**Nonlinear Functional Analysis and Applications**
**Vol. 15, No. 4 (2010), pp. 647-655**

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm


# ON THE HYERS-ULAM STABILITY OF THE QUADRATIC AND JENSEN FUNCTIONAL EQUATIONS ON A RESTRICTED DOMAIN

**Manar Youssef[1], Elqorachi Elhoucien[2] and Themistocles M. Rassias[3]**

[1]Department of Mathematics, Ibn Zohr University,
Faculty of Sciences, Agadir, Morocco
E-mail: manaryoussefr32@hotmail.fr

[2]Department of Mathematics, Ibn Zohr University,
Faculty of Sciences, Agadir, Morocco
E-mail: elqorachi@hotmail.com

[3]Department of Mathematics, National Technical University of Athens,
Zografou Campus, 15780, Athens Greece
E-mail: trassias@math.ntua.gr

**Abstract.** The aim of the present paper is to prove the Hyers-Ulam stability of the quadratic functional equation

$$f(x+y)+f(x+\sigma(y))=2f(x)+2f(y), \quad x,y\in E$$

and the Jensen functional equation

$$f(x+y)+f(x+\sigma(y))=2f(x), \quad x,y\in E$$

on a restricted domain, where $\sigma$ is an involution of the normed space $E$.

## 1. Introduction

In 1940, S. M. Ulam [27], during his talk before the mathematics club of the university of Wisconsin raised a question concerning the stability of functional equations: Given a group $(G_1,\cdot)$, a metric group $G_2$ with the metric $d$, a number $\epsilon>0$ and a mapping $f: G_1\to G_2$ which satisfies the inequality

$$d(f(x\cdot y), f(x)f(y))\le \epsilon$$

[0]Received August 30, 2009. Revised May 20, 2010.
[0]2000 Mathematics Subject Classification: 39B72, 39B52.
[0]Keywords: Hyers-Ulam stability, quadratic functional equation, Jensen functional equation.

for all $x, y$ in $G_1$, does there exists a homomorphism $a$: $G_1 \longrightarrow G_2$ and a constant $k > 0$ such that

$$d(a(x), f(x)) \leq k\epsilon$$

for all $x$ in $G_1$?

In 1941, D. H. Hyers [8] has given the following answer to Ulam's question.

**Theorem 1.1.** [8] *Let $f : E_1 \to E_2$ be a mapping between Banach spaces $E_1$, $E_2$ such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon \ \text{ for all } x, y \in E_1,$$

*for some $\epsilon > 0$. Then, there exists exactly one additive mapping $A : E_1 \to E_2$*

$$A(x+y) = A(x) + A(y) \ \text{ for all } x, y \in E_1$$

*such that*

$$\|A(x) - f(x)\| \leq \epsilon \ \text{ for all } x, y \in E_1,$$

*given by the formula*

$$A(x) = \lim_{n \to +\infty} 2^{-n} f(2^n x), \quad x \in E_1.$$

*Moreover, if $f(tx)$ is continuous in $t$ for each fixed $x \in E_1$, then $A$ is linear.*

In 1978, Th. M. Rassias [16] considerably weakened the condition for the bounded of the norm of the Cauchy difference

$$f(x+y) - f(x) - f(y)$$

by allowing growth of the form $\epsilon(\|x\|^p + \|y\|^p)$ for the norm of the Cauchy difference, where $0 \leq p < 1$, and still obtained the formula

$$A(x) = \lim_{n \to +\infty} 2^{-n} f(2^n x)$$

for the additive mapping approximating $f$.

The terminology Hyers-Ulam-Rassias stability originates from his historical background. Since then, a great deal of work has been done by a number of authors in various directions, (see for example [4], [5], [6], [7], [11], [10], [13], [17], [18], [19], [20], [21], [22]).

Concerning the stability of functional equations on restricted domains, F. Skof [24] was the first author to solve the Ulam problem for additive mappings. Given a real normed vector spaces $X$ and $E$, a function $f : X \to E$ will satisfy the functional equation

$$f(x+y) = f(x) + f(y) \ \text{ for all } x, y \in X$$

if and only if

$$\|f(x+y)-f(x)-f(y)\| \to 0 \text{ as } \|x\|+\|y\| \to +\infty.$$

In [9] D. H. Hyers, G. Isac and Th. M. Rassias considered the asymptotical aspect of Hyers-Ulam stability close to the asymptotic derivability.

In [12] S. -M. Jung investigated the Hyers-Ulam stability for quadratic mappings

$$f(x+y)+f(x-y)=2f(x)+2f(y), \quad x,y \in E \tag{1.1}$$

on restricted domain.

**Theorem 1.2.** [12] *Let $d$ and $\delta \geq 0$ be given. Assume that a mapping $f\colon E_1 \longrightarrow E_2$ satisfies the inequality*

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \leq \delta$$

*for all $x$ and $y$ in $E_1$ such that $\|x\|+\|y\| > d$. Then there exists a unique mapping $Q : E_1 \longrightarrow E_2$, solution of equation (1.1) such that*

$$\|f(x)-Q(x)\| \leq \frac{7}{2}\delta$$

*for all $x$ in $E_1$.*

In this paper, we consider the quadratic functional equation

$$f(x+y)+f(x+\sigma(y))=2f(x)+2f(y), \quad x,y \in E \tag{1.2}$$

and the Jensen functional equation

$$f(x+y)+f(x+\sigma(y))=2f(x), \quad x,y \in E \tag{1.3}$$

where $\sigma : E \to E$ is an involution of the normed space $E$, i.e. $\sigma(x+y) = \sigma(x)+\sigma(y)$ and $(\sigma \circ \sigma)(x) = x$ for all $x, y \in E$.

The functional equations (1.2) and (1.3) has been solved by Stetkær [26]. The stability problem for the quadratic equation (1.1) was proved firstly by Skof [25]. In [4] Cholewa extended the skof's result on an abelian group.

Recently the Hyers-Ulam-Rassias stability of the functional equations (1.2) and (1.3) has been investigated in [1], [2] and [3]. We refer also to [14] and [15] for the stability of equation (1.2) and (1.3), respectively. If we look carefully at the proof of the main part of the result obtained in ([1], Theorem 2.1), we easily recognize that the result is true if we replace the abelian group $G$ by a commutative semigroup $S$ not necessarily unitary.

**Theorem 1.3.** [1] *Let $\delta \geq 0$ be given. Assume that a mapping $f\colon E_1 \longrightarrow E_2$ satisfies the inequality*

$$\|f(x+y)+f(x+\sigma(y))-2f(x)-2f(y)\| \leq \delta$$

*for all $x$ and $y$ in $E_1$, where $E_1$ is a commutative semigroup without unity and $E_2$ is a Banach space. Then there exists a unique mapping $Q : E_1 \longrightarrow E_2$, solution of equation (1.2) such that*

$$\|f(x) - Q(x)\| \leq \frac{\delta}{2}$$

*for all $x$ in $E_1$.*

**Theorem 1.4.** [15] *Let $\delta \geq 0$ be given. Assume that a mapping $f\colon E_1 \longrightarrow E_2$ satisfies the inequality*

$$\|f(x+y) + f(x+\sigma(y)) - 2f(x)\| \leq \delta$$

*for all $x$ and $y$ in $E_1$. Then there exists a unique additive mapping $J : E_1 \longrightarrow E_2$, solution of equation (1.3) such that $J(\sigma(x)) = -J(x)$ and*

$$\|f(x) - f(0) - J(x)\| \leq \frac{3}{2}\delta$$

*for all $x$ in $E_1$.*

In the present paper, we establish new theorems about the Hyers-Ulam stability on restricted domains of the functional equations (1.2) and (1.3). Furthermore, we apply the results obtained to the asymptotic behavior of the above functional equations (1.2) and (1.3).

Throughout this paper, let $E$ be a real normed space and $F$ be a real Banach space.

## 2. Stability of equation (1.2) on a restricted domain

In the present section, we investigate the Hyers-Ulam stability of the quadratic functional equation (1.2) on a restricted domain. We present our main result in theorem below.

**Theorem 2.1.** *Let $d > 0$ and $\delta > 0$ be given. Assume that a mapping $f : E \to F$ satisfies the inequalities*

$$\|f(x+y) + f(x+\sigma(y)) - 2f(x) - 2f(y)\| \leq \delta, \tag{2.1}$$

*for all $x, y \in E$ with $\|y\| \geq d$. Then, there exists a unique mapping $q\colon E \longrightarrow F$ solution of equation (1.2) such that*

$$\|f(x) - q(x)\| \leq \frac{7}{2}\delta \tag{2.2}$$

*for all $x \in E$.*

*Proof.* Let $x, y \in E$ with $\|y\| < d$. If $y = 0$, then we have

$$\begin{aligned} & \|f(x+0) + f(x+\sigma(0)) - 2f(x) - 2f(0)\| \\ = & \|2f(0)\| \\ = & \|f(0+0) + f(0-0) - 2f(0) - 2f(0)\|. \end{aligned}$$

So in view of [12], we obtain

$$\|f(x+0) + f(x+\sigma(0)) - 2f(x) - 2f(0)\| \leq \frac{7\delta}{2}.$$

If $y \neq 0$, we choose $z = 2^n y$, with $n \in \mathbb{N}$.

**Case 1**: $\sigma(y) \neq y$. For $n$ large enough, we can easily verify that

$$\|z\| = 2^n \|y\| \geq d,$$

$$\|y + z - \sigma(z)\| = 2^n \|\frac{y}{2^n} + y - \sigma(y)\| \geq d$$

and

$$\|y - \sigma(z)\| = 2^n \|\frac{y}{2^n} - \sigma(y)\| \geq d.$$

Therefore, from (2.1), the triangle inequality and the following decomposition

$$\begin{aligned} & f(x+y) + f(x+\sigma(y)) - 2f(x) - 2f(y) \\ = & -[f(x+y+z-\sigma(z)) + f(x+\sigma(y)+\sigma(z)-z) \\ & -2f(x) - 2f(y+z-\sigma(z))] \\ & +2[f(x+z) + f(x+\sigma(z)) - 2f(x) - 2f(z)] \\ & +[f(x+y) + f(x+\sigma(z)+\sigma(y)-z) - 2f(x+\sigma(z)) - 2f(y-\sigma(z))] \\ & +[f(x+z+y-\sigma(z)) + f(x+\sigma(y)) - 2f(x+z) - 2f(y-\sigma(z))] \\ & -2[f(y-\sigma(z)+z) + f(y) - 2f(y-\sigma(z)) - 2f(z)], \end{aligned} \tag{2.3}$$

we get

$$\|f(x+y) + f(x+\sigma(y)) - 2f(x) - 2f(y)\| \leq 7\delta. \tag{2.4}$$

**Case 2:** $\sigma(y) = y$. By using (2.3), we obtain

$$\begin{aligned} & 2[f(x+y) + f(x+\sigma(y)) - 2f(x) - 2f(y)] \\ = & 2[f(x+z) + f(x+\sigma(z)) - 2f(x) - 2f(z)] \\ & +[f(x+y) + f(x+\sigma(z)+\sigma(y)-z) - 2f(x+\sigma(z)) - 2f(y-\sigma(z))] \\ & +[f(x+z+y-\sigma(z)) + f(x+\sigma(y)) - 2f(x+z) - 2f(y-\sigma(z))] \\ & -2[f(y-\sigma(z)+z) + f(y) - 2f(y-\sigma(z)) - 2f(z)]. \end{aligned}$$

So in view of inequality (2.1), the triangle inequality and for $n$ large enough, we get

$$2\|f(x+y)+f(x+\sigma(y))-2f(x)-2f(y)\| \le 6\delta.$$

Finally, the inequality (2.4) holds true for all $x, y \in E$. By applying Theorem 1.3 one gets that there exists a unique mapping $q : E \to F$ which is a solution of equation (1.2) and satisfies the inequality (2.2). This completes the proof. $\square$

The following corollary holds according to Theorem 2.1 and some computations used by S. -M. Jung in [12].

**Corollary 2.2.** *A mapping $f : E \to F$ is a solution of equation (1.2) if and only if*

$$\sup_{x\in E} \|f(x+y)+f(x+\sigma(y))-2f(x)-2f(y)\| \to 0 \quad \textit{as} \quad \|y\| \to +\infty. \tag{2.5}$$

**Corollary 2.3.** *A mapping $f : E \to F$ is a solution of equation (1.2) if and only if*

$$\|f(x+y)+f(x+\sigma(y))-2f(x)-2f(y)\| \to 0 \quad \textit{as} \quad \|x\|+\|y\| \to +\infty. \tag{2.6}$$

The following result is a generalization of the one obtained in [12]. We notice here that $\{(x,y)\in E^2 : \ \|y\| \ge d\} \subset \{(x,y)\in E^2 : \ \|x\|+\|y\| \ge d\}$.

**Corollary 2.4.** *($\sigma = -I$) Let $d > 0$ and $\delta > 0$ be given. Assume that a mapping $f : E \to F$ satisfies the inequality*

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \le \delta \tag{2.7}$$

*for all $x, y \in E$ with $\|y\| \ge d$. Then, there exists a unique mapping $q$: $E \longrightarrow F$ solution of the quadratic functional equation (1.1) such that*

$$\|f(x)-q(x)\| \le \frac{7}{2}\delta \tag{2.8}$$

*for all $x \in E$.*

**Corollary 2.5.** *A mapping $f : E \to F$ is a solution of (1.1) if and only if*

$$\sup_{x\in E} \|f(x+y)+f(x-y)-2f(x)-2f(y)\| \to 0 \quad \textit{as} \quad \|y\| \to +\infty. \tag{2.9}$$

**Corollary 2.6.** *A mapping $f : E \to F$ is a solution of (1.1) if and only if*

$$\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \to 0 \quad \textit{as} \quad \|x\|+\|y\| \to +\infty. \tag{2.10}$$

**Corollary 2.7.** *($\sigma = I$) Let $d > 0$ and $\delta > 0$ be given. Assume that a mapping $f : E \to F$ satisfies the inequality*

$$\|f(x+y)-f(x)-f(y)\| \le \delta \tag{2.11}$$

*for all $x, y \in E$ with $\|y\| \geq d$. Then, there exists a unique additive mapping $A\colon E \longrightarrow F$ such that*

$$\|f(x) - A(x)\| \leq 7\delta \tag{2.12}$$

*for all $x \in E$.*

**Corollary 2.8.** *A mapping $f : E \to F$ is additive if and only if*

$$\sup_{x \in E} \|f(x+y) - f(x) - f(y)\| \to 0 \quad as \quad \|y\| \to +\infty. \tag{2.13}$$

**Corollary 2.9.** *A mapping $f : E \to F$ is additive if and only if*

$$\|f(x+y) - f(x) - f(y)\| \to 0 \quad as \quad \|x\| + \|y\| \to +\infty. \tag{2.14}$$

## 3. Stability of equation (1.3) on a restricted domain

In this section, we will investigate the stability of Jensen functional equation (1.3) on a restricted domain.

**Theorem 3.1.** *Let $d > 0$ and $\delta > 0$ be given. Assume that a mapping $f : E \to F$ satisfies the inequality*

$$\|f(x+y) + f(x+\sigma(y)) - 2f(x)\| \leq \delta \tag{3.1}$$

*for all $x, y \in E$ with $\|y\| \geq d$. Then, there exists a unique additive mapping $J\colon E \longrightarrow F$ solution of equation (1.3) such that $J(\sigma(x)) = -J(x)$ and*

$$\|f(x) - f(0) - J(x)\| \leq 3\delta \tag{3.2}$$

*for all $x \in E$.*

*Proof.* Let $x, y \in E$ such that $0 < \|y\| < d$. We choose $z = 2^n y$, where $n$ is large enough, so $\|z\| \geq d$, $\|z + y\| \geq d$ and $\|\sigma(z) + y\| \geq d$. From (3.1), the triangle inequality and the following equation

$$\begin{aligned}
& 2[f(x+y) + f(x+\sigma(y)) - 2f(x)] \\
= \; & -[f(x+y+z) + f(x+y+\sigma(z)) - 2f(x+y)] \\
& -[f(x+\sigma(y)+z) + f(x+\sigma(y)+\sigma(z)) - 2f(x+\sigma(y))] \\
& +[f(x+y+z) + f(x+\sigma(y)+\sigma(z)) - 2f(x)] \\
& +[f(x+y+\sigma(z)) + f(x+\sigma(y)+z) - 2f(x)],
\end{aligned}$$

we have

$$\|f(x+y) + f(x+\sigma(y)) - 2f(x)\| \leq 2\delta. \tag{3.3}$$

Finally, the inequality (3.3) holds true for all $x, y \in E$. By applying now Theorem 1.4, we get the rest of the proof. $\square$

**Corollary 3.2.** *A mapping $f : E \to F$ is a solution of (1.3) if and only if*

$$\sup_{x\in E} \|f(x+y) + f(x+\sigma(y)) - 2f(x)\| \to 0 \quad as \quad \|y\| \to +\infty. \tag{3.4}$$

**Corollary 3.3.** *A mapping $f : E \to F$ is a solution of (1.3) if and only if*

$$\|f(x+y) + f(x+\sigma(y)) - 2f(x)\| \to 0 \quad as \quad \|x\| + \|y\| \to +\infty. \tag{3.5}$$

**Corollary 3.4.** *Let $d > 0$ and $\delta > 0$ be given. Assume that a mapping $f : E \to F$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x)\| \le \delta \tag{3.6}$$

*for all $x, y \in E$ with $\|y\| \ge d$. Then, there exists a unique additive mapping J: $E \longrightarrow F$ solution of equation (1.3) with $\sigma(x) = -x$ such that J(-x)=-J(x) and*

$$\|f(x) - f(0) - J(x)\| \le 3\delta \tag{3.7}$$

*for all $x \in E$.*

**Corollary 3.5.** *A mapping $f : E \to F$ is a solution of (1.3) with $\sigma(x) = -x$ if and only if*

$$\sup_{x\in E} \|f(x+y) + f(x-y) - 2f(x)\| \to 0 \quad as \quad \|y\| \to +\infty. \tag{3.8}$$

**Corollary 3.6.** *A mapping $f : E \to F$ is a solution of (1.3) with $\sigma(x) = -x$ if and only if*

$$\|f(x+y) + f(x-y) - 2f(x)\| \to 0 \quad as \quad \|x\| + \|y\| \to +\infty. \tag{3.9}$$

## References

[1] B. Bouikhalene, E. Elqorachi and Th. M. Rassias, On the generalized Hyers-Ulam stability of the quadratic functional equation with a general involution, *Nonlinear Funct. Anal. Appl.,* Vol. **12**, no 2 (2007), 247-262.

[2] B. Bouikhalene , E. Elqorachi and Th. M. Rassias, On the Hyers-Ulam stability of approximately Pexidetr mappings, *Math. Inequ. Appl.,* Vol. **11**, no 4 (2008), 805-818.

[3] A. Charifi, B. Bouikhalene and E. Elqorachi, Hyers-Ulam-Rassias stability of a generalized Pexider functional equation, *Banach J. Math. Anal.,* Vol. **1**, no 2 (2007), 176-185.

[4] P. W. Cholewa, Remarks on the stability of functional equations, *Aequationes Math.* **27** (1984), 76-86.

[5] S. Czerwik, On the stability of the quadratic mapping in normed spaces. *Abh. Math. Sem. Univ. Hamburg.,* **62** (1992), 59-64.

[6] Z. Gajda, On stability of additive mappings, *Internat. J. Math. Math. Sci.,* **14** (1991), 431-434.

[7] P. Găvruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. *J. Math. Anal. Appl.* **184** (1994), 431-436.

[8] D. H. Hyers, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci. U. S. A.* **27** (1941), 222-224.

[9] D. H. Hyers, G. Isac and Th. M. Rassias, On the asymptoticity aspect of Hyers-Ulam stability of mappings, *Proc. Amer. Math. Soc.,* **126** (1998),425-430.

[10] D. H. Hyers G. I. Isac and Th. M. Rassias, Stability of Functional Equations in Several Variables, *Birkhäuser, Basel,* 1998.

[11] D. H. Hyers and Th. M. Rassias, Approximate homomorphisms, *Aequationes Math.,* **44** (1992), 125-153.

[12] S.-M. Jung, On the Hyers-Ulam stability of the functional equation that have the quadratic property, *J. Math. Anal. Appl.,* **222** (1998), 126-137.

[13] S.-M. Jung, Stability of the quadratic equation of Pexider type, *Abh. Math. Sem. Univ. Hamburg,* **70** (2000), 175-190.

[14] S.-M. Jung and Z.-H. Lee, A fixed point approach to the stability of quadratic functional equation with involution, *Fixed Point Theory and Applications,* **2008** (2008), Article ID 732086, 11 pages.

[15] G. H. Kim and S. H. Lee, Stability of the d'Alembert type functional equations, *Nonlinear. Funct. Anal. Appl.,* **9** (2004), 593-604.

[16] Th. M. Rassias, On the stability of linear mapping in Banach spaces, *Proc. Amer. Math. Soc.,* **72** (1978), 297-300.

[17] Th. M. Rassias, On a modified Hyers-Ulam sequence, *J. Math. Anal. Appl.,* **158** (1991), 106-113.

[18] Th. M. Rassias, Functional Equations, Inequalities and Applications, *Kluwer Academic Publishers*, Dordercht, Boston, London, 2003.

[19] Th. M. Rassias, The problem of S. M. Ulam for approximately multiplicative mappings, *J. Math. Anal. Appl.,* **246** (2000), 352-378.

[20] Th. M. Rassias and P. Šemrl, On the behavior of mappings which do not satisfy Hyers-Ulam stability, *Proc. Amer. Math. Soc.,* **114** (1992), 989-993.

[21] Th. M. Rassias and P. Šemrl, On the Hyers-Ulam stability of linear mappings, *J. Math. Anal. Appl.,* **173** (1993), 325-338.

[22] Th. M. Rassias and J. Tabor, Stability of Mappings of Hyers-Ulam Type, *Hardronic Press, Inc., Palm Harbor, Florida*, 1994.

[23] F. Skof, Sull', approssimazione delle applicazioni localmente $\delta$-additive, *Atti. Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur.*, **117** (1983), 377-389.

[24] F. Skof, Approssimazione di funzioni $\delta$-quadratic su dominio restretto, *Atti. Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur.* **118** (1984), 58-70.

[25] F. Skof, local properties and approximations of operators, *Rend. Sem. Math. Fis. Milano.,* **53** (1983), 113-129.

[26] H. Stetkær, Functional equations on abelian groups with involution. *Aequationes Math.,* **54** (1997), 144-172.

[27] S. M. Ulam, A Collection of Mathematical Problems, *Interscience Publ. New York,* 1961. Problems in Modern Mathematics, *Wiley, New York* 1964.