + y) \odot (u + v) \geq (x \odot v) + (y \odot u)$;
(10) $\alpha x \oplus \beta x = |\alpha - \beta| x, \quad if \quad x \propto \theta$.

Proof. This directly follows from the definitions of the $\vee$, $\wedge$, $\oplus$ and $\odot$. □

Obviously, if M is a comparison mapping, then $M(x) \propto I$ for all $x \in X$.

1.2. **Ordered RME Mappings in Ordered Hilbert Spaces.**

Definition 1.6. *Let X be a real ordered Hilbert space, $M : X \to 2^X$ be a set-valued mapping and $M(x)$ be a nonempty closed subset in X, and $g : X \to X$ be a single-valued mapping.*

(i) *M is said to be a comparison mapping, if for any $v_x \in M(x)$, $x \propto v_x$, and if $x \propto y$, then for any $v_x \in M(x)$ and any $v_y \in M(y)$, $v_x \propto v_y (\forall x, y \in X)$.*

(ii) *M is said to be a comparison mapping with respect to g, if for any $v_x \in M(g(x))$, $x \propto v_x$, and if $x \propto y$, then for any $v_x \in M(g(x))$ and any $v_y \in M(g(y))$, $v_x \propto v_y (\forall x, y \in X)$.*

(iii) *a comparison mapping M is said to be ordered rectangular, if for each $x, y \in X$, $u_x \in M(x)$, and $u_y \in M(y)$ the following relation:*

$$\langle u_x \odot u_y, -(x \oplus y) \rangle = 0$$

holds.

(iv) *a comparison mapping M is said to be λ-ordered monotone, if there exists a constant $\lambda > 0$ such that*

$$\lambda(v_x - v_y) \geq x - y \quad \forall x, y \in X, \quad v_x \in M(x), \quad v_y \in M(y).$$

(v) *a comparison mapping M is said to be β-ordered extended, if there exists a constant $\beta > 0$ such that*

$$\beta(x \oplus y) \leq v_x \oplus v_y \quad \forall x, y \in X, \quad v_x \in M(x), \quad v_y \in M(y).$$

(vi) *a comparison mapping M with respect to $J_{M,\lambda}$ is said to be ordered RME with respect to $J_{M,\lambda}$, if M is rectangular and λ-ordered monotone with respect to $J_{M,\lambda}$ and β-ordered extended, and $(I+\lambda M)(X) = X$ for $\lambda, \beta > 0$.*

Obviously, if M is a comparison mapping, then $M(x) \propto I$ for all $x \in X$.

2. Main Results

2.1. Propositions of Resolvent Operator $J_{M,\lambda}$.

Lemma 2.1. *If M is a rectangular mapping, then a inverse mapping $J_{M,\lambda} = (I+\lambda M)^{-1} : X \to 2^X$ of $(I+\lambda M)$ is single-valued for $\lambda > 0$, where I is the identity mapping on X.*

Proof. Let $u \in X$, and x and y be two elements in $(I+\lambda M)^{-1}(u)$. Then, it follows that $u - I(x) \in \lambda M(x)$ and $u - I(y) \in \lambda M(y)$. From Lemma 2.1 in [5], we have

$$\begin{aligned}\frac{1}{\lambda}(u - I(x)) \odot \frac{1}{\lambda}(u - I(y)) &= \frac{1}{\lambda}(x \odot y) \\ &= -\frac{1}{\lambda}(x \oplus y).\end{aligned}$$

Since M is an rectangular mapping,

$$\begin{aligned}\langle \frac{1}{\lambda}(u - I(x)) \odot \frac{1}{\lambda}(u - I(y)), -(x \oplus y)\rangle &= \langle -\frac{1}{\lambda}(x \oplus y), -(x \oplus y)\rangle \\ &= \frac{1}{\lambda}\|x \oplus y\|^2 \\ &= 0.\end{aligned}$$

It follows that $x = y$. Thus $(I+\lambda M)^{-1}(u)$ is a single-valued mapping. The proof is completed. □

Definition 2.2. *Let X be a real ordered Hilbert space, $\boldsymbol{P}$ be a normal cone with normal constant N in X, and M be a rectangular mapping. The resolvent operator $J_{M,\lambda} : X \to X$ of the $M(x)$ is defined by*

$$J_{M,\lambda}(x) = (I+\lambda M)^{-1}(x)$$

for all $x \in X$, where $\lambda > 0$ is a constant.

Theorem 2.3. *Let X be an ordered Hilbert space, $\boldsymbol{P}$ be a normal cone with the normal constant N in X, $\leq$ be an ordered relation defined by the cone $\boldsymbol{P}$, the operator $\oplus$ be XOR. If $M : X \to 2^X$ is a rectangular, a comparison and λ-ordered monotone mapping with respect to $J_{M,\lambda}$, then the resolvent operator $J_{M,\lambda} : X \to X$ is comparison* [5].

Proof. Since $M : X \to 2^X$ is a comparison mapping with respect to $J_{M,\lambda}$ so that $x \propto J_{M,\lambda}(x)$. For any $x, y \in X$, let $x \propto y$, and $v_x = \frac{1}{\lambda}(x - J_{M,\lambda}(x)) \in M(J_{M,\lambda}(x))$ and $v_y = \frac{1}{\lambda}(y - J_{M,\lambda}(y)) \in M(J_{M,\lambda}(y))$. Setting

$$v_x - v_y = \frac{1}{\lambda}(x - y + J_{M,\lambda}(y) - J_{M,\lambda}(x)),$$

by using the λ-ordered monotonicity of M, we have

$$0 \leq \lambda(v_x - v_y) - (x - y) = J_{M,\lambda}(y) - J_{M,\lambda}(x)),$$

and $\lambda(v_x - v_y) - (x - y) \in \mathbf{P}$. Therefor $J_{M,\lambda}(y) \propto J_{M,\lambda}(x)$ [5]. □

Theorem 2.4. *Let X be an ordered Hilbert space, $\boldsymbol{P}$ be a normal cone with the normal constant N in X, $\leq$ be an ordered relation defined by the cone $\boldsymbol{P}$. If $M : X \to 2^X$ is an ordered RME set-valued mapping with respect to $J_{M,\lambda}$, then for the resolvent operator $J_{M,\lambda} : X \to X$, the following relation:*

$$(\lambda\beta - 1)J_{M,\lambda}(x) \oplus J_{M,\lambda}(y) \leq (x \oplus y) \tag{2.1}$$

holds.

Proof. Let M be an ordered RME set-valued mapping with respect to $J_{M,\lambda}$, and set $u_x = J_{M,\lambda}(x)$, $v_x = \frac{1}{\lambda}(x - J_{M,\lambda}(x))$, $u_y = J_{M,\lambda}(y)$ and $v_y = \frac{1}{\lambda}(y - J_{M,\lambda}(y))$. Then for $x, y \in X$ and $\lambda > 0$, $v_x \in M(J_{M,\lambda}(x))$, $v_y \in M(J_{M,\lambda}(y))$ and $J_{M,\lambda}$ is a comparison mapping.

Since $M : X \to 2^X$ is a λ-ordered monotone mapping with respect to $J_{M,\lambda}$ and β-ordered extended set-valued mapping, and $x \propto y$ so that for $x, y \in X, v_x \in M(u_x), v_y \in M(u_y)$, we have $u_x \propto u_y$, $v_x \propto v_y$. By (8) of the Proposition 1.5, we have

$$\begin{aligned} v_x \oplus v_y &= \frac{1}{\lambda}(x - u_x) \oplus \frac{1}{\lambda}(y - u_y) \\ &\leq \frac{1}{\lambda}(u_x \oplus u_y + (x \oplus y)). \end{aligned}$$

Therefore, it follows from $\beta(u_x \oplus u_y) \leq (v_x \oplus v_y)$ that

$$(\lambda\beta - 1)J_{M,\lambda}(x) \oplus J_{M,\lambda}(y) \leq (x \oplus y).$$

The proof is completed. □

2.2. **Convergence of Approximation Sequence for Solving the Problem(1.1).**

In this section, we will show the convergence of the approximation sequences for finding a solution of the problem (1.1), and discuss the relation between the first valued x_0 and a solution in the problem (1.1).

Lemma 2.5. *Let X be an ordered Hilbert space, $\boldsymbol{P}$ be a normal cone with the normal constant N in X, $\leq$ be an ordered relation defined by the cone $\boldsymbol{P}$, the operator $\oplus$ be XOR. If $M : X \to 2^X$ is a rectangular mapping, then the inclusion problem (1.1) has a solution x^* if and only if $x^* = J_{M,\lambda}(x^*)$ in X.*

Proof. This directly follows from the definition of the resolvent operator $J_{M,\lambda}$ of $M(x)$. □

Theorem 2.6. *Let X be an ordered Hilbert space, $\boldsymbol{P}$ be a normal cone with the normal constant N in X, $\leq$ be an ordered relation defined by the cone $\boldsymbol{P}$. If $M : X \to 2^X$ is a RME set-valued mapping, then a sequence $\{x_n\}$ convergence strongly to x^* solution of problem (1.1) for $\beta > \frac{2}{\lambda} > 0$, which is generated by following algorithm:*

For any given $x_0 \in X$, let $x_1 = J_{M,\lambda}(x_0)$, and for $n > 0$, set

$$x_{n+1} = J_{M,\lambda}(x_n).$$

And also, we have

$$\|x^* - x_0\| \leq \frac{2 + N - \lambda\beta}{\lambda\delta}\|J_{M,\lambda}(x_0) - x_0\|, \quad (\forall x_0 \in X). \tag{2.2}$$

Proof. For any $x_0 \in X$, let $x_1 = J_{M,\lambda}(x_0)$. By λ-monotonicity of M, $(I + \lambda M)(X) = X$, the comparison of $J_{M,\lambda}$, and Theorem 2.3, we know that $x_1 \propto x_0$. Further, we can obtain a sequence $\{x_n\}$, and $x_{n+1} \propto x_n$(where $n = 0, 1, 2, \cdots$). Using the Lemma 2.8 in [5], the Theorem 2.4 and condition $\beta > \frac{2}{\lambda} > 0$, we have

$$\begin{aligned} \theta &\leq x_{n+1} \oplus x_n \\ &\leq J_{M,\lambda}(x_n) \oplus J_{M,\lambda}(x_{n-1}) \\ &\leq \frac{1}{\lambda\beta - 1}(x_n \oplus x_{n-1}). \end{aligned} \tag{2.3}$$

By Lemma 2.5 and Definition 2.2 in [5], we obtain

$$\|x_n - x_{n-1}\| \leq \delta^n N\|x_1 - x_0\|, \tag{2.4}$$

where $\delta = \frac{1}{\lambda\beta - 1}$. Hence, for any $m > n > 0$, we have

$$\begin{aligned} \|x_m - x_n\| &\leq \sum_{i=n}^{m-1} \|x_{i+1} - x_i\| \\ &\leq N\|x_1 - x_0\| \sum_{i=n}^{m-1} \delta^i. \end{aligned}$$

It follows from the condition $\beta > \frac{2}{\lambda} > 0$ that $0 < \delta < 1$ and $\|x_m - x_n\| \to 0$, as $n, m \to \infty$, and hence the sequence $\{x_n\}$ is a Cauchy sequence in complete space X. We can set $x_n \to x^*$ as $n \to \infty (x^* \in X)$. By the conditions, we have

$$\begin{aligned} x^* &= \lim_{n\to\infty} x_{n+1} \\ &= \lim_{n\to\infty} J_{M,\lambda}(x_n) \\ &= J_{M,\lambda}(x^*). \end{aligned}$$

We know that x^* is a solution of the inclusion problem (1.1) by Lemma 2.5. From Lemma 2.6 in [5] and (2.2), we have $(J_{M,\lambda}(x_n)) \propto x^*(n = 0, 1, 2, \cdots)$ and

$$\begin{aligned} \|x^* - x_0\| &= \lim_{n\to\infty} \|x_n - x_0\| \\ &\leq \lim_{n\to\infty} \sum_{i=1}^{n} \|x_{i+1} - x_i\| \\ &\leq \lim_{n\to\infty} N \sum_{i=2}^{n} \delta^{n-1} \|x_1 - x_0\| + \|x_1 - x_0\| \\ &\leq \frac{2 + N - \lambda\beta}{\lambda\delta} \|J_{M,\lambda}(x_0) - x_0\|. \end{aligned}$$

This completes the proof. □

Remark 2.7. Though the method of solving problem by the resolvent operator is as same in [8], or [10] for nonlinear inclusion problem, but the character of ordered RME set-valued mapping is difference from one of (A, η)-accretive mapping [8], or monotone mapping [10].

References

[1] Y. P. Fang and N. J. Huang, H-accretive operators and resolvent operator technique for solving variational inclusions in Banach spaces, Appl. Math. Lett., **17**(6)(2004), 647-653.

[2] Y. H. Du, Fixed points of incresing operators in ordered Banach spaces and applications, Appl. Anal., **38**(1990), 1-20.

[3] D. J. Ge, Fixed points of mixed monotone operators with applications, Appl. Anal., **31**(1988), 215-224.

[4] D. J. Ge, V. Lakshmikantham, Couple fixed points of nonlinear operators with applictions, Nonlinear Anal. TMA,**11**(1987), 623-632.

[5] H. G. Li, Approximation Solution For Generalized Nonlinear Ordered Variational Inequality And Ordered Equation In Ordered Banach Space, Nonlinear Analysis Forum, **13(2)**(2008), 205-214.

[6] H. G. Li, Approximation Solution For A New Class Of General Nonlinear Ordered Variational Inequalities And Ordered Equations In Ordered Banach Spaces, Nonlinear Analysis Forum, **14(1)**(2009), 89-97.

[7] H. G. Li, Iterative Algorithm for A New Class of Generalized Nonlinear Fuzzy Set-Valude Variational Inclusions Involving (H, η)-monotone Mappings, Advances in Nonl. Vari. Ineq., **10(1)**(2007), 89-100.

[8] H. Y. Lan, Y. J Cho and R. U. Verma, On nonlinear relaxed cocoercive inclusions involving (A, η)-accretive mappings in Banach spaces, Comput. Math. Appl., **51** (2006), 1529-1538.
[9] H. H. Schaefer, Banach lattices and positive operators, Springer, 1974.
[10] R. U. Verma, A hybrid proximal point algorithm based on the (A, η)-maximal monotonicity framework, Applied Math. Lett., **21**(2008), 142-147.